Mid-surface scaling invariance of some bending strain measures

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Abstract

The mid-surface scaling invariance of bending strain measures proposed in [1] is discussed in light of the work of [3].

1 Introduction

This brief note discusses the mid-surface scaling invariance of three nonlinear measures of pure bending strain, as introduced in [1] and physically motivated therein more than 20 years ago, in light of the recent work of [3] where the said invariance is introduced.

It is shown that one of the strain measures introduced in [1] possesses scaling invariance, and the other two are easily modified to have the invariance as well. There has been a recent surge of interest in such matters, as can be seen from the works of [3, 5, 6].

We use the notation of [1]: a shell mid-surface is thought of as a 2D surface in ambient 3D space (the qualification 'mid-surface' will not be used in all instances; it is hoped that the meaning will be clear from the context). Both the reference and deformed shells are parametrized by the same coordinate system $((\xi^\alpha), \alpha = 1, 2)$ (convected coordinates). Points on the reference geometry are denoted generically by $\boldsymbol{X}$ and on the deformed geometry by $\boldsymbol{x}$. The reference unit normal is denoted by $\boldsymbol{N}$ and the unit normal on the deformed geometry by $\boldsymbol{n}$. A subscript comma refers to partial differentiation, e.g. $\frac{\partial()}{\partial \xi^\alpha} = ()_{,\alpha}$. Summation over repeated indices will be assumed. The convected coordinate basis vectors in the reference geometry will be referred to by the symbols $(\boldsymbol{E}_\alpha)$ and those in the deformed geometry by $(\boldsymbol{e}_\alpha)$, $\alpha = 1, 2$, with corresponding dual bases $(\boldsymbol{E}^\alpha), (\boldsymbol{e}^\alpha)$, respectively. A suitable number of dots placed between two tensors represent the operation of contraction, while the symbol $\otimes$ will represent a tensor product. The deformation gradient will be denoted by $\boldsymbol{f} = \boldsymbol{e}_\alpha \otimes \boldsymbol{E}^\alpha$ and admits the right polar decomposition $\boldsymbol{f} = \boldsymbol{r} \cdot \boldsymbol{U}$, where $\boldsymbol{U}(\boldsymbol{X}) : T_{\boldsymbol{X}} \to T_{\boldsymbol{X}}$ and $\boldsymbol{r}(\boldsymbol{X}) : T_{\boldsymbol{X}} \to T_{\boldsymbol{x}}$, where $T_{\boldsymbol{c}}$ represents the tangent space of the shell at the point $\boldsymbol{c}$. The curvature tensor on the deformed shell is denoted as $\boldsymbol{b} = \boldsymbol{n}_{,\beta} \otimes \boldsymbol{e}^\beta$ and that on the undeformed shell as $\boldsymbol{B} = \boldsymbol{N}_{,\beta} \otimes \boldsymbol{E}^\beta$.

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2 Some measures of pure bending and their invariance under mid-surface scaling

In [1] three measures of bending strain were proposed, given by

$$\widetilde{\boldsymbol{K}} = \left(\boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{r}^T \cdot \boldsymbol{n}_{,\beta} - \boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{N}_{,\beta}\right) \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta = \boldsymbol{f}^T \cdot \boldsymbol{b} \cdot \boldsymbol{f} - \boldsymbol{U} \cdot \boldsymbol{B} \tag{1a}$$

$$\check{\boldsymbol{K}} = \left(\boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{r}^T \cdot \boldsymbol{n}_{,\beta} - \frac{1}{2}\left(\boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{N}_{,\beta} + \boldsymbol{E}_\beta \cdot \boldsymbol{U} \cdot \boldsymbol{N}_{,\alpha}\right)\right) \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta$$
$$= \boldsymbol{f}^T \cdot \boldsymbol{b} \cdot \boldsymbol{f} - (\boldsymbol{U} \cdot \boldsymbol{B})_{sym} \tag{1b}$$

$$\overline{\boldsymbol{K}} = \left(\boldsymbol{E}_\alpha \cdot \boldsymbol{r}^T \cdot \boldsymbol{n}_{,\beta} - \boldsymbol{E}_\alpha \cdot \boldsymbol{N}_{,\beta}\right) \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta = \boldsymbol{r}^T \cdot \boldsymbol{b} \cdot \boldsymbol{f} - \boldsymbol{B}. \tag{1c}$$

Equation (1c) was unnumbered in that work, as the main emphasis was to obtain a nonlinear generalization of the Koiter-Sanders-Budiansky bending strain measure [4, 2]; $\widetilde{\boldsymbol{K}}$ is introduced as Equation (8) and $\check{\boldsymbol{K}}$ as Equation (10) in [1].

In [3] a physically natural requirement of invariance of bending strain measure under simple scalings of the form

$$\boldsymbol{x} \to a\boldsymbol{x}, \qquad 0 < a \in \mathbb{R}$$

is introduced (for plates, but the requirement is natural for shells as well) and it is shown that the measures $\widetilde{\boldsymbol{K}}, \check{\boldsymbol{K}}$ are not invariant under such a scaling. The measure $\overline{\boldsymbol{K}}$ is not discussed in [3].

It is straightforward to see that under the said scaling, the deformation gradient scales as

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{r} \cdot \boldsymbol{U} = \boldsymbol{f} \qquad \to \qquad a\boldsymbol{f} = \boldsymbol{r} \cdot (a\boldsymbol{U}) = a\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}},$$

resulting in the bending measures scaling as

$$\widetilde{\boldsymbol{K}} \to a\widetilde{\boldsymbol{K}}; \qquad \check{\boldsymbol{K}} \to a\check{\boldsymbol{K}}; \qquad \overline{\boldsymbol{K}} \to \overline{\boldsymbol{K}}.$$

Thus, the bending strain measure $\overline{\boldsymbol{K}}$ from [1], not discussed by [3], is actually ***invariant under scaling deformations of the deformed shell mid-surface***. Furthermore, the simple modifications of the measures $\widetilde{\boldsymbol{K}}, \check{\boldsymbol{K}}$ to

$$\widetilde{\boldsymbol{K}}_{mod} = \frac{1}{|\boldsymbol{U}|}\left(\boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{r}^T \cdot \boldsymbol{n}_{,\beta} - \boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{N}_{,\beta}\right) \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta \tag{2a}$$
$$= \left(tr\left(\boldsymbol{f}^T\boldsymbol{f}\right)\right)^{-\frac{1}{2}} \left(\boldsymbol{x}_{,\alpha} \cdot \boldsymbol{n}_{,\beta} - \boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{N}_{,\beta}\right) \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta$$

$$\check{\boldsymbol{K}}_{mod} = \frac{1}{|\boldsymbol{U}|}\left(\boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{r}^T \cdot \boldsymbol{n}_{,\beta} - \frac{1}{2}\left(\boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{N}_{,\beta} + \boldsymbol{E}_\beta \cdot \boldsymbol{U} \cdot \boldsymbol{N}_{,\alpha}\right)\right) \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta \tag{2b}$$
$$= \left(tr\left(\boldsymbol{f}^T\boldsymbol{f}\right)\right)^{-\frac{1}{2}} \left(\boldsymbol{x}_{,\alpha} \cdot \boldsymbol{n}_{,\beta} - \frac{1}{2}\left(\boldsymbol{E}_\alpha \cdot \boldsymbol{U} \cdot \boldsymbol{N}_{,\beta} + \boldsymbol{E}_\beta \cdot \boldsymbol{U} \cdot \boldsymbol{N}_{,\alpha}\right)\right) \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta,$$

where

$$|\boldsymbol{U}| = \sqrt{\boldsymbol{U} : \boldsymbol{U}} = \sqrt{tr\left(\boldsymbol{f}^T\boldsymbol{f}\right)},$$

make them mid-surface scaling invariant.

References

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