Second order structured deformations in the space of functions of bounded Hessian

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Abstract

This work addresses second order structured deformations in the framework of the space of special functions of bounded Hessian, BH. An integral representation result is obtained in BH in the vein of the global method for relaxation of Bouchitté, Fonseca, and Mascarenhas [4], and is applied to a relaxation problem in the context of structured deformations.

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1 Introduction

The macroscopic deformation of a continuous body does not need to coincide with the submacroscopic deformation. For instance, in a crystalline body deformed beyond the plastic regime the macroscopic deformation may be simply due to several slips of the crystallographic planes. Thus, submacroscopically the lattice of the crystalline body does not deform but simply undergoes "submacroscopic cracks" or disarrangements. This kind of multi-scale geometrical changes were addressed by Del Piero and Owen in [7] who introduced the notion of *structured deformation* (κ, u, G): κ being the macroscopic crack site, u the macroscopic deformation, and G a tensor associated with the submacroscopic geometrical changes and called deformation without disarrangements. In the example of the crystalline body, discussed above, we would have $\kappa = \emptyset$ since the submacroscopic cracks diffuse and do not generate a macroscopic crack, $G = I$ the identity tensor field since the lattice does not deform locally, and, in general the deformation gradient ∇u is different from $G = I$.

Del Piero and Owen, still in [7], showed that every structured deformation can be seen as the (appropriate) limit of sequences of piecewise-continuous "simple deformations". This result makes the theory even more interesting from a mechanical point of view, since, for instance, in the example of the crystalline body mentioned above, the "submacroscopic cracks" that form during the deformation can be thought as the jump sets of the piecewise-continuous "classical deformations" of an approximating sequence. This result also opens the way to define the energy of a structured deformation by using the "classical" energy of piecewise-continuous "classical deformations". Indeed, Choksi and Fonseca [5], following the belief that "Nature always minimizes actions", made the natural assumption that the structured deformation (κ, u, G) would be the limit, among all approximating sequences, of the approximating sequence that uses the least amount of energy. Choksi and Fonseca worked within a variational framework and described the macroscopic deformation by a function $u \in BV$ whose jump set represents the crack site κ of Del Piero and Owen, and with a deformation without disarrangements $G \in L^1$. In this framework, they proved the following approximation theorem: for any structured deformation (u, G) there exists a sequence $\{u_n\} \subset SBV$ such that

$$u_n \to u \text{ in } L^1, \qquad \nabla u_n \overset{*}{\rightharpoonup} G \text{ in the sense of measures,} \tag{1.1}$$

where ∇u_n denotes the absolutely continuous part of the distributional derivative of u_n; moreover, they defined the energy $\mathcal{E}(u, G)$ of (u, G) as

$$\mathcal{E}(u, G) := \inf_{\{u_n\}} \liminf_{n \to +\infty} \mathcal{E}_0(u_n), \tag{1.2}$$

where the inf is taken among all the sequences that generate, according to (1.1), the structured deformation (u, G), and $\mathcal{E}_0(u_n)$ is the energy associated to the "simple deformation" u_n. Thus, the energy $\mathcal{E}(u, G)$ is equal to the limit of the energies associated to the most economic approximating sequence from the energetic point of view.

The concept of structured deformation was extended in [13] by defining the *second-order structured deformation* (κ, u, G, U): where κ may be taken to be the set of points were the fields involved are discontinuous, u and G are as above, and U, called second-order deformation without disarrangements, is a third-order tensor field that allows one to describe the submacroscopic deformation up to the second-order; for instance, it allows one to describe the "bending" of the microstructure. Second-order structured deformation are important since they allow inclusion of the effects of limits of second gradients and jumps in the first gradients of approximating deformations: these jumps play a crucial role in the mechanics of phase-transitions. In [14] two different variational frameworks for second-order structured deformation are discussed: the primary difference being the function space on which the deformation fields are defined. The first framework employs a space named SBV^2 that allows jumps of the displacement as well as its gradient. A recent paper of Barroso, Matias, Morandotti, and Owen [2] provides relaxation and integral representation results for second-order structured deformations in the framework of SBV^2. The second framework considers the space SBH of special functions of bounded Hessian. Within this framework we have $u \in W^{1,1}, G = \nabla u$, and κ is simply the jump set of ∇u.

In the SBH framework, a second-order structured deformation is therefore described by the pair (u, U). We remark that the SBH setting is more constrained than the SBV^2 setting, since the functions may not have "jumps", and hence the techniques used in [2] cannot be directly applied to the SBH setting.

The goal of our paper is to obtain relaxation and integral representation results in the framework of SBH. We consider, over an open and bounded set $\Omega \subset \mathbb{R}^N$, the family of second-order structured deformations

$$SD_2(\Omega) := SBH(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{S}^{d\times N\times N}),$$

where $\mathbb{S}^{d\times N\times N} \subset \mathbb{R}^{d\times N\times N}$ denotes the set of tensors (M_{ijk}), $i \in \{1, \dots, d\}$, $j, k \in \{1, \dots, N\}$, such that $M_{ijk} = M_{ikj}$ for all $i, j \in \{1, \dots, N\}$, $d, N \in \mathbb{N}$. We first give a self-contained proof of the following approximation theorem: for every $(u, U) \in SD_2(\Omega)$ there exists a sequence $\{u_n\} \subset SBH(\Omega; \mathbb{R}^d)$ such that

$$u_n \to u \text{ in } W^{1,1}(\Omega; \mathbb{R}^d), \qquad \nabla^2 u_n \overset{*}{\rightharpoonup} U \text{ in } \mathcal{M}(\Omega).$$

We then prove a general integral representation result in the spirit of the global method of Bouchitté, Fonseca and Mascarenhas [4]. With $\mathcal{A}(\Omega)$ the family of open subsets of Ω, we consider a functional

$$\mathcal{F} : SD_2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$$

satisfying the following hypotheses:

(H1) $\mathcal{F}(u, U; \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure for every $(u, U) \in SD_2(\Omega)$.

(H2) $\mathcal{F}(\cdot, \cdot; A)$ is SD_2-lower semicontinuous, in the sense that if $(u, U) \in SD_2(\Omega)$, $\{(u_n, U_n)\} \subset SD_2(\Omega)$, $u_n \to u$ in $W^{1,1}(\Omega; \mathbb{R}^d)$ and $U_n \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$, then

$$\mathcal{F}(u, U; A) \le \liminf_{n\to+\infty} \mathcal{F}(u_n, U_n; A).$$

(H3) $\mathcal{F}$ is local, i.e., for all $A \in \mathcal{A}(\Omega)$ if $u = v$ and $U = V$ $\mathcal{L}^N$ a.e. $x \in A$ then $\mathcal{F}(u, U; A) = \mathcal{F}(v, V; A)$.

(H4) There exists a constant $C > 0$ such that

$$\frac{1}{C}(\|U\|_{L^1(A)} + |D^2 u|(A)) \le \mathcal{F}(u, U; A) \le C(\mathcal{L}^N(A) + \|U\|_{L^1(A)} + |D^2 u|(A))$$

for every $(u, U) \in SD_2(\Omega)$, $A \in \mathcal{A}(\Omega)$.

In Theorem 4.6 we prove an integral representation for $\mathcal{F}$ of the form

$$\mathcal{F}(u, U; A) = \int_A f(x, u, \nabla u, \nabla^2 u, U)\, dx + \int_{S(\nabla u)\cap A} h(x, u, \nabla u^+, \nabla u^-, \nu_{\nabla u})\, d\mathcal{H}^{N-1}.$$

In view of this result, we define the energy of a second-order structured deformation (u, U), in the same spirit of (1.2), as the limit of the energy of the most energetically convenient approximating sequence, i.e.,

$$\mathcal{F}(u, U) := \inf_{\{u_n\}} \liminf_{n\to+\infty} \mathcal{F}_0(u_n),$$

where the infimum is taken among all the sequences that generate the second-order structured deformation (u, U), and $\mathcal{F}_0(u_n)$ is the energy associated to the "simple deformation" u_n, see Theorem 5.4.

The general relaxation result proved has applications also outside the framework of structured deformations. Indeed, it has an immediate corollary to any functional defined on SBH: we can show, see Theorem 6.1, that for any $\mathcal{F} : SBH(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega)) \to [0, \infty)$ satisfying (H1)-(H4), we have the integral representation

$$\mathcal{F}(u; A) = \int_A f(x, u, \nabla u, \nabla^2 u)\, dx + \int_{S(\nabla u)\cap A} h(x, u, \nabla u^+, \nabla u^-, \nu_{\nabla u})\, d\mathcal{H}^{N-1}.$$

In the case of functionals defined in $BH(\Omega; \mathbb{R}^d)$, with the additional assumptions of affine invariance and area-strict continuity, results in [11] can be leveraged along with the SBH relaxation result to yield Corollary 6.4,

$$\mathcal{F}(u; A) = \int_A f(x, \nabla^2 u)\, dx + \int_A f^\infty \left(x, \frac{dD_s(\nabla u)}{d|D_s(\nabla u)|}\right) d|D_s(\nabla u)|(x).$$

The assumption of affine invariance is merely a technical detail due to the lack of a BH relaxation result as in [11] involving lower order terms. We motivate the assumption of area-strict continuity by comparison to the first order global method for relaxation [4]. In this situation, although we do not assume *a priori* that our abstract lower semicontinuous functional is area-strict continuous, once we have the integral representation result, area-strict continuity follows *a posteriori*, [12]. Thus, in the first-order case, nothing is lost by adding the additional assumption that the functional is area-strict continuous. We expect that the same holds in the second-order framework.

The paper is structured as follows. In Section 2 we collect some common notions and establish pointwise results about BH functions. In Section 3 we prove an approximation result in the SD_2 framework along the lines of the approximation theorems of [5] and [7]. In Section 4 we use the global method approach introduced in [4] on functionals defined on SD_2 in order to prove the main integral representation result. In Section 5 we apply the integral representation result to the problem of second order structured deformations to get a relaxation as in [5]. In Section 6 we find further application of the integral relaxation result in the spaces SBH and BH.

2 Preliminaries

In what follows, Ω is an open, bounded subset of $\mathbb{R}^N$. Given a smooth function $u : \mathbb{R}^N \to \mathbb{R}^d$ we denote by $\nabla^2 u \in \mathbb{R}^{d\times N\times N}$ the tensor field whose components are

$$(\nabla^2 u)_{ijk} = \frac{\partial^2 f_i}{\partial x_j \partial x_k},\ \ i \in \{1, \dots, d\}, j, k \in \{1, \dots, N\}.$$

Consider the space of functions of bounded Hessian

$$\begin{aligned} BH(\Omega;\mathbb{R}^d) : &= \{u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^d) : D(\nabla u) \text{ is a finite Radon measure}\} \\ &= \{u \in \mathrm{L}^1(\Omega;\mathbb{R}^d) : Du \in \mathrm{BV}(\Omega;\mathbb{R}^{d\times N})\}, \end{aligned}$$

where for locally integrable functions f, we write Df to denote its distributional derivative.

For $u \in BH(\Omega;\mathbb{R}^d)$, we set

$$\|u\|_{BH(\Omega)} := \|u\|_{W^{1,1}(\Omega)} + |D^2u|(\Omega),$$

where $|D^2u|(\Omega)$ is the variation of ∇u. Moreover, since $Du = \nabla u \in BV(\Omega;\mathbb{R}^{d\times N})$, we can express the Radon-Nikodym decomposition of $D(\nabla u) = D^2u$ as

$$D(\nabla u) = \nabla^2 u \mathcal{L}^N + D_s(\nabla u),$$

$$D_s(\nabla u) := [\nabla u] \otimes \nu_{\nabla u}\mathcal{H}^{N-1} \llcorner S(\nabla u) + D_c(\nabla u),$$

where $\nabla^2 u$ is a $\mathcal{L}^N$-measurable function with values in $\mathbb{S}^{d\times N\times N}$, $D_s(\nabla u)$ is the singular part with respect to $\mathcal{L}^N$, $S(\nabla u)$ denotes the jump set of ∇u, $\nu_{\nabla u}$ is the normal to the jump set, $[\nabla u] = (\nabla u)^+ - (\nabla u)^-$ is the jump of ∇u across $S(\nabla u)$, and $D_c(\nabla u)$ is the Cantor part of $D(\nabla u)$, which is singular with respect to $\mathcal{L}^N \llcorner \Omega + \mathcal{H}^{N-1} \llcorner S(\nabla u)$. Where the function u is clear in context, we will abuse notation by omitting the subscript ∇u and simply write ν.

We consider also the space of special functions of bounded Hessian

$$SBH(\Omega;\mathbb{R}^d) := \{u \in BH(\Omega;\mathbb{R}^d) : D_c(\nabla u) = 0\},$$

that is, BH functions with no Cantor part in the Hessian. This is distinct from the related space

$$SBV^2(\Omega;\mathbb{R}^d) := \{u \in BV(\Omega;\mathbb{R}^d) : \nabla u \in BV(\Omega;\mathbb{R}^{d\times N}),\ D_c(u) = 0,\ D_c(\nabla u) = 0\}.$$

We define the unit cube $Q := \{x \in \mathbb{R}^N : |x_i| \le \frac{1}{2} \text{ for all } 1 \le i \le N\}$. For $x_0 \in \mathbb{R}^N$ and $r > 0$, we consider the cube of side length r centered at x_0, $Q(x_0, r) := x_0 + rQ = \{x_0 + ry : y \in Q\}$.

In what follows, we fix a smooth, radially symmetric function $\phi \in C^\infty(\mathbb{R}^N;[0,\infty))$ such that $\mathrm{supp}(\phi) \subset B(0,1)$ and $\int_{\mathbb{R}^N} \phi(x)dx = 1$. We define the standard mollifiers $\phi_\varepsilon(x) := \frac{1}{\varepsilon^N}\phi(\varepsilon x)$, $\varepsilon > 0$.

We now establish some basic results concerning approximate differentiability properties of functions in the setting of BH.

Theorem 2.1. *If $u \in BH(\Omega;\mathbb{R}^d)$ then*

(i) for $\mathcal{L}^N$ a.e. $x \in \Omega$

$$\lim_{r\to 0^+} \frac{1}{r^2} \fint_{Q(x,r)} \left| u(y) - u(x) - \nabla u(x)(y-x) - \frac{1}{2}\nabla^2 u(x)(y-x, y-x) \right| dy = 0, \tag{2.1}$$

and

$$\lim_{r\to 0^+} \frac{1}{r} \fint_{Q(x,r)} \left| \nabla u(y) - \nabla u(x) - \nabla^2 u(x)(y-x) \right| dy = 0; \tag{2.2}$$

(ii) for $\mathcal{H}^{N-1}$ a.e. $x \in S(\nabla u)$ we have

$$\lim_{r\to 0^+} \frac{1}{r} \fint_{Q_\nu^\pm(x,r)} \left| u(y) - u(x) - \nabla u^\pm(x)(y-x) \right| \, dy = 0, \tag{2.3}$$

and

$$\lim_{r\to 0^+} \fint_{Q_\nu^\pm(x,r)} \left| \nabla u(y) - \nabla u^\pm(x) \right| \, dy = 0, \tag{2.4}$$

where $Q_\nu^+(x,r) = Q_\nu(x,r) \cap \{y : (y-x)\cdot\nu(x) > 0\}$ and $Q_\nu^-(x,r) = Q_\nu(x,r) \cap \{y : (y-x)\cdot\nu(x) < 0\}$.

Proof. A proof of (2.2) can be found in Theorem 6.1 in [8], applied to $f = \nabla u$. Similarly, a proof of (2.3) and (2.4) can be found in Theorem 5.19 in [8], applied to $f = u$ and $f = \nabla u$ respectively.

It remains to show (2.1), which involves a second-order approximation. Its proof uses arguments similar to those found in [8], and it is included below for completeness.

Fix $x_0 \in \Omega$ such that

$$\lim_{r\to 0^+} \fint_{Q(x_0,r)} |u(x) - u(x_0)| \, dx = 0, \tag{2.5}$$

$$\lim_{r\to 0^+} \fint_{Q(x_0,r)} |\nabla u(x) - \nabla u(x_0)| \, dx = 0, \tag{2.6}$$

$$\lim_{r\to 0^+} \fint_{Q(x_0,r)} \left|\nabla^2 u(x) - \nabla^2 u(x_0)\right| \, dx = 0, \tag{2.7}$$

and

$$\lim_{r\to 0^+} \frac{|D_s^2 u|(Q(x_0,r))}{r^N} = 0. \tag{2.8}$$

Since the above hold for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$, it suffices to show that for every such x_0 we have

$$\lim_{r\to 0^+} \frac{1}{r^2} \fint_{Q(x_0,r)} \left| u(x) - u(x_0) - \nabla u(x_0)(x-x_0) - \frac{1}{2}\nabla^2 u(x_0)(x-x_0, x-x_0) \right| \, dx = 0.$$

Without loss of generality, we can take $x_0 = 0$. Define smooth functions u_ε by $u * \phi_\varepsilon$, for $0 < \varepsilon << r << \operatorname{dist}(0, \partial\Omega)$. By (2.5), (2.6) and (2.7), note that

$$\lim_{\varepsilon\to 0^+} u_\varepsilon(0) = u(0), \lim_{\varepsilon\to 0^+} \nabla u_\varepsilon(0) = \nabla u(0), \lim_{\varepsilon\to 0^+} \nabla^2 u_\varepsilon(0) = \nabla^2 u(0).$$

For $x \in Q(0,r)$, consider now the function $g_\varepsilon(t)$ defined by

$$g_\varepsilon(t) := u_\varepsilon(tx), \; t \in [0,1].$$

By smoothness of the u_ε, applying the fundamental theorem of calculus twice, we see that

$$g(1) = g(0) + g'(0) + \int_0^1 (1-t) g''(t) \, dt$$

and thus

$$u_\varepsilon(x) = u_\varepsilon(0) + \nabla u_\varepsilon(0)x + \int_0^1 (1-t)\nabla^2 u_\varepsilon(tx)(x,x)\,dt.$$

Rearranging these terms and subtracting $\frac{1}{2}\nabla^2 u(0)(x,x)$ from both sides we obtain

$$u_\varepsilon(x) - u_\varepsilon(0) - \nabla u_\varepsilon(0)x - \frac{1}{2}\nabla^2 u_\varepsilon(0)(x,x) = \int_0^1 (1-t)(\nabla^2 u_\varepsilon(tx) - \nabla^2 u_\varepsilon(0))(x,x)\,dt,$$

and so

$$\begin{aligned}&\frac{1}{r^2}\fint_{Q(0,r)} |u_\varepsilon(x) - u_\varepsilon(0) - \nabla u_\varepsilon(0)x - \frac{1}{2}\nabla^2 u_\varepsilon(0)(x,x)|\,dx \\ &\quad \le \frac{1}{r^2}\fint_{Q(0,r)}\int_0^1 |(\nabla^2 u_\varepsilon(tx) - \nabla^2 u_\varepsilon(0))(x,x)|\,dt\,dx.\end{aligned}$$

By Fatou's lemma,

$$\begin{aligned}&\frac{1}{r^2}\fint_{Q(0,r)} |u(x) - u(0) - \nabla u(0)x - \frac{1}{2}\nabla^2 u(0)(x,x)|\,dx \\ &\le \liminf_{\varepsilon\to 0^+} \frac{1}{r^2}\fint_{Q(0,r)} |u_\varepsilon(x) - u_\varepsilon(0) - \nabla u_\varepsilon(0)x - \frac{1}{2}\nabla^2 u_\varepsilon(0)(x,x)|\,dx \\ &\le \liminf_{\varepsilon\to 0^+} \frac{1}{r^2}\fint_{Q(0,r)}\int_0^1 \left|\left(\nabla^2 u_\varepsilon(tx) - \nabla^2 u_\varepsilon(0)\right)(x,x)\right|\,dt\,dx. \qquad (2.9)\end{aligned}$$

Thus, it suffices to bound (2.9). Applying the change of variables $z = tx$, we have

$$\begin{aligned}&\int_0^1 \frac{1}{t^{N+2}}\frac{1}{r^{N+2}}\int_{Q(0,tr)} \left|\left(\nabla^2 u_\varepsilon(z) - \nabla^2 u_\varepsilon(0)\right)(z,z)\right|\,dz\,dt \\ &\quad \le \frac{N}{4}\int_0^1 \frac{1}{r^N t^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z) - \nabla^2 u_\varepsilon(0)|\,dz\,dt.\end{aligned}$$

Using the triangle inequality, we obtain

$$\begin{aligned}\int_0^1 \frac{1}{r^N t^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z) - \nabla^2 u_\varepsilon(0)|\,dz &\le \int_0^1 \frac{1}{r^N}\frac{1}{t^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z) - \nabla^2 u(z)|\,dzdt \\ &+ \int_0^1 \fint_{Q(0,tr)} |\nabla^2 u(z) - \nabla^2 u(0)|\,dzdt + \int_0^1 \fint_{Q(0,tr)} |\nabla^2 u(0) - \nabla^2 u_\varepsilon(0)|\,dz\,dt. \qquad (2.10)\end{aligned}$$

If we let ε tend to 0^+, the second term will be unchanged and the third term will vanish. We turn our attention to the first term, namely

$$\int_0^1 \frac{1}{t^N}\frac{1}{r^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z) - \nabla^2 u(z)|\,dz\,dt.$$

Set

$$h_\varepsilon(t) := \int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z) - \nabla^2 u(z)|\, dz, \text{ for } t \in (0,1),$$

and note that

$$h_\varepsilon(t) \le \int_{Q(0,tr)} |(\nabla^2 u) * \phi_\varepsilon(z) - \nabla^2 u(z)|\, dz + \int_{Q(0,tr)} |(D_s^2 u) * \phi_\varepsilon(z)|\, dz.$$

Sending $\varepsilon \to 0^+$, we have

$$\limsup_{\varepsilon\to 0^+} h_\varepsilon(t) \le \left|D_s^2 u\right| (\overline{Q(0,tr)}).$$

Observe that

$$\frac{h_\varepsilon(t)}{t^N} = \frac{1}{t^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z) - \nabla^2 u(z)|\, dz \le \frac{1}{t^N}\int_{Q(0,tr)} \left(|\nabla^2 u_\varepsilon(z)| + |\nabla^2 u(z)|\right)\, dz,$$

$$\frac{1}{t^N}\int_{Q(0,tr)} |\nabla^2 u(z)|\, dz \le C$$

for some constant C by (2.7), since r is fixed. On the other hand,

$$\begin{aligned}
\frac{1}{t^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z)|\, dz &\le \frac{1}{t^N}\int_{Q(0,tr)}\int_\Omega \phi_\varepsilon(z-y)\, d|D^2 u|(y)\, dz \\
&= \frac{1}{t^N}\int_\Omega\int_{Q(0,tr)} \phi_\varepsilon(z-y)\, dz\, d|D^2 u|(y) \\
&\le \frac{C}{\varepsilon^N t^N}\int_{Q(0,tr+\varepsilon)}\int_{Q(0,tr)\cap B(y,\varepsilon)} dz\, d|D^2 u|(y) \\
&\le \frac{C}{\varepsilon^N t^N} \min\{\varepsilon^N, t^N\} |D^2 u|(Q(0,tr+\varepsilon)).
\end{aligned}$$

Again by (2.7) and (2.8), we have

$$|D^2 u|(Q(0,tr+\varepsilon)) \le C(tr+\varepsilon)^N$$

so we conclude that $\frac{h_\varepsilon(t)}{t^N}$ is bounded by a constant for $t \in (0,1)$, and we may apply the Reverse Fatou Lemma to deduce

$$\limsup_{\varepsilon\to 0^+}\int_0^1 \frac{1}{t^N r^N}\int_{Q(0,tr)} |\nabla^2 u_\varepsilon(z) - \nabla^2 u(z)|\, dz dt \le \int_0^1 \frac{1}{t^N r^N}\left|D_s^2 u\right| (\overline{Q(0,tr)}) dt.$$

Thus from (2.9) and (2.10) we have

$$\frac{1}{r^2} ⨍_{Q(0,r)} |u(x) - u(0) - \nabla u(0)x - \frac{1}{2}\nabla^2 u(0)(x,x)|\, dx$$

$$\le \int_0^1 \left(\frac{|D_s^2 u|(\overline{Q(0,tr)})}{t^N r^N} + \frac{N}{4} ⨍_{Q(0,tr)} |\nabla^2 u(z) - \nabla^2 u(0)|\, dz \right) dt. \qquad (2.11)$$

For a given r there are only countably many $t \in (0,1)$ such that $|D_s^2 u|(\partial Q(0,tr)) > 0$. Thus, we can rewrite the upper bound in (2.11) as

$$\int_0^1 \left(\frac{|D_s^2 u|(Q(0,tr))}{t^N r^N} + \frac{N}{4} \fint_{Q(0,tr)} |\nabla^2 u(z) - \nabla^2 u(0)|\, dz \right) dt$$

We note that by (2.7) and (2.8) we can apply the dominated convergence theorem to conclude that

$$\lim_{r \to 0^+} \int_0^1 \left(\frac{|D_s^2 u|(Q(0,tr))}{t^N r^N} + \frac{N}{4} \fint_{Q(0,tr)} |\nabla^2 u(z) - \nabla^2 u(0)|\, dz \right) dt = 0.$$

□

The next result can be found in [10], Lemma 2.13.

Lemma 2.2. *Let λ be a nonnegative Radon measure in $\mathbb{R}^N$. For λ a.e. $x_0 \in \mathbb{R}^N$, for every $0 < \delta < 1$ and for every $\nu \in S^{N-1}$,*

$$\limsup_{\varepsilon \to 0^+} \frac{\lambda(Q_\nu(x_0, \delta\varepsilon))}{\lambda(Q_\nu(x_0, \varepsilon))} \geq \delta^N,$$

and, therefore,

$$\lim_{\delta \to 1^-} \limsup_{\varepsilon \to 0^+} \frac{\lambda(Q_\nu(x_0, \delta\varepsilon))}{\lambda(Q_\nu(x_0, \varepsilon))} = 1.$$

In order to establish that an abstract functional is the restriction of a Radon measure, we will apply the coincidence criterion of Dal Maso, Fonseca and Leoni (see [6] Corollary 5.2).

Lemma 2.3. *Let $\mathcal{A}(\Omega)$ be the family of open subsets of Ω and $\lambda : \mathcal{A}(\Omega) \to [0, \infty)$ be an increasing set function such that:*

(i) for all $A, B, C \in \mathcal{A}(\Omega)$ with $\overline{A} \subset B \subset C$ there holds

$$\lambda(C) \leq \lambda(C \setminus \overline{A}) + \lambda(B)\,,$$

(ii) $\lambda(A \cup B) = \lambda(A) + \lambda(B)$, for all $A, B \in \mathcal{A}(\Omega)$ with $A \cap B = \emptyset$,

(iii) there exists a measure $\mu : \mathcal{B}(\Omega) \to [0, \infty)$ such that

$$\lambda(A) \leq \mu(A)$$

for all $A \in \mathcal{A}(\Omega)$, where $\mathcal{B}(\Omega)$ denotes the family of Borel sets of Ω.

Then λ is the restriction to $\mathcal{A}(\Omega)$ of a measure defined on $\mathcal{B}(\Omega)$.

3 Second-order structured deformations

Let Ω be an open, bounded subset of $\mathbb{R}^N$. Set

$$\mathbb{S}^{d\times N\times N} = \{U \in \mathbb{R}^{d\times N\times N} : U_{ijk} = U_{ikj}, \forall i = 1,\dots,d, \quad j,k = 1,\dots,N\}.$$

Definition 3.1. The space of second-order structured deformations $SD_2(\Omega)$ consists of pairs (u,U) with $u \in SBH(\Omega;\mathbb{R}^d)$ and $U \in L^1(\Omega;\mathbb{S}^{d\times N\times N})$,

$$SD_2(\Omega) := SBH(\Omega;\mathbb{R}^d) \times L^1(\Omega;\mathbb{S}^{d\times N\times N}).$$

The approximation result stated next can be proved by applying the generalization of Alberti's theorem to BH functions contained in [9].

Theorem 3.2. *For every $(u,U) \in SD_2(\Omega)$ there exists a sequence $\{u_n\} \subset SBH(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in $W^{1,1}(\Omega;\mathbb{R}^d)$ and $\nabla^2 u_n \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$, with*

$$\sup_n \|u_n\|_{BH} \leq C(\|u\|_{BH} + \|U\|_{L^1})$$

for some constant $C > 0$.

For convenience of the reader, we give a self-contained proof of Theorem 3.2, for which we will make use of the following lemma.

Lemma 3.3. *Let $U \in L^1(\Omega, \mathbb{R}^{d\times N\times N})$, and for every $\delta > 0$ let $\{A_i^\delta\}_{i\in\mathbb{N}}$ be a countable family of open sets such that $A_i^\delta \subset \Omega$, $A_i^\delta \cap A_j^\delta = \emptyset$ for every $i,j \in \mathbb{N}$ with $i \neq j$, $\mathcal{L}^N(\Omega\setminus \cup_i A_i^\delta) = 0$, and $\sup_i \operatorname{diam} A_i^\delta \leq \delta$. For $i \in \mathbb{N}$ let $V_i^\delta : A_i^\delta \to \mathbb{R}^{d\times N\times N}$ be such that*

$$\int_{A_i^\delta} V_i^\delta \, dx = \int_{A_i^\delta} U \, dx,$$

and set

$$V^\delta := \sum_i \chi_{A_i^\delta} V_i^\delta.$$

If $\sup_\delta \|V^\delta\|_{L^1} < +\infty$ then $V^\delta \mathcal{L}^N \overset{*}{\rightharpoonup} U\mathcal{L}^N$.

Proof. Arguing componentwise, it suffices to prove the lemma for scalar fields, hence we suppose that $U \in L^1(\Omega)$. Define

$$\overline{V}^\delta := \sum_i \chi_{A_i^\delta} \fint_{A_i^\delta} V_i^\delta \, dx = \sum_i \chi_{A_i^\delta} \fint_{A_i^\delta} U \, dx.$$

Fix $\varepsilon > 0$ and choose $W \in L^1(\Omega) \cap C_0^\infty(\Omega)$ such that $\|U - W\|_{L^1(\Omega)} < \varepsilon/3$. Define

$$\overline{W}^\delta := \sum_i \chi_{A_i^\delta} \fint_{A_i^\delta} W \, dx.$$

Since W is uniformly continuous, there exists $\eta > 0$ such that if $|x-y| < \eta$ then $|W(x) - W(y)| \le \varepsilon/(3\mathcal{L}^N(\Omega))$. For $0 < \delta \le \eta$ we have

$$\begin{aligned}
\|\overline{W}^\delta - W\|_{L^1(\Omega)} &= \sum_i \int_{A_i^\delta} \left| W(x) - \fint_{A_i^\delta} W(y)\,dy \right| dx \\
&\le \sum_i \int_{A_i^\delta} \fint_{A_i^\delta} |W(x) - W(y)|\, dy\,dx \le \frac{\varepsilon}{3},
\end{aligned}$$

and

$$\|\overline{W}^\delta - \overline{V}^\delta\|_{L^1(\Omega)} = \sum_i \mathcal{L}^N(A_i^\delta) \left| \fint_{A_i^\delta} (W(y) - U(y))\, dy \right| \le \|U - W\|_{L^1(\Omega)} \le \frac{\varepsilon}{3}.$$

Thus

$$\|\overline{V}^\delta - U\|_{L^1(\Omega)} \le \|U - W\|_{L^1(\Omega)} + \|\overline{W}^\delta - W\|_{L^1(\Omega)} + \|\overline{W}^\delta - \overline{V}^\delta\|_{L^1(\Omega)} \le \varepsilon,$$

and we conclude that $\overline{V}^\delta \to U$ in $L^1(\Omega)$. For $\psi \in C_0(\Omega)$ we have

$$\lim_{\delta\to 0^+} \int_\Omega (V^\delta - U)\psi\,dx = \lim_{\delta\to 0^+} \int_\Omega (V^\delta - \overline{V}^\delta)\psi\,dx,$$

and hence it suffices to show that $\lim_{\delta\to 0^+} \int_\Omega (V^\delta - \overline{V}^\delta)\psi\,dx = 0$. Note that

$$\begin{aligned}
\int_\Omega \overline{V}^\delta \psi\,dx &= \sum_i \int_{A_i^\delta} \fint_{A_i^\delta} V^\delta(y)\,dy\,\psi(x)\,dx = \sum_i \int_{A_i^\delta} V^\delta(y)\,dy \fint_{A_i^\delta} \psi(x)\,dx \\
&= \sum_i \int_{A_i^\delta} V^\delta(y)\psi(y)\,dy + \sum_i \int_{A_i^\delta} V^\delta(y) \fint_{A_i^\delta} (\psi(x) - \psi(y))\,dx\,dy,
\end{aligned}$$

and therefore

$$\left| \int_\Omega (V^\delta - \overline{V}^\delta)\psi\,dx \right| = \left| \sum_i \int_{A_i^\delta} V^\delta(y) \fint_{A_i^\delta} (\psi(x) - \psi(y))\,dx\,dy \right| \le \sup_\delta \|V^\delta\|_{L^1(\Omega)} o(1),$$

since ψ is uniformly continuous. This concludes the proof. □

We now proceed to establish the approximation theorem.

Proof of Theorem 3.2 We claim that it suffices to prove that for every $V \in L^1(\Omega; \mathbb{S}^{d\times N\times N})$ there exists a sequence $\{f^\varepsilon\} \subset SBH(\Omega; \mathbb{R}^d)$ such that $f^\varepsilon \to 0$ in $W^{1,1}(\Omega; \mathbb{R}^d)$, $\nabla^2 f^\varepsilon \overset{*}{\rightharpoonup} V$ in $\mathcal{M}(\Omega)$ and $\sup_\varepsilon |D^2 f^\varepsilon|(\Omega) \le C\|V\|_{L^1(\Omega)}$. In fact, if the claim holds then we can define $u_n := u + f^{\varepsilon_n}$ where the sequence $\{f^\varepsilon\}$ is the one obtained by applying the claim to $V := U - \nabla^2 u$.

We now prove the claim. For simplicity of notation we will consider $N = 2$, however the same argument works for a generic N. Extend V outside Ω by 0 and denote this extension still by V. Fix $\varepsilon > 0$ and let $\{Q^{\varepsilon,l}\}_l$ be the family of open cubes whose side length is ε and whose centers $y^{\varepsilon,l}$ belong to the lattice $(\varepsilon\mathbb{Z})^2$. Let

$$\phi^\varepsilon(x) := \left(1 - \frac{2|x_1|}{\varepsilon}\right)\chi_{\{|x_2|<\varepsilon/2,|x_1|<|x_2|\}} + \left(1 - \frac{2|x_2|}{\varepsilon}\right)\chi_{\{|x_1|<\varepsilon/2,|x_2|<|x_1|\}}$$

i.e., ϕ^ε is the function whose graph is the pyramid over the cube $Q(0,\varepsilon)$ of height one. Let $\{A^{\varepsilon,l}\}_l$ be a family of symmetric tensors in $\mathbb{S}^{d\times 2\times 2}$ to be defined later and let $f^\varepsilon \in SBH(\Omega;\mathbb{R}^d)$ be given by

$$f^\varepsilon(x) := \sum_l \frac{1}{2}\phi^\varepsilon(x - y^{\varepsilon,l})A^{\varepsilon,l}(x - y^{\varepsilon,l}, x - y^{\varepsilon,l}).$$

We now define $A^{\varepsilon,l}$ as the symmetric tensor for which

$$\int_{Q^{\varepsilon,l}} \nabla^2 f^\varepsilon \, dx = \int_{Q^{\varepsilon,l}} V \, dx. \tag{3.1}$$

Note that, since $\nabla^2\phi^\varepsilon = 0$ and $A^{\varepsilon,l}$ is symmetric,

$$(\nabla^2 f^\varepsilon)_{irs} = (\nabla\phi^\varepsilon)_s A^{\varepsilon,l}_{ijr}(x_j - y^{\varepsilon,l}_j) + (\nabla\phi^\varepsilon)_r A^{\varepsilon,l}_{ijs}(x_j - y^{\varepsilon,l}_j) + \phi^\varepsilon A^{\varepsilon,l}_{irs},$$

where the summation convention is adopted throughout this proof. Define

$$Z^\varepsilon_{js} := \int_{Q(0,\varepsilon)} (\nabla\phi^\varepsilon)_s(x)x_j \, dx = \int_{Q^{\varepsilon,l}} (\nabla\phi^\varepsilon)_s(x - y^{\varepsilon,l})(x_j - y^{\varepsilon,l}_j) \, dx,$$

$$\widetilde{z}^\varepsilon := \int_{Q(0,\varepsilon)} \phi^\varepsilon(x) \, dx = \int_{Q^{\varepsilon,l}} \phi^\varepsilon(x - y^{\varepsilon,l}) \, dx, \qquad \text{and} \qquad \widetilde{V}^{\varepsilon,l} := \int_{Q^{\varepsilon,l}} V \, dx,$$

and rewrite (3.1) as

$$A^{\varepsilon,l}_{ijr} Z^\varepsilon_{js} + A^{\varepsilon,l}_{ijs} Z^\varepsilon_{jr} + A^{\varepsilon,l}_{irs}\widetilde{z}^\varepsilon = \widetilde{V}^{\varepsilon,l}_{irs}. \tag{3.2}$$

It turns out that $Z^\varepsilon = -\varepsilon^2 I$, where I is the identity matrix, indeed,

$$Z^\varepsilon_{11} = \int_{Q(0,\varepsilon)} x_1 \frac{-2\,\mathrm{sgn}(x_1)}{\varepsilon}\chi_{\{|x_1|<|x_2|\}} \, dx = 2\int_{\varepsilon/2}^{\varepsilon/2}\int_{-x_1}^{x_1} \frac{-2x_1}{\varepsilon} \, dx_2 dx_1 = -\varepsilon^2$$

and, similarly,

$$Z^\varepsilon_{22} = \int_{Q(0,\varepsilon)} x_2 \frac{-2\,\mathrm{sgn}(x_2)}{\varepsilon}\chi_{\{|x_2|<|x_1|\}} \, dx = -\varepsilon^2.$$

On the other hand,

$$Z^\varepsilon_{12} = \int_{Q(0,\varepsilon)} x_2 \frac{-2\,\mathrm{sgn}(x_1)}{\varepsilon}\chi_{\{|x_1|<|x_2|\}} \, dx = 0$$

since the integrand is odd in x_2 and x_1 and the region of integration is symmetric in both variables, and the same is true for Z^ε_{21}. We can also calculate $\widetilde{z}^\varepsilon$ as the volume of a pyramid with base ε^2 and height 1 to find $\widetilde{z}^\varepsilon = \frac{1}{3}\varepsilon^2$.

From this, (3.2) becomes

$$-\frac{5}{3}\varepsilon^2 A^{\varepsilon,l} = \widetilde{V}^{\varepsilon,l}, \tag{3.3}$$

We now prove that $f^\varepsilon \to 0$ in $W^{1,1}(\Omega;\mathbb{R}^d)$. We have

$$\begin{aligned}
\int_\Omega |f^\varepsilon|\,dx &= \frac{1}{2}\sum_l \int_{Q^{\varepsilon,l}\cap\Omega} |\phi^\varepsilon(x-y^{\varepsilon,l})A^{\varepsilon,l}(x-y^{\varepsilon,l},x-y^{\varepsilon,l})|\,dx \\
&\le C\sum_l |A^{\varepsilon,l}|\varepsilon^2\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega) \le C\varepsilon^2\sum_l |\widetilde{V}^{\varepsilon,l}| \\
&\le C\varepsilon^2\sum_l \int_{Q^{\varepsilon,l}} |V|\,dx \le C\varepsilon^2\|V\|_{L^1(\Omega)}
\end{aligned}$$

where we have used (3.3). Furthermore, again by (3.3) we obtain

$$\begin{aligned}
\int_\Omega |\nabla f^\varepsilon|\,dx &\le C\sum_l \Big[\|\nabla\phi^\varepsilon\|_{L^\infty}|A^{\varepsilon,l}|\varepsilon^2\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega) + \|\phi^\varepsilon\|_{L^\infty}|A^{\varepsilon,l}|\varepsilon\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega)\Big] \\
&\le C\sum_l \left[\frac{1}{\varepsilon}\varepsilon^2\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega) + |A^{\varepsilon,l}|\varepsilon\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega)\right] \le C\varepsilon\|V\|_{L^1(\Omega)}.
\end{aligned}$$

Next, we show that $\sup_\varepsilon |D^2 f^\varepsilon|(\Omega) \le C\|V\|_{L^1(\Omega)}$. Indeed, by (3.3)

$$\int_\Omega |\nabla^2 f^\varepsilon|\,dx \le C\sum_l \left[|A^{\varepsilon,l}|\varepsilon\frac{1}{\varepsilon}\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega) + |A^{\varepsilon,l}|\mathcal{L}^2(Q^{\varepsilon,l}\cap\Omega)\right] \le C\|V\|_{L^1(\Omega)},$$

and

$$\begin{aligned}
\int_{\Omega\cap S(\nabla f^\varepsilon)} |[\nabla f^\varepsilon]|\,d\mathcal{H}^1 &\le \sum_l \int_{\partial Q^{\varepsilon,l}\cap\Omega} |[\nabla f^\varepsilon]|\,d\mathcal{H}^1 + \int_{d^{\varepsilon,l}} |[\nabla f^\varepsilon]|\,d\mathcal{H}^1 \\
&\le C\sum_l \left(\frac{1}{\varepsilon}\left(|A^{\varepsilon,l}|\varepsilon^2\right)\varepsilon + |A^{\varepsilon,l}|\varepsilon^2\right) \\
&\le C\sum_l |A^{\varepsilon,l}|\mathcal{L}^2(Q^{\varepsilon,l}) \le C\|V\|_{L^1(\Omega)},
\end{aligned}$$

where $d^{\varepsilon,l}$ is the union of the diagonals of $Q^{\varepsilon,l}$, and we used the estimate

$$\int_{d^{\varepsilon,l}} |[\nabla f^\varepsilon]|\,d\mathcal{H}^1 \le C\int_0^\varepsilon \frac{1}{\varepsilon}|A^{\varepsilon,l}||(t,t)|^2\,dt \le C|A^{\varepsilon,l}|\varepsilon^2.$$

That $\nabla^2 f^\varepsilon \overset{*}{\rightharpoonup} V$ in $\mathcal{M}(\Omega)$ follows from (3.1), the inequalities above and from Lemma 3.3. □

4 The global method

Recall that $\mathcal{A}(\Omega)$ is the family of open subsets of Ω. Consider a functional

$$\mathcal{F} : SD_2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty] \tag{4.1}$$

satisfying the following hypotheses:

(H1) $\mathcal{F}(u, U; \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure for every $(u, U) \in SD_2(\Omega)$.

(H2) $\mathcal{F}(\cdot, \cdot; A)$ is SD_2-lower semicontinuous, in the sense that if $(u, U) \in SD_2(\Omega)$, $\{(u_n, U_n)\} \subset SD_2(\Omega)$, $u_n \to u$ in $W^{1,1}(\Omega; \mathbb{R}^d)$ and $U_n \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$, then

$$\mathcal{F}(u, U; A) \leq \liminf_{n \to +\infty} \mathcal{F}(u_n, U_n; A).$$

(H3) $\mathcal{F}$ is local, i.e., for all $A \in \mathcal{A}(\Omega)$, if $u = v$ and $U = V$ $\mathcal{L}^N$ a.e. $x \in A$ then $\mathcal{F}(u, U; A) = \mathcal{F}(v, V; A)$.

(H4) There exists a constant $C > 0$ such that

$$\frac{1}{C}(\|U\|_{L^1(A)} + |D^2 u|(A)) \leq \mathcal{F}(u, U; A) \leq C(\mathcal{L}^N(A) + \|U\|_{L^1(A)} + |D^2 u|(A))$$

for every $(u, U) \in SD_2(\Omega)$, $A \in \mathcal{A}(\Omega)$.

In the spirit of the global method for relaxation [3, 4], given $(u, U; A) \in SD_2(\Omega) \times \mathcal{A}(\Omega)$ we define

$$\mathfrak{A}(u, U; A) := \left\{ (v, V) \in SD_2(\Omega) : \text{spt } (u - v) \subset\subset A, \int_A (U - V)\, dx = 0 \right\}, \tag{4.2}$$

and

$$m(u, U; A) := \inf \left\{ \mathcal{F}(v, V; A) : (v, V) \in \mathfrak{A}(u, U; A) \right\}. \tag{4.3}$$

Lemma 4.1. *If (H1) and (H4) hold, then for every $(u, U) \in SD_2(\Omega)$ and $A \in \mathcal{A}(\Omega)$*

$$\limsup_{\delta \to 0^+} m(u, U; A_\delta) \leq m(u, U; A),$$

where $A_\delta = \{x \in A : \text{dist}(x, \partial A) > \delta\}$.

Proof. Let $\varepsilon > 0$. Choose $(\widetilde{u}, \widetilde{U}) \in \mathfrak{A}(u, U; A)$ such that

$$\mathcal{F}(\widetilde{u}, \widetilde{U}; A) \leq m(u, U; A) + \varepsilon.$$

Let $\delta_0 :=$ dist(spt$(u - \widetilde{u}), \partial A) > 0$. For $0 < \delta < \delta_0/2$ define

$$\widehat{U} = \begin{cases} \widetilde{U} & \text{in } A_{2\delta}, \\ (\mathcal{L}^N(A_\delta \backslash A_{2\delta}))^{-1}(\int_{A_\delta} U\, dx - \int_{A_{2\delta}} \widetilde{U}\, dx) & \text{on } A_\delta \setminus A_{2\delta}. \end{cases}$$

Since $(\widetilde{u},\widehat{U}) \in \mathfrak{A}(u,U;A_\delta)$, for every compact set $K \subset A_{2\delta}$ we have by (H1) and (H4),

$$\begin{aligned}
m(u,U;A_\delta) &\le \mathcal{F}(\widetilde{u},\widehat{U};A_\delta)\\
&\le \mathcal{F}(\widetilde{u},\widetilde{U};A_{2\delta}) + \mathcal{F}(u,\widehat{U};A_\delta \backslash K)\\
&\le \mathcal{F}(\widetilde{u},\widetilde{U};A) + C\left(\mathcal{L}^N(A\backslash K) + \int_{A_\delta\backslash K} |\widehat{U}|\,dx + |D^2u|(A\backslash K)\right)\\
&\le m(u,U;A) + \varepsilon + C\Bigg(\mathcal{L}^N(A\backslash K) + |D^2u|(A\backslash K)\\
&\qquad + \frac{\mathcal{L}^N(A_\delta\backslash K)}{\mathcal{L}^N(A_\delta\backslash A_{2\delta})}\left|\int_{A_\delta} U\,dx - \int_{A_{2\delta}} \widetilde{U}\,dx\right|\Bigg).
\end{aligned}$$

Using inner regularity and letting $K \nearrow A_{2\delta}$, we have

$$\begin{aligned}
m(u,U;A_\delta) &\le m(u,U;A) + \varepsilon\\
&\qquad + C\left(\mathcal{L}^N(A\backslash A_{2\delta}) + |D^2u|(A\backslash A_{2\delta}) + \left|\int_{A_\delta} U\,dx - \int_{A_{2\delta}} \widetilde{U}\,dx\right|\right)
\end{aligned}$$

and since $\int_A U\,dx = \int_A \widetilde{U}\,dx$, we obtain

$$\limsup_{\delta\to 0^+} m(u,U;A_\delta) \le m(u,U;A) + \varepsilon$$

and by letting ε go to zero we finish the proof. □

Again by analogy with [3, 4], for a fixed $(u,U) \in SD_2(\Omega)$ we set $\mu := \mathcal{L}^N \llcorner \Omega + |D_s^2 u|$, we define

$$\mathcal{A}^*(\Omega) := \{Q_\nu(x,\varepsilon) : x \in \Omega, \nu \in S^{N-1}, \varepsilon > 0\},$$

and for $A \in \mathcal{A}(\Omega)$ and $\delta > 0$,

$$\begin{aligned}
m^\delta(u,U;A) := \inf\Bigg\{ \sum_{i=1}^{\infty} m(u,U;Q_i) : Q_i \in \mathcal{A}^*(\Omega),\, Q_i \cap Q_j = \emptyset,\, Q_i \subset A,\\
\operatorname{diam}(Q_i) < \delta,\, \mu(A \setminus \cup_{i=1}^{\infty} Q_i) = 0\Bigg\}.
\end{aligned}$$

Since m^δ increases as δ goes to 0, we can define

$$m^*(u,U;A) := \sup_{\delta>0} m^\delta(u,U;A) = \lim_{\delta\to 0^+} m^\delta(u,U;A).$$

Lemma 4.2. *Assume that hypotheses (H1)-(H4) hold. Then for all $A \in \mathcal{A}(\Omega)$*

$$\mathcal{F}(u,U;A) = m^*(u,U;A).$$

Proof. Fix $A \in \mathcal{A}(\Omega)$. For every $\delta > 0$ and every collection of cubes $\{Q_i\}_{i=1}^\infty$ admissible in the definition of m^δ we obtain

$$m^\delta(u, U; A) \le \sum_{i=1}^\infty m(u, U; Q_i) \le \sum_{i=1}^\infty \mathcal{F}(u, U; Q_i) \le \mathcal{F}(u, U; A)$$

where we used (H1) in the last inequality. Hence $m^*(u, U; A) \le \mathcal{F}(u, U; A)$.

Conversely, fix $\delta > 0$ and choose a family $\{Q_i^\delta\}_{i=1}^\infty$ such that

$$\sum_{i=1}^\infty m(u, U; Q_i^\delta) \le m^\delta(u, U; A) + \delta.$$

For each Q_i^δ let $(v_i^\delta, V_i^\delta) \in \mathfrak{A}(u, U; Q_i^\delta)$ be such that

$$\mathcal{F}(v_i^\delta, V_i^\delta; Q_i^\delta) \le m(u, U; Q_i^\delta) + \delta \mathcal{L}^N(Q_i^\delta).$$

Now, we stitch together these v_i^δ and V_i^δ to define

$$v^\delta := \sum_{i=1}^\infty v_i^\delta \chi_{Q_i^\delta} + u\chi_{N_\delta}, \; V^\delta := \sum_{i=1}^\infty V_i^\delta \chi_{Q_i^\delta} + U\chi_{N_\delta},$$

where $N_\delta := \Omega \setminus \cup_{i=1}^\infty Q_i^\delta$. By the coercivity hypothesis (H4), we have $v^\delta \in BH(\Omega)$ and $V^\delta \in L^1(\Omega)$. By (H1) and (H3),

$$\mathcal{F}(v^\delta, V^\delta; A) = \sum_{i=1}^\infty \mathcal{F}(v_i^\delta, V_i^\delta; Q_i^\delta) + \mathcal{F}(u, U; N_\delta \cap A),$$

and since $\mu(N_\delta \cap A) = 0$, by (H4) we have $\mathcal{F}(u, U; N_\delta \cap A) = 0$, and so

$$\mathcal{F}(v^\delta, V^\delta; A) \le \sum_{i=1}^\infty \left[m(u, U; Q_i^\delta) + \delta \mathcal{L}^N(Q_i^\delta) \right] \le m^\delta(u, U; A) + \delta + \delta \mathcal{L}^N(A).$$

If we prove that $v^\delta \to u$ in $W^{1,1}(\Omega; \mathbb{R}^d)$ and $V^\delta \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$, then by lower semicontinuity of $\mathcal{F}$ (see (H2)), we will have

$$\mathcal{F}(u, U; A) \le \liminf_{\delta \to 0^+} \mathcal{F}(v^\delta, V^\delta; A) \le \liminf_{\delta \to 0^+} m^\delta(u, U; A) = m^*(u, U; A),$$

thus proving the lemma. To see that $v^\delta \to u$ in $W^{1,1}$, by the BV Poincaré inequality (see Theorem 5.10 in [8]) applied to $\left(\nabla u - \nabla v^\delta\right)$ we obtain

$$\begin{aligned} \|\nabla u - \nabla v^\delta\|_{L^1(\Omega)} = \sum_{i=1}^\infty \|\nabla u - \nabla v^\delta\|_{L^1(Q_i^\delta)} &\le \sum_{i=1}^\infty C\delta |D^2 u - D^2 v^\delta|(Q_i^\delta) \\ &\le C\delta(|D^2 u|(A) + |D^2 v^\delta|(A)). \end{aligned}$$

By coercivity of $\mathcal{F}$ we have that $\{|D^2 v^\delta|(A)\}$ is bounded, so this term goes to 0 with δ. By Poincaré's inequality applied now to $u - v^\delta$, we see that since $\|\nabla u - \nabla v^\delta\|_{L^1(\Omega)} \to 0$ we have that $\|u - v^\delta\|_{W^{1,1}(\Omega)} \to 0$. Finally, again by (H4)

$$\sup_\delta \|V^\delta\|_{L^1(\Omega)} < \infty$$

and applying Lemma 3.3 we conclude that $V^\delta \overset{*}{\rightharpoonup} U$ in $\mathcal{M}(\Omega)$. □

Theorem 4.3. *If (H1), (H2) and (H4) hold then for every $(u, U) \in SD_2(\Omega)$ and for all $\nu \in S^{N-1}$*

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} = \lim_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))}$$

for μ a.e. $x_0 \in \Omega$ where $\mu := \mathcal{L}^N \llcorner \Omega + |D^2_s u|$.

Proof. By (H4), $\mathcal{F}(u, U; \cdot)$ is absolutely continuous with respect to μ. Therefore, by Besicovitch's derivation theorem,

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))}$$

exist for μ-almost every $x_0 \in \Omega$. Since $m(u, U; \cdot) \leq \mathcal{F}(u, U; \cdot)$, we have trivially that

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} \geq \limsup_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))}$$

whenever the lefthand limit exists. Thus, it suffices to show that

$$\liminf_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} \geq \lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))}$$

for μ-almost every $x_0 \in \Omega$. Fix $t > 0$ and let

$$\begin{aligned} E_t := \{x \in \Omega : \exists\ \varepsilon_n \to 0 \text{ such that } \mu(\partial Q_\nu(x, \varepsilon_n)) = 0 \text{ and} \\ \mathcal{F}(u, U; Q_\nu(x, \varepsilon_n)) > m(u, U; Q_\nu(x, \varepsilon_n)) + t\mu(Q_\nu(x, \varepsilon_n)) \text{ for every } n\}. \end{aligned}$$

First, we observe that the condition that μ does not charge the boundary of the cubes is innocuous: for every $x \in \Omega$ such that there is a sequence $\{\varepsilon_n\}$ converging to 0 with

$$\mathcal{F}(u, U; Q_\nu(x, \varepsilon_n)) > m(u, U; Q_\nu(x, \varepsilon_n)) + t\mu(Q_\nu(x, \varepsilon_n))$$

for every n, we can find another sequence $\{\varepsilon'_n\}$ such that for every n

$$\mathcal{F}(u, U; Q_\nu(x, \varepsilon'_n)) > m(u, U; Q_\nu(x, \varepsilon'_n)) + t\mu(Q_\nu(x, \varepsilon'_n)),\ \mu(\partial Q_\nu(x, \varepsilon'_n)) = 0. \tag{4.4}$$

Indeed, for every n we can find $\varepsilon_n^k \nearrow \varepsilon_n$ so that $\mu(\partial Q_\nu(x, \varepsilon_n^k)) = 0$. By inner regularity we have

$$\lim_{k \to \infty} \mathcal{F}(u, U; Q_\nu(x, \varepsilon_n^k)) = \mathcal{F}(u, U; Q_\nu(x, \varepsilon_n)),\ \lim_{k \to \infty} \mu(Q_\nu(x, \varepsilon_n^k)) = \mu(Q_\nu(x, \varepsilon_n)),$$

and by Lemma 4.1

$$\limsup_{k\to\infty} m(u,U;Q_\nu(x,\varepsilon_n^k)) \le m(u,U;Q_\nu(x,\varepsilon_n)).$$

Hence for k large enough we have

$$\mathcal{F}(u,U;Q_\nu(x,\varepsilon_n^k)) > m(u,U;Q_\nu(x,\varepsilon_n^k)) + t\mu(Q_\nu(x,\varepsilon_n^k)).$$

Extracting a diagonal subsequence of $\{\varepsilon_n^k\}$ we obtain a suitable subsequence $\{\varepsilon_n' := \varepsilon_n^{k(n)}\}$ for which (4.4) holds. Thus we see that without loss of generality we can take the ε_n so that μ does not charge the boundary.

Fix a compact set $K \subset \Omega$ such that $K \subset E_t$. For $\delta > 0$, define the families of cubes

$$\begin{aligned} X^\delta := \{Q_\nu(x,\varepsilon) : \varepsilon < \delta,\ \overline{Q_\nu(x,\varepsilon)} \subset \Omega,\ \mu(\partial Q_\nu(x,\varepsilon)) = 0, \\ \mathcal{F}(u,U;Q_\nu(x,\varepsilon)) > m(u,U;Q_\nu(x,\varepsilon)) + t\mu(Q_\nu(x,\varepsilon))\}, \end{aligned}$$

$$Y^\delta = \{Q_\nu(x,\varepsilon) : \varepsilon < \delta,\ \overline{Q_\nu(x,\varepsilon)} \subset \Omega\setminus K,\ \mu(\partial Q_\nu(x,\varepsilon)) = 0\}.$$

Since $K \subset E_t$, for every $x \in K$ there exists $Q_\nu(x,\varepsilon) \in X^\delta$ for some $\varepsilon < \delta$, and, similarly, if $x \in \Omega \setminus K$ there exists a cube $Q_\nu(x,\varepsilon) \in Y^\delta$. Hence, we can write

$$\Omega = \bigcup_{Q\in X^\delta} Q \cup \bigcup_{Q'\in Y^\delta} Q'$$

and applying the Vitali-Besicovitch covering theorem, we can find a countable collection of $Q_i^{X^\delta} \in X^\delta, Q_j^{Y^\delta} \in Y^\delta$, all mutually disjoint, such that

$$\Omega = \bigcup_{i=1}^\infty Q_i^{X^\delta} \cup \bigcup_{j=1}^\infty Q_j^{Y^\delta} \cup E$$

where $\mu(E) = 0$, and, as a consequence $\mathcal{F}(u,U;E) = 0$. Note that since $Q_j^{Y^\delta} \subset \Omega \setminus K$ for all j, we have

$$\mu(K) = \mu(\Omega\cap K) = \mu\left(\bigcup_{i=1}^\infty Q_i^{X^\delta}\right),$$

and thus

$$\begin{aligned} \mathcal{F}(u,U;\Omega) &= \sum_{i=1}^\infty \mathcal{F}(u,U;Q_i^{X^\delta}) + \sum_{j=1}^\infty \mathcal{F}(u,U;Q_j^{Y^\delta}) \\ &\ge \sum_{i=1}^\infty \left[m(u,U;Q_i^{X^\delta}) + t\mu(Q_i^{X^\delta})\right] + \sum_{j=1}^\infty m(u,U;Q_j^{Y^\delta}) \\ &\ge m^\delta(u,U;\Omega) + t\sum_{i=1}^\infty \mu(Q_i^{X^\delta}) = m^\delta(u,U;\Omega) + t\mu(K). \end{aligned}$$

Sending $\delta \to 0$, we can apply Lemma 4.2 to obtain

$$\mathcal{F}(u, U; \Omega) \geq m^*(u, U, \Omega) + t\mu(K) = \mathcal{F}(u, U; \Omega) + t\mu(K)$$

and so $\mu(K) = 0$ for every compact $K \subset E_t$. By inner regularity we conclude that $\mu(E_t) = 0$, i.e., for μ-almost every $x \in \Omega$, if ε is sufficiently small,

$$\mathcal{F}(u, U; Q_\nu(x, \varepsilon)) \leq m(u, U; Q_\nu(x, \varepsilon)) + t\mu(Q_\nu(x, \varepsilon))$$

and thus

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} \leq \liminf_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} + t.$$

Sending $t \to 0$, we assert our claim. □

Lemma 4.4. *Assume that hypotheses (H1), (H3) and (H4) hold. Let* $\{(v_\varepsilon, V_\varepsilon)\} \subset SD_2(\Omega)$, $(u, U) \in SD_2(\Omega), x_0 \in \Omega, \nu \in S^{N-1}$, *and let* λ *be a nonnegative Radon measure on* Ω. *Let* $x_0 \in \Omega$ *and suppose that*

$$\lim_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\lambda(Q_\nu(x_0, \varepsilon))}$$

exists. Then,

$$\begin{aligned}
&\limsup_{\varepsilon \to 0^+} \frac{m(v_\varepsilon, V_\varepsilon; Q_\nu(x_0, \varepsilon))}{\lambda(Q_\nu(x_0, \varepsilon))} - \lim_{\varepsilon \to 0^+} \frac{m(u, U; Q_\nu(x_0, \varepsilon))}{\lambda(Q_\nu(x_0, \varepsilon))} \leq \\
&\limsup_{\delta \to 1^-} \limsup_{\varepsilon \to 0^+} \frac{C}{\lambda(Q_\nu(x_0, \varepsilon))} \Big\{ \varepsilon^{N+1} + \varepsilon^N (1 - \delta^N) + |D^2 u|(Q_\nu(x_0, \varepsilon) \setminus \overline{Q_\nu(x_0, \delta\varepsilon)}) \\
&\quad + |D^2 v_\varepsilon|(Q_\nu(x_0, \varepsilon) \setminus \overline{Q_\nu(x_0, \delta\varepsilon)}) + \frac{1}{\varepsilon^2 (1-\delta)^2} \int_{Q_\nu(x_0, \varepsilon)} |u(x) - v_\varepsilon(x)| \, dx \\
&\quad + \frac{1}{\varepsilon(1-\delta)} \int_{Q_\nu(x_0, \varepsilon)} |\nabla u(x) - \nabla v_\varepsilon(x)| \, dx + \left| \int_{Q_\nu(x_0, \varepsilon)} V_\varepsilon \, dx - \int_{Q_\nu(x_0, \delta\varepsilon)} U \, dx \right| \Big\}.
\end{aligned}$$

Proof. Fix $\delta \in (0, 1)$ and let $\varepsilon > 0$ be so small that $Q_\nu(x_0, \varepsilon) \subset \Omega$. Choose a cut-off function $\phi \in C_c^\infty(Q_\nu(x_0, \varepsilon))$ such that $\phi = 1$ in a neighborhood of $Q_\nu(x_0, \varepsilon\delta)$,

$$\|\nabla \phi\|_{L^\infty} \leq \frac{2}{\varepsilon(1-\delta)}, \qquad \text{and} \qquad \|\nabla^2 \phi\|_{L^\infty} \leq \frac{4}{\varepsilon^2 (1-\delta)^2}.$$

Define

$$w_\varepsilon := \begin{cases} \phi u + (1 - \phi) v_\varepsilon & \text{in } Q_\nu(x_0, \varepsilon), \\ v_\varepsilon & \text{otherwise,} \end{cases}$$

and choose $(\widetilde{u}, \widetilde{U}) \in \mathfrak{A}(u, U; Q_\nu(x_0, \varepsilon\delta))$ such that

$$\frac{1}{2}\varepsilon^{N+1} + m(u, U; Q_\nu(x_0, \varepsilon\delta)) \geq \mathcal{F}(\widetilde{u}, \widetilde{U}; Q_\nu(x_0, \varepsilon\delta)).$$

By outer regularity of $\mathcal{F}(\widetilde{u}, \widetilde{U}; \cdot)$ (see (H1)) we can find $\delta' \in (\delta, 1)$ such that

$$\mathcal{F}(\widetilde{u}, \widetilde{U}; Q_\nu(x_0, \varepsilon\delta')) - \frac{1}{2}\varepsilon^{N+1} \leq \mathcal{F}(\widetilde{u}, \widetilde{U}; Q_\nu(x_0, \varepsilon\delta)).$$

Set

$$\widetilde{v}_\varepsilon := \begin{cases} \widetilde{u} & \text{in } Q_\nu(x_0, \varepsilon\delta), \\ w_\varepsilon & \text{on } \Omega \setminus Q_\nu(x_0, \varepsilon\delta), \end{cases}$$

and

$$\widetilde{V}_\varepsilon := \begin{cases} \widetilde{U} & \text{in } Q_\nu(x_0, \varepsilon\delta), \\ \frac{1}{\mathcal{L}^N(Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,\varepsilon\delta))}\left(\int_{Q_\nu(x_0,\varepsilon)} V_\varepsilon\, dx - \int_{Q_\nu(x_0,\varepsilon\delta)} U\, dx\right) & \text{on } \Omega \setminus Q_\nu(x_0, \varepsilon\delta). \end{cases}$$

Recalling that $\int_{Q_\nu(x_0,\varepsilon\delta)} U\, dx = \int_{Q_\nu(x_0,\varepsilon\delta)} \widetilde{U}\, dx$, we have $(\widetilde{v}_\varepsilon, \widetilde{V}_\varepsilon) \in \mathfrak{A}(v_\varepsilon, V_\varepsilon; Q_\nu(x_0, \varepsilon))$, and by (H3) and (H4) we obtain

$$\begin{aligned} m(v_\varepsilon, V_\varepsilon; Q_\nu(x_0,\varepsilon)) &\leq \mathcal{F}(\widetilde{v}_\varepsilon, \widetilde{V}_\varepsilon; Q_\nu(x_0,\varepsilon)) \\ &\leq \mathcal{F}(\widetilde{u}, \widetilde{U}; Q_\nu(x_0,\varepsilon\delta')) + \mathcal{F}(w_\varepsilon, \widetilde{V}_\varepsilon; Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) \\ &\leq \varepsilon^{N+1} + m(u, U; Q_\nu(x_0,\varepsilon\delta)) + \\ &\qquad C\Bigg(\varepsilon^N(1-\delta^N) + |D^2 w_\varepsilon|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) + \\ &\qquad\qquad \left|\int_{Q_\nu(x_0,\varepsilon)} V_\varepsilon\, dx - \int_{Q_\nu(x_0,\varepsilon\delta)} U\, dx\right|\Bigg). \end{aligned} \tag{4.5}$$

Since

$$\nabla w_\varepsilon = (u - v_\varepsilon) \otimes \nabla\phi + \phi\nabla u + (1-\phi)\nabla v_\varepsilon,$$

we obtain

$$\begin{aligned} |D^2 w_\varepsilon|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) \leq C\Bigg\{ & |D^2 u|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) \\ & + |D^2 v_\varepsilon|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\varepsilon\delta)}) \\ & + \frac{1}{\varepsilon^2(1-\delta)^2}\int_{Q_\nu(x_0,\varepsilon)} |u(x) - v_\varepsilon(x)|\, dx \\ & + \frac{1}{\varepsilon(1-\delta)}\int_{Q_\nu(x_0,\varepsilon)} |\nabla u(x) - \nabla v_\varepsilon(x)|\, dx. \Bigg\} \end{aligned} \tag{4.6}$$

From Lemma 2.2 we deduce that

$$\begin{aligned} \overline{\lim_{\delta\to 1^-}}\,\overline{\lim_{\varepsilon\to 0^+}} \frac{m(u, U; Q_\nu(x_0,\varepsilon\delta))}{\lambda(Q_\nu(x_0,\varepsilon))} &= \overline{\lim_{\delta\to 1^-}}\,\overline{\lim_{\varepsilon\to 0^+}} \left(\frac{m(u, U; Q_\nu(x_0,\varepsilon\delta))}{\lambda(Q_\nu(x_0,\varepsilon\delta))}\frac{\lambda(Q_\nu(x_0,\varepsilon\delta))}{\lambda(Q_\nu(x_0,\varepsilon))}\right) \\ &\leq \lim_{\varepsilon\to 0^+} \frac{m(u, U; Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))}, \end{aligned}$$

and hence to complete the proof it suffices to substitute (4.6) into (4.5), divide the resulting inequality by $\lambda(Q_\nu(x_0,\varepsilon))$ and take the lim sup as $\varepsilon \to 0^+$ and $\delta \to 1^-$. □

The next corollary is an immediate consequence of the previous lemma.

Corollary 4.5. *Assume that hypotheses (H1), (H3) and (H4) hold. Let* $(v,V),(u,U) \in SD_2(\Omega), x_0 \in \Omega, \nu \in S^{N-1}$*, and let* λ *be a nonnegative Radon measure on* Ω *be given. Let* $x_0 \in \Omega$ *and suppose that*

$$\lim_{\varepsilon\to 0^+} \frac{m(u,U;Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))} \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{m(v,V;Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))}$$

exist. Then

$$\begin{aligned}
&\left|\lim_{\varepsilon\to 0^+} \frac{m(v,V;Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))} - \lim_{\varepsilon\to 0^+} \frac{m(u,U;Q_\nu(x_0,\varepsilon))}{\lambda(Q_\nu(x_0,\varepsilon))}\right| \\
&\leq \limsup_{\delta\to 1^-}\limsup_{\varepsilon\to 0^+} \frac{C}{\lambda(Q_\nu(x_0,\varepsilon))}\Big\{\varepsilon^{N+1} + \varepsilon^N(1-\delta^N) + |D^2u|(Q_\nu(x_0,\varepsilon)\setminus\overline{Q_\nu(x_0,\delta\varepsilon)}) \\
&\quad + |D^2v|(Q_\nu(x_0,\varepsilon)\setminus\overline{Q_\nu(x_0,\delta\varepsilon)}) + \frac{1}{\varepsilon^2(1-\delta)^2}\int_{Q_\nu(x_0,\varepsilon)} |u(x)-v(x)|\,dx \\
&\quad + \frac{1}{\varepsilon(1-\delta)}\int_{Q_\nu(x_0,\varepsilon)} |\nabla u(x)-\nabla v(x)|\,dx + \left|\int_{Q_\nu(x_0,\varepsilon)} V\,dx - \int_{Q_\nu(x_0,\varepsilon)} U\,dx\right| \\
&\quad + \left|\int_{Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,\delta\varepsilon)} V\,dx\right| + \left|\int_{Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,\delta\varepsilon)} U\,dx\right|\Big\}.
\end{aligned}$$

Theorem 4.6. *Under hypotheses (H1), (H2), (H3),and (H4), for every* $(u,U) \in SD_2(\Omega)$ *and* $A \in \mathcal{A}(\Omega)$ *we have*

$$\mathcal{F}(u,U;A) = \int_A f(x,u,\nabla u,\nabla^2 u,U)\,dx + \int_{S(\nabla u)\cap A} h(x,u,\nabla u^+,\nabla u^-,\nu_{\nabla u})\,d\mathcal{H}^{N-1},$$

where

$$f(x_0,r,\xi,G,H) := \lim_{\varepsilon\to 0^+} \frac{m(r+\xi(\cdot-x_0)+1/2G(\cdot-x_0,\cdot-x_0),H;Q(x_0,\varepsilon))}{\varepsilon^N},$$

$$h(x_0,r,\eta,\zeta,\nu) := \lim_{\varepsilon\to 0^+} \frac{m(r+u_{\eta,\zeta,\nu}(\cdot-x_0),O;Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}},$$

for all $x_0 \in \Omega, r \in \mathbb{R}^N, \xi,\eta,\zeta \in \mathbb{R}^{d\times N}, G,H \in \mathbb{R}^{d\times N\times N}, \nu \in S^{N-1}$*, with* $O \in \mathbb{R}^{d\times N\times N}$ *being the matrix with all entries equal to zero, and*

$$u_{\eta,\zeta,\nu}(y) := \begin{cases} \eta y & \text{if } y\cdot\nu > 0, \\ \zeta y & \text{otherwise.}\end{cases}$$

Proof. We first show that

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{L}^N}(x_0) = f(x_0,u(x_0),\nabla u(x_0),\nabla^2 u(x_0),U(x_0)) \tag{4.7}$$

for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$. Define

$$v_a(x) := u(x_0) + \nabla u(x_0)(x-x_0) + \frac{1}{2}\nabla^2 u(x_0)(x-x_0,x-x_0).$$

By Theorem 2.1 and Theorem 4.3, for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$,

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2} \fint_{Q(x_0,\varepsilon)} |u(x) - v_a(x)| \, dx = 0, \qquad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \fint_{Q(x_0,\varepsilon)} |\nabla u(x) - \nabla v_a(x)| \, dx = 0, \tag{4.8}$$

$$\lim_{\varepsilon\to 0^+} \frac{|D^2 u|(Q(x_0;\varepsilon))}{\mathcal{L}^N(Q(x_0;\varepsilon))} = |\nabla^2 u(x_0)|, \tag{4.9}$$

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+} \frac{m(u,U;Q(x_0;\varepsilon))}{\mathcal{L}^N(Q(x_0;\varepsilon))}, \tag{4.10}$$

$$\frac{d\mathcal{F}(v_a,U(x_0);\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+} \frac{m(v_a,U(x_0);Q(x_0;\varepsilon))}{\mathcal{L}^N(Q(x_0;\varepsilon))}, \tag{4.11}$$

$$\lim_{\varepsilon\to 0^+} \fint_{Q(x_0,\varepsilon)} |U(x) - U(x_0)| \, dx = 0 \tag{4.12}$$

Select a point $x_0 \in \Omega$ with the above properties. Apply Corollary 4.5 with $v := v_a, V := U(x_0)$ and $\lambda := \mathcal{L}^N \llcorner \Omega$ to find

$$\begin{aligned}&\left| \lim_{\varepsilon\to 0^+} \frac{m(v_a,U(x_0);Q_\nu(x_0,\varepsilon))}{\mathcal{L}^N(Q_\nu(x_0,\varepsilon))} - \lim_{\varepsilon\to 0^+} \frac{m(u,U;Q_\nu(x_0,\varepsilon))}{\mathcal{L}^N(Q_\nu(x_0,\varepsilon))} \right| \\ &\qquad \le C \limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \mathcal{G}(\varepsilon,\delta,u,v_a,U),\end{aligned}$$

where

$$\begin{aligned}\mathcal{G}(\varepsilon,\delta,u,v_a,U) \;:=\; & C\Bigg\{ \varepsilon + (1-\delta^N) + \frac{|D^2u|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\delta\varepsilon)})}{\varepsilon^N} \\ & + |\nabla^2 u(x_0)|(1-\delta^N) + \frac{1}{(1-\delta)^2} \frac{1}{\varepsilon^2} \fint_{Q(x_0,\varepsilon)} |u(x) - v_a(x)| \, dx \\ & + \frac{1}{(1-\delta)} \frac{1}{\varepsilon} \fint_{Q(x_0,\varepsilon)} |\nabla u(x) - \nabla v_a(x)| \, dx + \left| \fint_{Q(x_0,\varepsilon)} U \, dx - U(x_0) \right| \\ & + \left| \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)\setminus Q(x_0,\varepsilon\delta)} U \, dx \right| + (1-\delta^N) U(x_0) \Bigg\}.\end{aligned}$$

By (4.9) we find

$$0 \le \limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \frac{|D^2u|(Q_\nu(x_0,\varepsilon) \setminus \overline{Q_\nu(x_0,\delta\varepsilon)})}{\varepsilon^N} \le \limsup_{\delta\to 1^-} |\nabla^2 u(x_0)|(1-\delta^N) = 0,$$

and by (4.12) we obtain

$$\begin{aligned}&\limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \left| \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)\setminus Q(x_0,\varepsilon\delta)} U \, dx \right| \\ &\qquad = \limsup_{\delta\to 1^-} \limsup_{\varepsilon\to 0^+} \left| \fint_{Q(x_0,\varepsilon)} U \, dx - \delta^N \fint_{Q(x_0,\varepsilon\delta)} U \, dx \right| \\ &\qquad = \limsup_{\delta\to 1^-} |U(x_0) - \delta^N U(x_0)| = 0,\end{aligned}$$

which, together with (4.8), yields

$$\limsup_{\delta\to 1^-}\limsup_{\varepsilon\to 0^+}\mathcal{G}(\varepsilon,\delta,u,v_a,U)=0$$

and, consequently,

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{L}^N}(x_0)=\lim_{\varepsilon\to 0^+}\frac{m(u,U;Q_\nu(x_0,\varepsilon))}{\mathcal{L}^N(Q_\nu(x_0,\varepsilon))}=\lim_{\varepsilon\to 0^+}\frac{m(v_a,U(x_0);Q_\nu(x_0,\varepsilon))}{\mathcal{L}^N(Q_\nu(x_0,\varepsilon))},$$

concluding the proof of (4.7).

Now we show that

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(\nabla u)}(x_0)=g(x_0,u(x_0),\nabla u^+(x_0),\nabla u^-(x_0),\nu_{\nabla u}(x_0)),$$

for $\mathcal{H}^{N-1}\lfloor S(\nabla u)$ a.e. $x_0\in\Omega$. Hereafter, for simplicity, we will just write ν in place of $\nu_{\nabla u}$. Define

$$v_J(x):=u(x_0)+\begin{cases}\nabla u^+(x_0)(x-x_0) & \text{if } (x-x_0)\cdot\nu(x_0)>0,\\ \nabla u^-(x_0)(x-x_0) & \text{if } (x-x_0)\cdot\nu(x_0)<0.\end{cases}$$

Again by Theorem 2.1 and Theorem 4.3, for $\mathcal{H}^{N-1}$ a.e. $x_0\in S(\nabla u)$ we have

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\fint_{Q_\nu(x_0,\varepsilon)}|u(x)-v_J(x)|\,dx=0,\quad \lim_{\varepsilon\to 0^+}\fint_{Q_\nu(x_0,\varepsilon)}|\nabla u(x)-\nabla v_J(x)|\,dx=0, \tag{4.13}$$

$$\lim_{\varepsilon\to 0^+}\frac{|D^2u|(Q_\nu(x_0;\varepsilon))}{\varepsilon^{N-1}}=|[\nabla u](x_0)|, \tag{4.14}$$

$$\frac{d\mathcal{F}(u,U;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(\nabla u)}(x_0)=\lim_{\varepsilon\to 0^+}\frac{m(u,U;Q_\nu(x_0;\varepsilon))}{\varepsilon^{N-1}}, \tag{4.15}$$

$$\frac{d\mathcal{F}(v_J,O;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(\nabla u)}(x_0)=\lim_{\varepsilon\to 0^+}\frac{m(v_J,O;Q_\nu(x_0;\varepsilon))}{\varepsilon^{N-1}}, \tag{4.16}$$

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N-1}}\int_{Q_\nu(x_0;\varepsilon)}|U|dx=0. \tag{4.17}$$

Select a point $x_0\in S(\nabla u)$ such that the above properties hold. Apply Corollary 4.5 with $v:=v_J, V:=O$ and $\lambda:=\mathcal{H}^{N-1}\llcorner S(\nabla u)$ to deduce that

$$\left|\lim_{\varepsilon\to 0^+}\frac{m(v_J,O;Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}-\lim_{\varepsilon\to 0^+}\frac{m(u,U;Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}\right|$$
$$\le C\limsup_{\delta\to 1^-}\limsup_{\varepsilon\to 0^+}\mathcal{G}_J(\varepsilon,\delta,u,v_J,U),$$

where

$$\begin{aligned}
\mathcal{G}_J(\varepsilon,\delta,u,v_J,U) &= C\bigg\{\varepsilon^2+\varepsilon(1-\delta^N)+\frac{|D^2u|(Q_\nu(x_0,\varepsilon)\setminus\overline{Q_\nu(x_0,\delta\varepsilon)})}{\varepsilon^{N-1}}\\
&\quad+|[\nabla u](x_0)|(1-\delta^{N-1})+\frac{1}{(1-\delta)^2}\frac{1}{\varepsilon}\fint_{Q_\nu(x_0,\varepsilon)}|u(x)-v_J(x)|\,dx\\
&\quad+\frac{1}{(1-\delta)}\fint_{Q_\nu(x_0,\varepsilon)}|\nabla u(x)-\nabla v_J(x)|\,dx+\varepsilon\left|\fint_{Q_\nu(x_0,\varepsilon)}U\,dx\right|\\
&\quad+\left|\frac{1}{\varepsilon^{N-1}}\int_{Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,\varepsilon\delta)}U\,dx\right|\bigg\}.
\end{aligned}$$

By (4.14) we find

$$0\le\limsup_{\delta\to1^-}\limsup_{\varepsilon\to0^+}\frac{|D^2u|(Q_\nu(x_0,\varepsilon)\setminus\overline{Q_\nu(x_0,\delta\varepsilon)})}{\varepsilon^{N-1}}\le\limsup_{\delta\to1^-}|[\nabla u](x_0)|(1-\delta^{N-1})=0,$$

while from (4.17) we obtain

$$\limsup_{\varepsilon\to0^+}\left|\frac{1}{\varepsilon^{N-1}}\int_{Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,\varepsilon\delta)}U\,dx\right|\le\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N-1}}\int_{Q_\nu(x_0,\varepsilon)}|U|\,dx=0,$$

and thus, using Eq. (4.13), we conclude that

$$\limsup_{\delta\to1^-}\limsup_{\varepsilon\to0^+}\mathcal{G}_J(\varepsilon,\delta,u,v_J,U)=0,$$

and hence the proof is completed. □

5 Applications (SD_2 integral representation)

We consider the functional defined for each $A\in\mathcal{A}(\Omega)$ by

$$\mathcal{F}_0(u;A):=\begin{cases}\int_A f_0(x,u,\nabla u,\nabla^2u)\,dx\\ \qquad+\int_{S(\nabla u)\cap A}g_0(x,u,\nabla u^+,\nabla u^-,\nu_{\nabla u})\,d\mathcal{H}^{N-1} & \text{if } u\in SBH(\Omega;\mathbb{R}^d),\\ +\infty & \text{otherwise,}\end{cases}\tag{5.1}$$

where the densities f_0 and g_0 satisfy the following hypotheses:

(G1) $f_0:\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}\times\mathbb{R}^{d\times N\times N}\to[0,+\infty)$ is measurable in x and continuous in all other variables and

$$\frac{1}{C}|\Lambda|\le f_0(x,u,\xi,\Lambda)\le C(1+|\Lambda|)$$

for all $(x,u,\xi,\Lambda)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}\times\mathbb{R}^{d\times N\times N}$ and for some $C>0$;

(G2) the function $g_0:\Omega\times\mathbb{R}^d\times(\mathbb{R}^{d\times N})^2\times S^{N-1}\to[0,+\infty)$ is continuous and

$$\frac{1}{C}|\xi-\eta|\le g_0(x,u,\xi,\eta,\nu)\le C(1+|\xi-\eta|)$$

for all $(x, u, \xi, \eta, \nu) \in \Omega \times \mathbb{R}^d \times (\mathbb{R}^{d\times N})^2 \times S^{N-1}$ and for some $C > 0$;

The functional $\mathcal{F} : SD_2(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty)$ is defined by

$$\mathcal{F}(u, U; A) := \inf \Bigg\{ \liminf_{n\to+\infty} \mathcal{F}_0(u_n; A) : u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^d), \\ \nabla^2 u_n \overset{*}{\rightharpoonup} U \text{ in } \mathcal{M}(\Omega; \mathbb{R}^{d\times N\times N}) \Bigg\}. \tag{5.2}$$

Lemma 5.1. *For every $(u, U) \in SD_2(\Omega)$, $A \in \mathcal{A}(\Omega)$ and every sequence $\{(u_n, U_n)\} \subset SD_2(\Omega)$ such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and $U_n \overset{*}{\rightharpoonup} U$ in in $\mathcal{M}(\Omega; \mathbb{R}^{d\times N\times N})$,*

$$\mathcal{F}(u, U; A) \le \liminf_{n\to\infty} \mathcal{F}(u_n, U_n; A).$$

Proof. Fix a sequence $\{(u_n, U_n)\} \subset SD_2(\Omega)$ such that $u_n \to u$ in L^1 and $U_n \overset{*}{\rightharpoonup} U$. For every (u_n, U_n) we can pick a sequence $\{(u_{n,k}, U_{n,k})\} \subset SD_2(\Omega)$ such that $u_{n,k} \to u_n$ in L^1 and $U_{n,k} \overset{*}{\rightharpoonup} U_n$ as $k \to \infty$ and

$$\liminf_{k\to\infty} \mathcal{F}_0(u_{n,k}, U_{n,k}; A) \le \mathcal{F}(u_n, U_n; A) + \frac{1}{n}.$$

By diagonalizing we find sequences $v_n := u_{n,k_n}$ and $V_n := v_{n,k_n}$ such that $v_n \to u$ in L^1, $V_n \overset{*}{\rightharpoonup} U$ as $n \to \infty$, and

$$\liminf_{n\to\infty} \mathcal{F}_0(v_n, V_n; A) \le \liminf_{n\to\infty} \mathcal{F}(u_n, U_n; A)$$

and thus

$$\mathcal{F}(u, U; A) \le \liminf_{n\to\infty} \mathcal{F}(u_n, U_n; A).$$

□

Lemma 5.2. *The functional $\mathcal{F}$ is local, i.e. for all $A \in \mathcal{A}(\Omega)$, if $u = v$ and $U = V$ $\mathcal{L}^N$ a.e. $x \in A$ then $\mathcal{F}(u, U; A) = \mathcal{F}(v, V; A)$.*

Proof. Let A, u, U, v and V be as in the statement of the lemma. For every sequence $\{(u_n, U_n)\} \subset SD_2(\Omega)$ such that $u_n \to u$ in $L^1(A)$ and $U_n \overset{*}{\rightharpoonup} U$, we also have $u_n \to v$ in $L^1(A)$ and $V_n \overset{*}{\rightharpoonup} V$. Thus

$$\mathcal{F}(u, U; A) \ge \mathcal{F}(v, V; A),$$

and by symmetry we conclude that

$$\mathcal{F}(u, U; A) = \mathcal{F}(v, V; A).$$

□

Lemma 5.3. *Assume hypotheses (G1) and (G2) hold. For every $(u, U) \in SD_2(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$ we have*

$$\frac{1}{C}\left(\|U\|_{L^1(A)} + |D^2u|(A)\right) \le \mathcal{F}(u; A) \le C\left(\mathcal{L}^N(A) + \|U\|_{L^1(A)} + |D^2u|(A)\right)$$

where $C > 0$. Moreover, for every $u \in BH(\Omega; \mathbb{R}^d)$ the functional $\mathcal{F}(u; \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure.

Proof. We note that hypotheses (G1) and (G2) imply that

$$\frac{1}{C}|D^2u|(A) \le \mathcal{F}_0(u;A) \le C(\mathcal{L}^N(A) + |D^2u|(A))$$

for every $u \in SBH(\Omega;\mathbb{R}^N)$ and $A \in \mathcal{A}(\Omega)$. For any $(u,U) \in SD_2(\Omega)$ and any $\delta > 0$, we can find $u_n \in SBH(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in L^1, $\nabla^2 u_n \overset{*}{\rightharpoonup} U$, and

$$\mathcal{F}(u,U;A) \ge \liminf_{n\to\infty} \mathcal{F}_0(u_n;A) - \delta.$$

On one hand, this implies that

$$\mathcal{F}(u,U;A) \ge \liminf_{n\to\infty} \frac{1}{C}|D^2u_n|(A) - \delta \ge \frac{1}{C}|D^2u|(A) - \delta,$$

and letting $\delta \to 0$ we have

$$\mathcal{F}(u,U;A) \ge \frac{1}{C}|D^2u|(A). \tag{5.3}$$

On the other hand, we obtain

$$\mathcal{F}(u,U;A) \ge \liminf_{n\to\infty} \frac{1}{C}|D^2u_n|(A) - \delta \ge \liminf_{n\to\infty} \frac{1}{C}\|\nabla^2 u_n\|_{L^1(A)} - \delta \ge \frac{1}{C}\|U\|_{L^1(A)} - \delta$$

and, again letting $\delta \to 0$, we have

$$\mathcal{F}(u,U;A) \ge \frac{1}{C}\|U\|_{L^1(A)}.$$

Averaging this with (5.3), we deduce that

$$\mathcal{F}(u,U;A) \ge \frac{1}{C}\left(\|U\|_{L^1(A)} + |D^2u|(A)\right).$$

To prove the upper bound, we consider the sequence $\{u_n\}$ constructed in the Approximation Theorem, Theorem 3.2, which satisfies $u_n \to u$ in L^1, $\nabla^2 u_n \overset{*}{\rightharpoonup} U$ and

$$\sup_n |D(\nabla u_n)|(A) \le C\left(|D(\nabla u)|(A) + \|U\|_{L^1(A)}\right).$$

Then we have

$$\begin{aligned}\mathcal{F}(u,U;A) &\le \liminf_{n\to\infty} \mathcal{F}_0(u_n;A) \le \liminf_{n\to\infty} C\left(\mathcal{L}^N(A) + |D(\nabla u_n)|(A)\right)\\ &\le C\left(\mathcal{L}^N(A) + |D(\nabla u)|(A) + \|U\|_{L^1(A)}\right).\end{aligned}$$

Finally, we will prove that for $(u,U) \in SD_2(\Omega)$, $\mathcal{F}(u,U;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure. We will apply the coincidence criterion, Lemma 2.3. Since item (ii) follows directly from the fact that $\mathcal{F}_0(u,\cdot)$ is a Radon measure and item (iii) follows from the growth condition that we have just proved, it only remains to prove that for any open sets $A,B,C \in \mathcal{A}(\Omega)$ with $\overline{A} \subset B \subset C$ we have

$$\mathcal{F}(u,U;C) \le \mathcal{F}(u,U;C \setminus \overline{A}) + \mathcal{F}(u,U;B).$$

To see this, for $\varepsilon > 0$ we choose $v_n \in BH(\Omega;\mathbb{R}^d)$ and $w_n \in BH(\Omega;\mathbb{R}^d)$ as in the definition of $\mathcal{F}(u,U;\cdot)$ (perhaps along a subsequence) so that

$$\lim_{n\to\infty} \mathcal{F}_0(v_n, C\setminus\overline{A}) \le \mathcal{F}(u,U;C\setminus\overline{A}) - \varepsilon \tag{5.4}$$

and

$$\lim_{n\to\infty} \mathcal{F}_0(w_n, B) \le \mathcal{F}(u,U;B) - \varepsilon.$$

We will use a slicing argument in order to construct (up to a subsequence) a sequence $\{u_n\} \subset BH(C;\mathbb{R}^d)$ as in the definition of $\mathcal{F}(u,U;\cdot)$ so that

$$\liminf_{n\to\infty} \mathcal{F}_0(u_n;C) \le \lim_{n\to\infty} \mathcal{F}_0(v_n, C\setminus\overline{A}) + \lim_{n\to\infty} \mathcal{F}_0(w_n, B).$$

Let $\delta > 0$ be so small that

$$S_\delta := \{x \in B : \operatorname{dist}(x,A) < \delta\} \subset\subset B.$$

Given $k \in \mathbb{N}$ we can decompose $S_\delta \setminus A$ into a disjoint union of strips, to be precise we write

$$S_\delta \setminus A = \bigcup_{i=1}^{k} L_{i,k},$$

where

$$L_{i,k} = \left\{ x \in S_\delta : \frac{(i-1)\delta}{k} < \operatorname{dist}(x,A) \le \frac{i\delta}{k} \right\}.$$

By coercivity of $\mathcal{F}_0$, we have

$$\sup_n |D(\nabla v_n)|(C\setminus\overline{A}) + \sup_n |D(\nabla w_n)|(B) \le M$$

for some $M < \infty$, and thus

$$\sup_n \sum_{i=1}^{k} \left(|D(\nabla v_n)| + |D(\nabla w_n)|\right)(L_{i,k}) \le M.$$

We remark that since there are only finitely many values of i and infinitely many values of n, there must be some fixed i such that

$$\left(|D(\nabla v_n)| + |D(\nabla w_n)|\right)(L_{i,k}) \le \frac{M}{k}$$

for infinitely many $n \in \mathbb{N}$. Thus for any k, there is a $i_k \in \{1,\dots,k\}$ and a subsequence $\{n_j^{(k)}\} \subset \{n\}$ such that

$$\left(|D(\nabla v_{n_j^{(k)}})| + |D(\nabla w_{n_j^{(k)}})|\right)(L_{i_k,k}) \le \frac{M}{k}, \quad \forall j,k \in \mathbb{N}.$$

We consider a smoooth cutoff function $\phi_k \in C_c^\infty(B;[0,1])$ such that $\{0<\phi_k<1\} \subset L_{i_k,k}$, $\phi_k(x)=0$ if $\operatorname{dist}(x,A) \le \frac{i_k-1}{k}\delta$, $\phi_k(x)=1$ if $\operatorname{dist}(x,A)\ge \frac{i_k}{k}\delta$ and

$$\|\nabla\phi_k\|_\infty \le Ck,\ \|\nabla^2\phi_k\|_\infty \le Ck^2.$$

For $x \in C$, we define

$$u_{j,k} = \phi_k v_{n_j^{(k)}} + (1-\phi_k) w_{n_j^{(k)}}.$$

Then we have

$$\mathcal{F}_0(u_{j,k};C) \le \mathcal{F}_0(v_{n_j^{(k)}}; C\setminus\overline{A}) + \mathcal{F}_0(w_{n_j^{(k)}};B) + \mathcal{F}_0(u_{j,k};L_{i_k,k}),$$

and the last term is bounded by

$$\begin{aligned}\mathcal{F}_0(u_{j,k};L_{i_k,k}) \le & C\bigg(\mathcal{L}^N(L_{i_k,k}) + k^2\int_{L_{i_k,k}} |v_{n_j^{(k)}} - w_{n_j^{(k)}}|dx + k\int_{L_{i_k,k}} |\nabla v_{n_j^{(k)}} - \nabla w_{n_j^{(k)}}|dx \\ & + |D(\nabla v_{n_j^{(k)}})|(L_{i_k,k}) + |D(\nabla w_{n_j^{(k)}})|(L_{i_k,k})\bigg) \\ \le & C\bigg(\frac{1}{k} + k^2\int_{L_{i_k,k}} |v_{n_j^{(k)}} - w_{n_j^{(k)}}|dx + k\int_{L_{i_k,k}} |\nabla v_{n_j^{(k)}} - \nabla w_{n_j^{(k)}}|dx\bigg).\end{aligned}$$

Since $v_n \to u$ and $w_n \to u$ in $W^{1,1}(B\setminus\overline{A})$, for any k we can choose an element $n_{j_k}^{(k)}$ of $n_j^{(k)}$ so that the map $k \mapsto n_{j_k}^{(k)}$ is increasing and

$$\int_{B\setminus\overline{A}} |v_{n_{j_k}^{(k)}} - w_{n_{j_k}^{(k)}}|dx = o(1/k^2)$$

and

$$\int_{B\setminus\overline{A}} |\nabla v_{n_{j_k}^{(k)}} - \nabla w_{n_{j_k}^{(k)}}|dx = o(1/k).$$

With this choice we have that

$$\liminf_{k\to\infty} \mathcal{F}_0(u_{j_k,k};L_{i_k,k}) = 0.$$

Since $v_n \to u$ in $L^1(C\setminus\overline{A})$, $w_n \to u$ in $L^1(B)$ and $\nabla^2 v_n \overset{*}{\rightharpoonup} U$ in $C\setminus\overline{A}$, $\nabla^2 w_n \overset{*}{\rightharpoonup} U$ in B, we must have that $u_{j_k,k} \to u$ in $L^1(C)$ and $\nabla^2 u_{j_k,k} \overset{*}{\rightharpoonup} U$ in C. Thus, by definition of $\mathcal{F}(u,U;\cdot)$, we conclude

$$\begin{aligned}\mathcal{F}(u,U;C) \le \liminf_{k\to\infty}\mathcal{F}_0(u_{j_k,k};C) &\le \lim_{k\to\infty}\mathcal{F}_0(v_{n_{j_k}^{(k)}};C\setminus\overline{A}) + \lim_{k\to\infty}\mathcal{F}_0(w_{n_{j_k}^{(k)}};B) \\ &\le \mathcal{F}(u,U;C\setminus\overline{A}) + \mathcal{F}(u,U;B) - 2\varepsilon.\end{aligned}$$

Sending $\varepsilon \to 0$, we are done. □

Theorem 5.4. *Assume (G1) and (G2) hold and let $\mathcal{F}$ be the functional defined by (5.2). Then, there exist functions $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N\times N} \times \mathbb{R}^{d\times N\times N} \to [0,\infty)$ and $g : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \times S^{N-1} \to [0,\infty)$ such that*

$$\begin{aligned}\mathcal{F}(u,U;A) \quad := \quad & \int_A f(x,u(x),\nabla u(x),\nabla^2 u(x),U)\,dx \\ & + \int_{S(\nabla u)\cap A} g(x,u(x),\nabla u^+(x),\nabla u^-(x),\nu_{\nabla u}(x))\,d\mathcal{H}^{N-1}(x),\end{aligned}$$

for all $u \in SBH(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$.

Proof. We note that by Lemmas 5.1, 5.2 and 5.3, the functional $\mathcal{F}$ satisfies the hypotheses of Theorem 4.6, and so the integral representation result follows immediately. □

6 Applications (SBH, BH integral representation)

In this section we obtain integral representation results for abstract lower semicontinuous functionals on SBH and BH. Consider a functional

$$\mathcal{F} : BH(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,+\infty] \tag{6.1}$$

satisfying the following hypotheses:
(I1) $\mathcal{F}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure,
(I2) $\mathcal{F}(\cdot;A)$ is $L^1(A,\mathbb{R}^d)$-lower semicontinuous,
(I3) $\mathcal{F}$ is local, i.e., for all $A \in \mathcal{A}(\Omega)$ if $u = v$ $\mathcal{L}^N$ a.e. in A then $\mathcal{F}(u;A) = \mathcal{F}(v;A)$,
(I4) there exists a constant $C > 0$ such that

$$\frac{1}{C}|D^2u|(A) \le \mathcal{F}(u;A) \le C(\mathcal{L}^N(A) + |D^2u|(A)).$$

Given $(u,A) \in BH(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega)$ we introduce

$$\mathfrak{A}(u;A) := \left\{v \in BH(\Omega;\mathbb{R}^d) : \text{spt }(u-v) \subset\subset A\right\}, \tag{6.2}$$

and

$$m(u;A) := \inf\{\mathcal{F}(v,A) : v \in \mathfrak{A}(u;A)\}. \tag{6.3}$$

As a corollary of Theorem 4.6, we have the following SBH representation theorem.

Theorem 6.1. *Under hypotheses (I1), (I2), (I3) and (I4), for every $u \in SBH(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have*

$$\mathcal{F}(u;A) = \int_A f(x,u,\nabla u,\nabla^2 u)\,dx + \int_{S(\nabla u)\cap A} h(x,u,\nabla u^+,\nabla u^-,\nu_{\nabla u})\,d\mathcal{H}^{N-1},$$

where

$$f(x_0,g,G,\Sigma) := \lim_{\varepsilon\to 0^+} \frac{m(g + G(\cdot - x_0) + 1/2\Sigma(\cdot - x_0,\cdot - x_0);Q(x_0,\varepsilon))}{\varepsilon^N},$$

$$h(x_0,g,L,H,\nu) := \lim_{\varepsilon\to 0^+} \frac{m(g+u_{L,H,\nu}(\cdot - x_0); Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}},$$

for all $x_0 \in \Omega, g \in \mathbb{R}^d, G, H, L \in \mathbb{R}^{d\times N}, \Sigma \in \mathbb{R}^{d\times N\times N}, \nu \in S^{N-1}$, *and where*

$$u_{L,H,\nu}(y) := \begin{cases} Ly & \text{if } y\cdot\nu > 0, \\ Hy & \text{otherwise.} \end{cases}$$

In the case where the functional $\mathcal{F}$ is invariant under affine translations of u, we can use this result to upper bound $\mathcal{F}$ on the space BH.

Corollary 6.2. *Let* $\mathcal{F}$ *satisfy hypotheses* $(I1), (I2), (I3), (I4)$, *and further assume that for every affine function*

$$v(x) := p + Ax$$

for $p \in \mathbb{R}^d, A \in \mathbb{R}^{d\times N}$, *we have*

$$\mathcal{F}(u;\cdot) = \mathcal{F}(u+v;\cdot).$$

Then for every $u \in SBH(\Omega;\mathbb{R}^d)$ *and* $A \in \mathcal{A}(\Omega)$ *we have*

$$\mathcal{F}(u;A) = \int_A f(x,\nabla^2 u)\,dx + \int_{S(\nabla u)\cap A} h(x, \nabla u^+ - \nabla u^-, \nu_{\nabla u})\,d\mathcal{H}^{N-1}$$

where, with an abuse of notation, we write $f(x,\Sigma) := f(x,0,0,\Sigma)$ *and* $h(x,J,\nu) := h(x,0,0,J,\nu)$. *Moreover, for* $u \in BH(\Omega;\mathbb{R}^d)$ *and* $A \in \mathcal{A}(\Omega)$ *we have*

$$\mathcal{F}(u,A) \le \int_A f(x,\nabla^2 u)\,dx + \int_A f^\infty\left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)}\right) d|D_s(\nabla u)|$$

where f^∞ *is the recession function defined by*

$$f^\infty(x,\Sigma) = \lim_{t\to\infty} \frac{f(x,t\Sigma)}{t}.$$

Proof. The assumption that $\mathcal{F}$ is affine invariant implies that m is also affine invariant. Thus for any $x_0 \in \Omega$, $g \in \mathbb{R}^d$, $G \in \mathbb{R}^{d\times N}$, $\Sigma \in \mathbb{R}^{d\times N\times N}$, $\nu \in S^{N-1}$, we have

$$f(x_0,g,G,\Sigma) = f(x_0,0,0,\Sigma)$$

and for any $x_0 \in \Omega, g \in \mathbb{R}^d, L, H \in \mathbb{R}^{d\times N}, \nu \in S^{N-1}$ we have

$$h(x_0,g,L,H,\nu) = g(x_0,0,0,H-L,\nu).$$

In particular, we deduce that for every $u \in W^{2,1}(\Omega;\mathbb{R}^d)$

$$\mathcal{F}(u;A) = \int_A f(x,\nabla^2 u)\;dx.$$

The relaxation of such functionals to BH is the subject of [11], where we get an integral representation of the relaxation, to be precise

$$\begin{aligned}\inf&\left\{\liminf_{n\to\infty}\mathcal{F}(u_n;A):,\ u_n\in W^{2,1}(\Omega;\mathbb{R}^d), u_n\to u, \sup_n \|u_n\|_{W^{2,1}}<\infty\right\}\\ &=\int_A \mathcal{Q}_2 f(x,\nabla^2 u)\,dx+\int_A (\mathcal{Q}_2 f)^\infty\left(x,\frac{dD_s(\nabla u)}{d|D_s|(\nabla u)}\right)d|D_s(\nabla u)| \qquad (6.4)\end{aligned}$$

for every $u\in BH(\Omega;\mathbb{R}^d)$, $A\in\mathcal{A}(\Omega)$, where $\mathcal{Q}_2 f$ is the 2-quasiconvex envelope of f. In this case, since $\mathcal{F}$ is lower semicontinuous, we must have that f is 2-quasiconvex as shown in [1], and thus $\mathcal{Q}_2 f=f$. Thus for every $u\in BH(\Omega;\mathbb{R}^d)$ we may take a recovery sequence for the relaxation $\{u_n\}\subset W^{2,1}(\Omega;\mathbb{R}^d)$ such that $u_n\to u$ in L^1 and

$$\lim_{n\to\infty}\mathcal{F}(u_n;A)=\int_A f(x,\nabla^2 u)\,dx+\int_A f^\infty\left(x,\frac{dD_s(\nabla u)}{d|D_s|(\nabla u)}\right)d|D_s(\nabla u)|$$

to conclude from lower semicontinuity of $\mathcal{F}$ that

$$\mathcal{F}(u;A)\le\int_A f(x,\nabla^2 u)\,dx+\int_A f^\infty\left(x,\frac{dD_s(\nabla u)}{d|D_s|(\nabla u)}\right)d|D_s(\nabla u)|.$$

□

We can push this further under a stronger continuity assumption on $\mathcal{F}$. If $\mathcal{F}$ is *continuous* with respect to *area-strict* convergence, (see [12]), then this upper bound is actually sharp.

Definition 6.3. We say that a sequence of signed measures $\{\mu^{(n)}\}\subset\mathcal{M}(\Omega;\mathbb{R}^N)$ converges area-strictly to μ if $\mu^{(n)}\overset{*}{\rightharpoonup}\mu$ and

$$\lim_{n\to\infty}\left(\int_\Omega\sqrt{1+\left|\frac{d\mu^{(n)}}{d\mathcal{L}^N}\right|^2}\,dx+|\mu_s^{(n)}|(\Omega)\right)=\int_\Omega\sqrt{1+\left|\frac{d\mu}{d\mathcal{L}^N}\right|^2}\,dx+|\mu|(\Omega).$$

This condition is actually very natural for BH lower semicontinuous integral functionals. Indeed, Theorem 2.14 in [11] shows that 2-quasiconvex potentials along with their recession function of the form (6.4) are automatically area-strict continuous. In the first order global method result [4], the area-strict continuity assumption is not needed, but once we have a potential function for the integral relaxation, we can see that it is automatically area-strict continuous by using the results in [12]. Thus in the first order case, an assumption of area-strict continuity is innocuous, which motivates our assumption here. With the assumption of area-strict continuity, we have the following:

Corollary 6.4. *Let $\mathcal{F}$ satisfy hypotheses $(I1),(I2),(I3),(I4)$ and further assume that for every affine function*

$$v(x):=p+Ax$$

for $p\in\mathbb{R}^d, A\in\mathbb{R}^{d\times N}$, we have

$$\mathcal{F}(u;\cdot)=\mathcal{F}(u+v;\cdot)$$

and that for every $u \in BH(\Omega;\mathbb{R}^d)$ and every sequence $\{u_n\} \subset BH(\Omega;\mathbb{R}^d)$ so that $u_n \to u$ in L^1 and $D(\nabla u_n) \to D(\nabla u)$ area-strictly, we have that

$$\lim_{n\to\infty} \mathcal{F}(u_n;\Omega) = \mathcal{F}(u;\Omega).$$

Then for every $u \in BH(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}(u,A) = \int_A f(x,\nabla^2 u)\,dx + \int_A f^\infty\left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)}\right) d|D_s(\nabla u)|.$$

Proof. Following the proof of Corollary 6.4, (I1), (I2), (I3), (I4) and the affine invariance property give us a representation of $\mathcal{F}$ on $W^{2,1}(\Omega;\mathbb{R}^d)$. For any $u \in BH$, we can use Corollary 5.8 in [11] to construct a sequence $\{u_n\} \subset W^{2,1}(\Omega;\mathbb{R}^d)$ so that $u_n \to u$ in L^1 and $D(\nabla u_n) \to D(\nabla u)$ area-strictly. Thus, by area-strict continuity, we have

$$\mathcal{F}(u;\Omega) = \lim_{n\to\infty} \mathcal{F}(u_n;\Omega). \tag{6.5}$$

On the other hand, the functional

$$u \in BH(\Omega;\mathbb{R}^d) \mapsto \int_\Omega f(x,\nabla^2 u)\,dx + \int_\Omega f^\infty\left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)}\right) d|D_s(\nabla u)| =: I(u;\Omega)$$

is area-strict continuous on BH by Theorem 2.14 in [11] and agrees with $\mathcal{F}$ on $W^{2,1}$, therefore

$$\lim_{n\to\infty} \mathcal{F}(u_n;\Omega) = \lim_{n\to\infty} I(u_n;\Omega) = I(u;\Omega).$$

This, together with (6.5), yields

$$\mathcal{F}(u;\Omega) = \int_\Omega f(x,\nabla^2 u)\,dx + \int_\Omega f^\infty\left(x, \frac{dD_s(\nabla u)}{d|D_s|(\nabla u)}\right) d|D_s(\nabla u)|$$

for every $u \in BH(\Omega;\mathbb{R}^d)$.

□

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