Asymptotic Analysis of Second Order Nonlocal Cahn-Hilliard-Type Functionals

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Abstract

In this paper the study of a nonlocal second order Cahn–Hilliard-type singularly perturbed family of functions is undertaken. The kernels considered include those leading to Gagliardo fractional seminorms for gradients. Using Γ convergence the integral representation of the limit energy is characterized leading to an anisotropic surface energy on interfaces separating different phases.

1 Introduction

In the van der Waals–Cahn–Hilliard theory of phase transitions [15], [38], [47], [28], the total energy is given by

$$\frac{1}{\varepsilon}\int_{\Omega} W(u(x))\,dx + \varepsilon\int_{\Omega} |\nabla u(x)|^2\,dx, \tag{1.1}$$

where the open bounded set $\Omega \subset \mathbb{R}^n$ represents a container, $u : \Omega \to \mathbb{R}$ is the fluid density, and $W : \mathbb{R} \to [0, +\infty)$ is a double-well potential vanishing only at the phases -1 and 1. The perturbation $\varepsilon \int_\Omega |\nabla u(x)|^2 \, dx$ penalizes rapid changes of the density u, and it plays the role of an interfacial energy. This problem has been extensively studied in the last four decades (see, e.g., [8], [9], [10], [24], [34], [35], [37], [36], [44], [45]).

Higher order perturbations were considered in the study of shape deformation of unilamellar membranes undergoing inplane phase separation (see, e.g., [30], [46], [31, 40]). A simplified local version of that model (see [40]) leads to the study of a Ginzburg-Landau-type energy

$$\frac{1}{\varepsilon}\int_\Omega W(u(x))\, dx + q\varepsilon \int_\Omega |\nabla u(x)|^2 \, dx + \varepsilon^3 \int_\Omega \left|\nabla^2 u(x)\right|^2 dx \ , \tag{1.2}$$

where $q \in \mathbb{R}$. This functional is also related to the Swift–Hohenberg equation (see [43]). When $q = 0$, the functional reduces to the second order version of (1.1), to be precise,

$$\frac{1}{\varepsilon}\int_\Omega W(u(x))\, dx + \varepsilon^3 \int_\Omega \left|\nabla^2 u(x)\right|^2 dx \ , \tag{1.3}$$

which was studied in [23]. The case $q > 0$ in was treated in [29], with $|\nabla^2 u|^2$ replaced by $|\Delta u|^2$. The case $q < 0$ is more delicate and was considered in [16] and [17]. The original energy functional proposed in [30], [46], [31], [40]) involved also a nonlocal perturbation and was addressed in [22].

A nonlocal local version of (1.1) was studied in [1], [2], [3], with the perturbation $\varepsilon \int_\Omega |\nabla u(x)|^2 \, dx$ replaced by a nonlocal term, leading to the energy

$$\frac{1}{\varepsilon}\int_\Omega W(u(x))\, dx + \varepsilon \int_\Omega \int_\Omega J_\varepsilon(x-y)|u(x)-u(y)|^2 dxdy \ , \tag{1.4}$$

where

$$J_\varepsilon(x) := \frac{1}{\varepsilon^n} J\Big(\frac{x}{\varepsilon}\Big) \tag{1.5}$$

and the kernel $J : \mathbb{R}^n \to [0, +\infty)$ is an even measurable function such that

$$\int_{\mathbb{R}^n} J(x)(|x| \wedge |x|^2)\; dx =: M_J < +\infty \ , \tag{1.6}$$

with $a \wedge b := \min\{a, b\}$. Functionals of the form (1.4) arise in equilibrium statistical mechanics as free energies of continuum limits of Ising spin systems on lattices. In that setting, u is a macroscopic magnetization density and J stands for a ferromagnetic Kac potential (see [3]). Note that (1.6) is satisfied if J is integrable and has compact support. Another important case is when

$$J(x) = |x|^{-n-2s} \quad \text{with } \frac{1}{2} < s < 1 \ , \tag{1.7}$$

so that $J_\varepsilon(x) = \varepsilon^{2s}|x|^{-n-2s}$, which leads to Gagliardo's seminorm for the fractional Sobolev space $H^s(\mathbb{R}^n)$ (see [20], [25] [32]). A functional related to (1.4) with kernel (1.7) has been studied in [4], [5], and [39] for $0 < s < 1$ (see also [27] for an L^p version in dimension $n = 1$).

The motivation in [39] was the renewed interest in the fractional Laplacian (see, e.g., [14] and the references therein), and nonlocal characterizations of fractional Sobolev spaces ([6], [11], [12], [33] and the references therein).

Another important application of this type of nonlocal singular perturbation functionals is in the study of dislocations in elastic materials exhibiting microstructure (see, e.g., [13], [18], [26]).

In this paper we consider a nonlocal version of (1.3), to be precise, we study the functional

$$\mathcal{F}_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx + \varepsilon \int_\Omega\int_\Omega J_\varepsilon(x-y)|\nabla u(x) - \nabla u(y)|^2 dxdy \tag{1.8}$$

for $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open set with Lipschitz boundary, the double-well potential $W : \mathbb{R} \to [0, +\infty)$ is a continuous function with $W^{-1}(\{0\}) = \{-1, +1\}$ satisfying appropriate coercivity and growth conditions, and J_ε is given by (1.5). We assume a non-degeneracy hypothesis (see (2.2)) on the even measurable kernel $J : \mathbb{R}^n \to [0, +\infty)$, and that (1.6) holds.

We establish compactness in $L^2(\Omega)$ for energy bounded sequences, and in order to study the asymptotic behavior of (1.8) as $\varepsilon \to 0^+$, we use the notion of Γ-convergence (see [19]) with respect to the metric in $L^2(\Omega)$ and we identify the Γ-limit of $\mathcal{F}_\varepsilon$. As it is usual, we extend $\mathcal{F}_\varepsilon(u)$ to be $+\infty$ for $u \in L^2(\Omega) \setminus W^{1,2}_{\mathrm{loc}}(\Omega)$. Our first main result is the following theorem.

Theorem 1.1 (Compactness) *Assume that W and J satisfy (2.3)–(2.6) and (1.6), (2.2), respectively. Let $\{u_\varepsilon\} \subset W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^2(\Omega)$ be such that*

$$M := \sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty\ . \tag{1.9}$$

Then there exists a sequence $\varepsilon_j \to 0^+$ such that $\{u_{\varepsilon_j}\}$ converges in $L^2(\Omega)$ to some function $u \in BV(\Omega; \{-1, 1\})$.

The proof of this theorem is more involved than the corresponding one in [2] due to the presence of gradients in the nonlocal term. This prevents us from using standard arguments in which discontinuities in u may be allowed. We first prove compactness in $n = 1$, and then use a slicing technique to treat the higher dimensional case.

To state the Γ convergence result, we need to introduce some notation. Given $n \geq 2$ and $\nu \in \mathbb{S}^{n-1} := \partial B_1(0)$, let $\nu_1, \ldots, \nu_n$ be an orthonormal basis in $\mathbb{R}^n$ with $\nu_n = \nu$. Here, and in what follows, we denote by $B_r(x)$ the open ball in $\mathbb{R}^n$ centered at x and with radius r. Let

$$V^\nu := \{x \in \mathbb{R}^n :\ |x \cdot \nu_i| < 1/2 \text{ for } i = 1, \ldots, n-1\}\ , \tag{1.10}$$
$$Q^\nu := \{x \in \mathbb{R}^n :\ |x \cdot \nu_i| < 1/2 \text{ for } i = 1, \ldots, n\}\ , \tag{1.11}$$

let $W^{1,2}_{\nu_1,\ldots,\nu_{n-1}}$ be the set of all functions $v \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ such that $v(x+\nu_i) = v(x)$ for a.e. $x \in \mathbb{R}^n$ and for every $i = 1, \ldots, n-1$, and let

$$X^\nu := \{v \in W^{1,2}_{\nu_1,\ldots,\nu_{n-1}} : \ v(x) = \pm 1 \text{ for a.e. } x \in \mathbb{R}^n \text{ with } \pm x\cdot\nu \geq 1/2\} \tag{1.12}$$

When $n = 1$ take $\nu = \pm 1$, $V^\nu := \mathbb{R}$, $Q^\nu := (-1/2, 1/2)$, and let X^ν be the space of all functions $v \in W^{1,2}_{\text{loc}}(\mathbb{R})$ such that $v(x) = \pm 1$ for a.e. $x \in \mathbb{R}$ with $\pm x \geq 1/2$. We define the anisotropic surface energy density

$$\psi(\nu) := \inf_{0<\varepsilon<1} \inf_{v \in X^\nu} \mathcal{F}^\nu_\varepsilon(v)\,, \tag{1.13}$$

where

$$\mathcal{F}^\nu_\varepsilon(u) := \frac{1}{\varepsilon}\int_{Q^\nu} W(u(x))\,dx + \varepsilon\int_{V^\nu}\int_{\mathbb{R}^n} J_\varepsilon(x-y)|\nabla u(x) - \nabla u(y)|^2 dxdy\ .$$

Finally, we define $\mathcal{F} : L^2(\Omega) \to [0, +\infty]$ by

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_{S_u} \psi(\nu_u)\ d\mathcal{H}^{n-1} & \text{if } u \in BV(\Omega; \{-1, 1\})\ , \\ +\infty & \text{otherwise in } L^2(\Omega)\ , \end{cases} \tag{1.14}$$

where S_u is the jump set of u, ν_u is the approximate normal to S_u, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (see [7] for a detailed description of these notions).

Theorem 1.2 (Γ-Limit) *Assume that W and J satisfy (2.2)–(2.6) and (1.6), respectively. Then for every $\varepsilon_j \to 0^+$ the sequence $\{\mathcal{F}_{\varepsilon_j}\}$ Γ-converges to $\mathcal{F}$ in $L^2(\Omega)$.*

Although the general structure of the proof is standard, there are remarkable technical difficulties due to the nonlocality of the perturbation and the presence of gradients.

This paper is organized as follows. After a brief section on preliminaries, on Section 3 in order to establish compactness in dimension $n = 1$, we prove an interpolation result, which allows us to control the L^2 norm of u' in terms of the full energy (see Lemma 3.5). Section 4 is devoted to compactness in higher dimensions, and here again we obtain the equivalent to the interpolation Lemma 3.5 (see Lemma 4.3). As it is classical in this type of problems, it is important to be able to modify admissible sequences near the boundary of their domain without increasing the limit energy. We address this in Theorem 5.1 in Section 5. Section 6 concerns the Γ-liminf inequality, and in Section 7 we construct the recovery sequence for the Γ-limsup inequality.

2 Preliminaries

In what follows, in addition to (1.6) we also assume that the kernel $J : \mathbb{R}^n \to [0, +\infty)$ has the following property: there exist $\gamma_J > 0$, $\delta_J \in (0, 1)$, $c_J > 0$, such that for all $\xi \in \mathbb{S}^{n-1}$ there are $\alpha(\xi) < \beta(\xi)$ satisfying

$$-\gamma_J \leq \alpha(\xi) \leq \alpha(\xi) + \delta_J \leq \beta(\xi) \leq \gamma_J \tag{2.1}$$

and

$$\int_{\alpha(\xi)}^{\beta(\xi)} \frac{1}{J(t\xi)|t|^{n-1}}\, dt \leq c_J \ . \tag{2.2}$$

Remark 2.1 *For example, condition (2.2) holds if there exist $0 < r < R$ and $a > 0$ such that $J(x) \geq a$ for every $x \in \mathbb{R}^n$ with $r < |x| < R$. Indeed, it is enough to set $\gamma_J = R$, $\delta_J = R - r$, $\alpha(\xi) = r$, $\beta(\xi) = R$, and $c_J = (na)^{-1}(r^{-n} - R^{-n})$.*

We assume that the double-well potential is a continuous function $W : \mathbb{R} \to [0, +\infty)$ such that

$$W^{-1}(\{0\}) = \{-1, 1\} \ , \tag{2.3}$$

$$(|s| - 1)^2 \leq c_W W(s) \quad \text{for all } s \in \mathbb{R} \ , \tag{2.4}$$

$$W \text{ is increasing on } [1, +\infty) \text{ and on } [-1, -1 + a_W] \ , \tag{2.5}$$

$$W \text{ is decreasing on } (-\infty, -1] \text{ and on } [1 - a_W, 1] \ , \tag{2.6}$$

for some constants $c_W > 0$ and $a_W \in (0, 1)$.

If $s \leq 0$ and $|s + 1| \geq \frac{1}{2}$, then $|s - 1| = |s| - 1 + 2$, hence $(s - 1)^2 \leq 2(|s| - 1)^2 + 4 \leq 2c_W W(s) + \frac{4}{m_W} W(s)$, where

$$m_W := \min_{\{||s|-1| \geq \frac{1}{2}\}} W(s) > 0 \ . \tag{2.7}$$

Together with (2.4) this leads to the estimate

$$(s - 1)^2 \leq \hat{c}_W W(s) \quad \text{for all } s \in \mathbb{R} \text{ with } |s + 1| \geq \frac{1}{2} \ , \tag{2.8}$$

where $\hat{c}_W := 2c_W + \frac{4}{m_W}$. Similarly, it can be shown that

$$(s + 1)^2 \leq \hat{c}_W W(s) \quad \text{for all } s \in \mathbb{R} \text{ with } |s - 1| \geq \frac{1}{2} \ . \tag{2.9}$$

We recall that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary. For every $\varepsilon > 0$ and $u \in L^2(\Omega)$ consider the functional

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \mathcal{W}_\varepsilon(u) + \mathcal{J}_\varepsilon(u) & \text{if } u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^2(\Omega) \ , \\ +\infty & \text{otherwise,} \end{cases} \tag{2.10}$$

where

$$\mathcal{W}_\varepsilon(u) := \frac{1}{\varepsilon} \int_\Omega W(u(x))\, dx \quad \text{for } u \in L^2(\Omega) \ , \tag{2.11}$$

and

$$\mathcal{J}_\varepsilon(u) := \varepsilon \int_\Omega \int_\Omega J_\varepsilon(x - y) |\nabla u(x) - \nabla u(y)|^2 dx dy \quad \text{for } u \in W^{1,2}_{\text{loc}}(\Omega). \tag{2.12}$$

In the sequel, we will use a localized version of (2.10). To be precise, given two open sets A, $B \subset \mathbb{R}^n$ we define

$$\mathcal{W}_\varepsilon(u, A) := \frac{1}{\varepsilon}\int_A W(u(x))\ dx \tag{2.13}$$

for $u \in L^2(A)$, and

$$\mathcal{J}_\varepsilon(u, A, B) := \varepsilon \int_A \int_B J_\varepsilon(x-y)|\nabla u(x) - \nabla u(y)|^2 dxdy \tag{2.14}$$

for $u \in W^{1,2}_{\mathrm{loc}}(A \cup B)$. When $A = B$ we set

$$\mathcal{F}_\varepsilon(u, A) := \mathcal{W}_\varepsilon(u, A) + \mathcal{J}_\varepsilon(u, A, A) \quad \text{and} \quad \mathcal{J}_\varepsilon(u, A) := \mathcal{J}_\varepsilon(u, A, A) \tag{2.15}$$

for $u \in W^{1,2}_{\mathrm{loc}}(A) \cap L^2(A)$.

Since J is even, by Fubini's theorem for all $u \in W^{1,2}_{\mathrm{loc}}(A \cup B)$ we have that

$$\mathcal{J}_\varepsilon(u, A, B) = \mathcal{J}_\varepsilon(u, B, A)\ . \tag{2.16}$$

Moreover, if $A \cap B = \emptyset$ we have

$$\mathcal{J}_\varepsilon(u, A \cup B) = \mathcal{J}_\varepsilon(u, A) + 2\mathcal{J}_\varepsilon(u, A, B) + \mathcal{J}_\varepsilon(u, B)\ . \tag{2.17}$$

In the compactness theorem we use a slicing argument based on the following preliminary result. Given a vector $\xi \in \mathbb{S}^{n-1}$, the hyperplane through the origin orthogonal to ξ is denoted by Π^ξ, that is,

$$\Pi^\xi := \{x \in \mathbb{R}^n :\ x \cdot \xi = 0\}\ . \tag{2.18}$$

If $E \subset \mathbb{R}^n$ and $y \in \Pi^\xi$, then we define

$$E^\xi_y := \{t \in \mathbb{R} :\ y + t\xi \in E\}\ . \tag{2.19}$$

The next result is a particular case of the affine Blaschke–Petkantschin formula, for which we refer to [41, Theorem 7.2.7].

Proposition 2.2 *Let $E \subset \mathbb{R}^n$ be a Borel set and let $g : E \times E \to [0, +\infty]$ be a Borel function. Then*

$$\int_E \int_E g(x, y)\ dxdy$$
$$= \frac{1}{2}\int_{\mathbb{S}^{n-1}} \int_{\Pi^\xi} \int_{E^\xi_z} \int_{E^\xi_z} g(z + s\xi, z + t\xi)|t - s|^{n-1} dsdtd\mathcal{H}^{n-1}(z)d\mathcal{H}^{n-1}(\xi)\ .$$

Proof. For the convenience of the reader we present a proof. We extend g to be zero outside $E \times E$. Using the change of variables $\tau = t - s$, we obtain

$$\int_{\mathbb{R}} g(z + s\xi, z + t\xi)|t - s|^{n-1} ds = \int_{\mathbb{R}} g(z + t\xi - \tau\xi, z + t\xi)|\tau|^{n-1} d\tau\ ,$$

and by Fubini's theorem we get

$$\int_{\Pi^\xi}\int_{\mathbb{R}}\int_{\mathbb{R}} g(z+s\xi, z+t\xi)|t-s|^{n-1} ds dt d\mathcal{H}^{n-1}(z) = \int_{\mathbb{R}^n}\int_{\mathbb{R}} g(y-\tau\xi, y)|\tau|^{n-1} d\tau dy \ .$$

Exchanging the order of integration and using integration in spherical coordinates we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\Pi^\xi}\int_{\mathbb{R}}\int_{\mathbb{R}} & g(z+s\xi, z+t\xi)|t-s|^{n-1} ds dt d\mathcal{H}^{n-1}(z) d\mathcal{H}^{n-1}(\xi) \\ &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}} g(y-\tau\xi, y)|\tau|^{n-1} d\tau d\mathcal{H}^{n-1}(\xi) dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x,y)\ dx dy\ ,\end{aligned}$$

which concludes the proof. ∎

For $\xi \in \mathbb{S}^{n-1}$ and $\varepsilon > 0$ define $J^\xi : \mathbb{R} \to [0, +\infty)$ by

$$J^\xi(t) := J(t\xi)|t|^{n-1} \quad \text{and} \quad J^\xi_\varepsilon(t) := \frac{1}{\varepsilon} J^\xi\left(\frac{t}{\varepsilon}\right)\ . \tag{2.20}$$

By (1.6) and using spherical coordinates, we have

$$\int_{\mathbb{R}} J^\xi(t)(|t| \wedge |t|^2)\ dt < +\infty \tag{2.21}$$

for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$, and in view of (2.2) we obtain

$$\int_{\alpha(\xi)}^{\beta(\xi)} \frac{1}{J^\xi(t)}\ dt \le c_J\ . \tag{2.22}$$

Moreover,

$$J^\xi_\varepsilon(t) = \frac{1}{\varepsilon} J^\xi\left(\frac{t}{\varepsilon}\right) = \frac{1}{\varepsilon} J\left(\frac{t\xi}{\varepsilon}\right)\left|\frac{t}{\varepsilon}\right|^{n-1} = J_\varepsilon(t\xi)|t|^{n-1}\ . \tag{2.23}$$

For $\xi \in \mathbb{S}^{n-1}$, $A \subset \mathbb{R}$, and $\varepsilon > 0$, we define

$$\mathcal{F}^\xi_\varepsilon(v, A) := \frac{1}{\sigma_{n-1}\varepsilon}\int_A W(v(t))\ dt + \frac{\varepsilon}{2}\int_A\int_A J^\xi_\varepsilon(s-t)(v'(s)-v'(t))^2 ds dt \tag{2.24}$$

for $v \in W^{1,2}_{\mathrm{loc}}(A) \cap L^2(A)$, where $\sigma_{n-1} := \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$.

3 Compactness and interpolation in dimension one

For a set A contained in $\mathbb{R}^n$ and for $\eta > 0$ we define

$$\begin{array}{l}(A)^\eta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) < \eta\} \ , \\ (A)_\eta := \{x \in \ A : \ \operatorname{dist}(x, \partial A) > \eta\} \ .\end{array} \tag{3.1}$$

The main result of this section is the following theorem.

Theorem 3.1 *Let $\xi \in \mathbb{S}^{n-1}$, let $A \subset \mathbb{R}$ be a bounded open set, and let $\{u_\varepsilon\} \subset W^{1,2}_{\mathrm{loc}}(A) \cap L^2(A)$ be such that*

$$M := \sup_\varepsilon \mathcal{F}^\xi_\varepsilon(u_\varepsilon, A) < +\infty \ , \tag{3.2}$$

where $\mathcal{F}^\xi_\varepsilon$ is defined in (2.24). Then there exists a sequence $\varepsilon_j \to 0^+$ such that $\{u_{\varepsilon_j}\}$ converges in $L^2(A)$ to some function $u \in BV(A; \{-1, 1\})$. Moreover, there exists a constant $c_{J,W} > 0$, independent of ξ, A, and $\{u_\varepsilon\}$, such that

$$\# S_u \leq \frac{M}{c_{J,W}} \ , \tag{3.3}$$

where $\# S_u$ denotes the number of jump points of u.

Next we introduce some auxiliary lemmas that will be used in the proof of Theorem 3.1.

Lemma 3.2 *Let $\xi \in \mathbb{S}^{n-1}$, let $A \subset \mathbb{R}$ be an open set, let $\varepsilon > 0$, let $\alpha < \beta$, and let $u \in W^{1,2}_{\mathrm{loc}}((A)^{\varepsilon\gamma_J})$, where γ_J is the constant in (2.1). Then for a.e. $t \in A$,*

$$\begin{aligned}&\varepsilon \int_{t-\varepsilon\beta}^{t-\varepsilon\alpha} J^\xi_\varepsilon(t-s)(u'(t) - u'(s))^2 ds \\ &\quad\geq \varepsilon(\beta-\alpha)^2 \left(\int_\alpha^\beta \frac{1}{J^\xi(z)} \, dz\right)^{-1} \left(u'(t) - \frac{u(t-\varepsilon\alpha) - u(t-\varepsilon\beta)}{\varepsilon(\beta-\alpha)}\right)^2 ,\end{aligned} \tag{3.4}$$

where J^ξ and J^ξ_ε are defined in (2.20).

Proof. It is enough to show that for every $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}&\varepsilon \int_{t-\varepsilon\beta}^{t-\varepsilon\alpha} J^\xi_\varepsilon(t-s)(\lambda - u'(s))^2 ds \\ &\quad\geq \varepsilon(\beta-\alpha)^2 \left(\int_\alpha^\beta \frac{1}{J^\xi(z)} \, dz\right)^{-1} \left(\lambda - \frac{u(t-\varepsilon\alpha) - u(t-\varepsilon\beta)}{\varepsilon(\beta-\alpha)}\right)^2 .\end{aligned}$$

This inequality follows by considering the Euler–Lagrange equation of the minimum problem

$$\min \int_{t-\varepsilon\beta}^{t-\varepsilon\alpha} J^\xi_\varepsilon(t-s)(\lambda - v'(s))^2 ds$$

over all $v \in W^{1,2}((t-\varepsilon\beta, t-\varepsilon\alpha))$ satisfying $v(t-\varepsilon\beta) = u(t-\varepsilon\beta)$ and $v(t-\varepsilon\alpha) = u(t-\varepsilon\alpha)$. ■

Remark 3.3 *Under the same assumptions of Lemma 3.2, it follows from (2.1), (2.2), and (3.4) that*

$$\varepsilon(u'(t))^2 \leq \frac{2}{\delta_J^2}\frac{1}{\varepsilon}\big(u(t-\varepsilon\alpha(\xi)) - u(t-\varepsilon\beta(\xi))\big)^2 + 2c_J\varepsilon\int_{t-\varepsilon\gamma_J}^{t+\varepsilon\gamma_J} J_\varepsilon^\xi(t-s)(u'(t)-u'(s))^2 ds$$

for a.e. $t \in A$.

Lemma 3.4 *Let* γ_J *be the constant in (2.1). Then there exists a constant* $c_{J,W} > 0$ *such that*

$$\varepsilon\int_\sigma^\tau\int_{\sigma-\varepsilon\gamma_J}^{\tau+\varepsilon\gamma_J} J_\varepsilon^\xi(t-s)(u'(t)-u'(s))^2 dsdt + \frac{1}{\varepsilon}\int_{\sigma-\varepsilon\gamma_J}^{\tau+\varepsilon\gamma_J} W(u(t))\, dt \geq c_{J,W} \tag{3.5}$$

for every $\xi \in \mathbb{S}^{n-1}$*, for every* $\varepsilon > 0$*, for every* σ*,* τ*, with* $\sigma < \tau$*, and for every* $u \in W^{1,2}_{\mathrm{loc}}((\sigma-\varepsilon\gamma_J, \tau+\varepsilon\gamma_J))$ *such that*

$$u(t) \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \text{ for every } t \in (\sigma, \tau)\ , \tag{3.6}$$

and either

$$u(\sigma) = -\tfrac{1}{2} \quad \text{and} \quad u(\tau) = \tfrac{1}{2} \tag{3.7}$$

or

$$u(\sigma) = \tfrac{1}{2} \quad \text{and} \quad u(\tau) = -\tfrac{1}{2}\ . \tag{3.8}$$

Proof. Fix ξ, ε, σ, τ, and u as in the statement of the lemma, and let $\hat\alpha$ and $\hat\beta$ be such that $\alpha(\xi) < \hat\alpha < \hat\beta < \beta(\xi)$, and

$$\alpha(\xi) - \hat\alpha > \frac{1}{4}\delta_J, \quad \hat\beta - \hat\alpha > \frac{1}{4}\delta_J, \quad \beta(\xi) - \hat\beta > \frac{1}{4}\delta_J\ , \tag{3.9}$$

where δ_J is the constant in (2.1). By (2.4) and (3.6), we have $W(u(t)) \geq \frac{1}{4C_W}$ for every $t \in (\sigma, \tau)$. Therefore, if $\tau - \sigma > \varepsilon\delta_J/2^6$, then

$$\frac{1}{\varepsilon}\int_\sigma^\tau W(u_\varepsilon(t))\, dt > \frac{\delta_J}{2^8 C_W}\ . \tag{3.10}$$

If $\tau - \sigma \leq \varepsilon\delta_J/2^6$, define

$$A_0 := \left\{t \in (\sigma, \tau) :\ |u'(t)| \geq \frac{1}{2}\frac{1}{\tau-\sigma}\right\}\ . \tag{3.11}$$

We consider now two cases.

Case 1: Assume that for every $t \in A_0$ there exist $\alpha \in [\alpha(\xi), \hat{\alpha}]$ and $\beta \in [\hat{\beta}, \beta(\xi)]$ such that

$$\frac{|u(t-\varepsilon\alpha) - u(t-\varepsilon\beta)|}{\varepsilon(\beta-\alpha)} < \frac{1}{2}|u'(t)| \ .$$

Then

$$\left(u'(t) - \frac{u(t-\varepsilon\alpha) - u(t-\varepsilon\beta)}{\varepsilon(\beta-\alpha)}\right)^2 \geq \frac{1}{4}(u'(t))^2 \ .$$

Therefore, by Lemma 3.2,

$$\begin{aligned}\varepsilon \int_{t-\varepsilon\beta}^{t-\varepsilon\alpha} & J_\varepsilon^\xi(t-s)(u'(t)-u'(s))^2 ds \\ & \geq \frac{\varepsilon(\beta-\alpha)^2}{4} \left(\int_\alpha^\beta \frac{1}{J^\xi(z)}\ dz\right)^{-1} (u'(t))^2 \ ,\end{aligned}$$

and integrating over A_0, using (2.22) and (3.9), we obtain

$$\varepsilon \int_{A_0} \int_{t-\varepsilon\beta(\xi)}^{t-\varepsilon\alpha(\xi)} J_\varepsilon^\xi(t-s)(u'(t)-u'(s))^2 ds dt \geq \frac{\varepsilon\delta_J^2}{2^6 c_J} \int_{A_0} (u'(t))^2 dt \ . \tag{3.12}$$

By (3.7), (3.8), and (3.11) using Jensen's inequality and $\tau - \sigma \leq \frac{\delta_J}{2^6}\varepsilon$, we have

$$\int_{A_0} (u'(t))^2 dt = \int_\sigma^\tau (u'(t))^2 dt - \int_{(\sigma,\tau)\setminus A_0} (u'(t))^2 dt \geq \frac{1}{\tau-\sigma} - \frac{1}{4}\frac{1}{\tau-\sigma} \geq \frac{3\cdot 2^4}{\varepsilon\delta_J} \ .$$

Hence, from (3.12) we deduce that

$$\varepsilon \int_\sigma^\tau \int_{\sigma-\varepsilon\beta(\xi)}^{\tau-\varepsilon\alpha(\xi)} J_\varepsilon^\xi(t-s)(u'(t)-u'(s))^2 ds dt \geq \frac{3}{4}\frac{\delta_J}{c_J} \ . \tag{3.13}$$

Case 2: It remains to study the case in which there exists $t_0 \in A_0$ such that

$$\frac{|u(t_0-\varepsilon\alpha) - u(t_0-\varepsilon\beta)|}{\varepsilon(\beta-\alpha)} \geq \frac{1}{2}|u_\varepsilon'(t_0)|$$

for every $\alpha \in [\alpha(\xi), \hat{\alpha}]$ and for every $\beta \in [\hat{\beta}, \beta(\xi)]$. By (3.11) and the inequality $\tau - \sigma \leq \varepsilon\delta_J/2^6$, we have

$$\frac{|u(t_0-\varepsilon\alpha) - u(t_0-\varepsilon\beta)|}{\varepsilon(\beta-\alpha)} \geq \frac{1}{4(\tau-\sigma)} \geq \frac{16}{\varepsilon\delta_J} \,,$$

hence by (3.9),

$$|u(t_0-\varepsilon\alpha) - u(t_0-\varepsilon\beta)| \geq \frac{16(\hat{\beta}-\hat{\alpha})}{\delta_J} \geq 4 \ .$$

If $|u(t_0 - \varepsilon\alpha)| \geq 2$ for every $\alpha \in [\alpha(\xi), \hat{\alpha}]$, then by (2.4) we have $W(u(t_0 - \varepsilon\alpha)) \geq \frac{1}{c_W}$ for every $\alpha \in [\alpha(\xi), \hat{\alpha}]$. This leads to $W(u(t)) \geq \frac{1}{c_W}$ for every $t \in [t_0 - \varepsilon\hat{\alpha}, t_0 - \varepsilon\alpha(\xi)]$, hence

$$\frac{1}{\varepsilon}\int_{\sigma-\varepsilon\gamma_J}^{\tau+\varepsilon\gamma_J} W(u(t))\, dt \geq \frac{1}{\varepsilon}\int_{t_0-\varepsilon\hat{\alpha}}^{t_0-\varepsilon\alpha(\xi)} W(u(t))\, dt \geq \frac{\hat{\alpha}-\alpha(\xi)}{c_W} \geq \frac{\delta_J}{4c_W}\ , \tag{3.14}$$

where in the last inequality we used (3.9).

If there exists $\alpha \in [\alpha(\xi), \hat{\alpha}]$ such that $|u(t_0 - \varepsilon\alpha)| < 2$, then $|u(t_0 - \varepsilon\beta)| > 2$ for every $\beta \in [\hat{\beta}, \beta_J]$ (if not, there exists $\beta \in [\hat{\beta}, \beta(\xi)]$ such that $|u(t_0 - \varepsilon\beta)| \leq 2$, which gives $|u(t_0 - \varepsilon\alpha) - u(t_0 - \varepsilon\beta)| < 4$, a contradiction). Consequently, for every $\beta \in [\hat{\beta}, \beta(\xi)]$ we have $W(u(t_0 - \varepsilon\beta)) \geq \frac{1}{c_W}$. This leads to $W(u(t)) \geq \frac{1}{c_W}$ for every $t \in [t_0 - \varepsilon\beta(\xi), t_0 - \varepsilon\hat{\beta}]$, hence

$$\frac{1}{\varepsilon}\int_{\sigma-\varepsilon\gamma_J}^{\tau+\varepsilon\gamma_J} W(u(t))\, dt \geq \frac{1}{\varepsilon}\int_{t_0-\varepsilon\beta(\xi)}^{t_0-\varepsilon\hat{\beta}} W(u(t))\, dt \geq \frac{\beta(\xi)-\hat{\beta}}{c_W} \geq \frac{\delta_J}{4c_W}\ , \tag{3.15}$$

where in the last inequality we used (3.9). The conclusion follows now from (3.10), (3.13), (3.14), and (3.15). ■

Lemma 3.5 (Interpolation inequality in dimension one) *There exists a constant $c^{(1)}_{J,W}$ such that*

$$\varepsilon\int_A (u'(t))^2 dt \leq c^{(1)}_{J,W}\mathcal{F}^\xi_\varepsilon(u, (A)^{2\varepsilon\gamma_J})\ . \tag{3.16}$$

for every $\xi \in \mathbb{S}^{n-1}$, for every $\varepsilon > 0$, for every open set $A \subset \mathbb{R}$, and for every $u \in W^{1,2}_{\mathrm{loc}}((A)^{2\varepsilon\gamma_J})$, where γ_J is the constant in (2.1).

Proof. Fix ξ, ε, A, and u as in the statement of the lemma, and define

$$\begin{aligned} U &:= \{t \in A : u(t-\varepsilon\alpha(\xi)), u(t-\varepsilon\beta(\xi)) \notin [\tfrac{1}{2}, \tfrac{3}{2}]\}\ , \\ V &:= \{t \in A : u(t-\varepsilon\alpha(\xi)), u(t-\varepsilon\beta(\xi)) \notin [-\tfrac{3}{2}, -\tfrac{1}{2}]\}\ . \end{aligned} \tag{3.17}$$

If $t \in V$, then by (2.8),

$$\begin{aligned} (u(t-\varepsilon\alpha(\xi)) - u(t-\varepsilon\beta(\xi)))^2 &\leq 2(u(t-\varepsilon\alpha(\xi))-1)^2 + 2(u(t-\varepsilon\beta(\xi))-1)^2 \\ &\leq 2\hat{c}_W\big(W(u(t-\varepsilon\alpha(\xi))) + W(u(t-\varepsilon\beta(\xi)))\big)\ . \end{aligned}$$

Using (2.9) we prove the same inequality for $t \in U$. Integrating and using Remark 3.3, we obtain

$$\varepsilon\int_{U\cup V} (u'(t))^2 dt \leq \big(8\frac{\hat{c}_W}{\delta_J^2} + 2c_J\big)\mathcal{F}^\xi_\varepsilon(u, (A)^{\varepsilon\gamma_J})\ . \tag{3.18}$$

If $t \in A \setminus (U \cup V)$, then either

$$u(t-\varepsilon\alpha(\xi)) \in [-\tfrac{3}{2}, -\tfrac{1}{2}] \quad \text{and} \quad u(t-\varepsilon\beta(\xi)) \in [\tfrac{1}{2}, \tfrac{3}{2}]$$

or

$$u(t-\varepsilon\beta(\xi)) \in [-\tfrac{3}{2},-\tfrac{1}{2}] \quad \text{and} \quad u(t-\varepsilon\alpha(\xi)) \in [\tfrac{1}{2},\tfrac{3}{2}] \ .$$

Then

$$(u(t-\varepsilon\alpha(\xi)) - u(t-\varepsilon\beta(\xi)))^2 \leq 9 \ . \tag{3.19}$$

Moreover there exist σ and τ, satisfying

$$t-\varepsilon\gamma_J \leq t-\varepsilon\beta(\xi) \leq \sigma < \tau \leq t-\varepsilon\alpha(\xi) \leq t+\varepsilon\gamma_J \tag{3.20}$$

and such that

$$u(t) \in \left(-\tfrac{1}{2},\tfrac{1}{2}\right) \text{ for every } t \in (\sigma,\tau)$$

and either

$$u(\sigma) = \tfrac{1}{2} \quad \text{and} \quad u(\tau) = -\tfrac{1}{2}$$

or

$$u(\sigma) = -\tfrac{1}{2} \quad \text{and} \quad u(\tau) = \tfrac{1}{2} \ .$$

By Lemma 3.4 and by (3.20), there exists $c_{J,W} > 0$ such that

$$c_{J,W} \leq \varepsilon \int_{t-\varepsilon\gamma_J}^{t+\varepsilon\gamma_J} \int_{t-2\varepsilon\gamma_J}^{t+2\varepsilon\gamma_J} J_\varepsilon^\xi(r-s)(u_\varepsilon'(r)-u_\varepsilon'(s))^2 dsdr + \frac{1}{\varepsilon}\int_{t-2\varepsilon\gamma_J}^{t+2\varepsilon\gamma_J} W(u_\varepsilon(r)) \ dr \ .$$

Therefore by (3.19) we have

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{A\setminus(U\cup V)} (u(t-\varepsilon\alpha(\xi)) - u(t-\varepsilon\beta(\xi)))^2 dt \\ &\qquad \leq \frac{9}{c_{J,W}} \int_A \int_{t-\varepsilon\gamma_J}^{t+\varepsilon\gamma_J} \int_{t-2\varepsilon\gamma_J}^{t+2\varepsilon\gamma_J} J_\varepsilon^\xi(r-s)(u_\varepsilon'(r)-u_\varepsilon'(s))^2 dsdrdt \\ &\qquad\quad + \frac{9}{c_{J,W}}\frac{1}{\varepsilon^2}\int_A \int_{t-2\varepsilon\gamma_J}^{t+2\varepsilon\gamma_J} W(u_\varepsilon(r)) \ drdt \ . \end{aligned} \tag{3.21}$$

Since

$$\frac{1}{2\eta}\int_A \int_{t-\eta}^{t+\eta} f(r) \ drdt \leq \int_{(A)_\eta} f(t) \ dt$$

for every $\eta > 0$ and for every integrable function $f\colon A \to [0,+\infty]$, from (3.21) we obtain

$$\frac{1}{\varepsilon}\int_{A\setminus(U\cup V)} (u(t-\varepsilon\alpha(\xi)) - u(t-\varepsilon\beta(\xi)))^2 dt \leq \tilde{c}_{J,W}\mathcal{F}_\varepsilon^\xi(u,(A)^{2\varepsilon\gamma_J}) \ . \tag{3.22}$$

for a suitable constant $\tilde{c}_{J,W}$ depending only on J and W. The conclusion follows from (3.18) and (3.22) using Remark 3.3. ■

Proof of Theorem 3.1. By (3.2) we have that

$$\int_A W(u_\varepsilon(t)) \ dt \leq M\varepsilon \ . \tag{3.23}$$

By (2.3) and (2.4) this implies that $\{u_\varepsilon^2\}$ converges to 1 in $L^1(A)$ and, up to a subsequence (not relabeled) pointwise a.e. in A.

Let $\gamma_J > 0$ be the constant given in (2.1). Consider the collection $\mathcal{I}_\varepsilon$ of all intervals $(\sigma - \varepsilon\gamma_J, y_\varepsilon + \varepsilon\gamma_J)$ such that (σ, τ) is contained in $(A)^{\varepsilon\gamma_J}$, and u_ε satisfies (3.6) and either (3.7) or (3.8) in (σ, τ). Note that by the intermediate value theorem for all $\varepsilon > 0$ sufficiently small there exist such intervals. Moreover, by construction, all intervals in $\mathcal{I}_\varepsilon$ are contained in A. It follows from (2.4) and (3.23) that

$$M\varepsilon \geq \int_\sigma^\tau W(u_\varepsilon(t))\, dt \geq \frac{\tau - \sigma}{4c_W} ,$$

hence

$$\tau - \sigma \leq 4c_W M\varepsilon . \tag{3.24}$$

In particular, for every $I \in \mathcal{I}_\varepsilon$ we have

$$\operatorname{diam} I \leq (4c_W M + 2\gamma_J)\varepsilon . \tag{3.25}$$

Moreover, by (3.2) and (3.5), if $I_1, \ldots, I_k$ are pairwise disjoint intervals in $\mathcal{I}_\varepsilon$, then

$$k \leq \frac{M}{c_{J,W}} . \tag{3.26}$$

Let B_ε be the union of all intervals in $\mathcal{I}_\varepsilon$ and let $\mathcal{C}_\varepsilon$ be the collection of its connected components. Observe that distinct elements of $\mathcal{C}_\varepsilon$ must contain disjoint intervals of $\mathcal{I}_\varepsilon$, and so by (3.26) the number of elements of $\mathcal{C}_\varepsilon$ is uniformly bounded. To be precise,

$$\#\mathcal{C}_\varepsilon \leq \frac{M}{c_{J,W}} . \tag{3.27}$$

Next we claim that if $C \in \mathcal{C}_\varepsilon$, then

$$\operatorname{diam} C \leq 2(4C_W M + 2\gamma_J)\left(\frac{M}{c_{J,W}} + 1\right)\varepsilon . \tag{3.28}$$

Assume by contradiction that (3.28) fails. Let k be the integer such that $\frac{M}{c_{J,W}} < k \leq \frac{M}{c_{J,W}} + 1$ and partition C into k subintervals $C_1, \ldots, C_k$ of equal length larger that $2(4C_W M + 2\gamma_J)\varepsilon$. The middle point of each C_i belongs to some interval $I_i \in \mathcal{I}_\varepsilon$. By (3.25), we have that $I_i \subset C_i$ and so $I_1, \ldots, I_k$ are pairwise disjoint. In turn k satisfies (3.26), which contradicts its definition. This concludes the proof of (3.28).

In view of (3.27) there exist a sequence $\varepsilon_j \to 0^+$ and a nonnegative integer $k \leq \frac{M}{c_{J,W}}$ such that $\#\mathcal{C}_{\varepsilon_j} = k$ for all $j \in \mathbb{N}$. Write $\mathcal{C}_{\varepsilon_j} = \{C_j^1, \ldots, C_j^k\}$ and choose $t_j^i \in C_j^i$. Up to a subsequence (not relabeled) we may assume that $t_j^i \to t^i \in \overline{A}$ for all $i = 1, \ldots, k$. By (3.28) for every $\eta > 0$ we have that $C_j^i \subset [t^i - \eta, t^i + \eta]$ for all j sufficiently large. Let $S := \{t^1, \ldots, t^k\}$ and let K be a closed interval contained in $A \setminus S$. Then $B_{\varepsilon_j} \cap K = \emptyset$ for all j sufficiently large. We claim that for all such j either $\inf_K u_{\varepsilon_j} \geq -\frac{1}{2}$ or $\sup_K u_{\varepsilon_j} \leq \frac{1}{2}$. Indeed, if this does not hold then we can find σ_j and τ_j in K for which u_{ε_j} satisfies (3.6) and either (3.7) or (3.8). On the one hand $(\sigma_j, \tau_j) \subset B_{\varepsilon_j}$ by the definition of B_{ε_j}. On the other hand $(\sigma_j, \tau_j) \subset K$ since K an interval. Therefore $(\sigma_j, \tau_j) \subset B_{\varepsilon_j} \cap K$ and this contradicts the fact that $B_{\varepsilon_j} \cap K = \emptyset$.

We extract a subsequence, possibly depending on K, not relabelled, such that, either $\inf_K u_{\varepsilon_j} \geq -\frac{1}{2}$ for all j or $\sup_K u_{\varepsilon_j} \leq \frac{1}{2}$ for all j. Since $u^2_{\varepsilon_j}(t) \to 1$ for a.e. $t \in K$, we conclude that $u_{\varepsilon_j}(t) \to 1$ for a.e. $t \in K$ in the former case while $u_{\varepsilon_j}(t) \to -1$ for a.e. $t \in K$ in the latter. By iterating this argument with an increasing sequence of compact intervals K whose union is a connected component of $A \setminus S$, it follows by a diagonal argument that a subsequence $\{u_{\varepsilon_j}\}$ (not relabeled) converges pointwise a.e in $A \setminus S$ to a function u constantly equal to -1 or 1 in each connected component of $A \setminus S$. This implies that $u \in BV(A; \{-1, 1\})$ with $S_u \subset S$, hence $\# S_u \leq \# S \leq k \leq \frac{M}{c_{J,W}}$. The L^2 convergence of $\{u_{\varepsilon_j}\}$ to u now follows from (2.4) and (3.23). ■

4 Compactness and interpolation for $n \geq 2$

Given $a \in \mathbb{R}$ we define

$$a^{(1)} := (-1) \vee (a \wedge 1) \ . \tag{4.1}$$

Lemma 4.1 *Let $\{u_\varepsilon\} \subset L^2(\Omega)$ be such that*

$$M := \sup_\varepsilon \mathcal{W}_\varepsilon(u_\varepsilon) < +\infty \ . \tag{4.2}$$

Then $u_\varepsilon - u_\varepsilon^{(1)} \to 0$ strongly in $L^2(\Omega)$.

Proof. By (2.11) and (4.2) we have that

$$\int_\Omega W(u_\varepsilon(x)) \, dx \to 0 \tag{4.3}$$

as $\varepsilon \to 0^+$. By (2.3) and (2.4) this implies that, up to a subsequence, $|u_\varepsilon(x)| \to 1$ for a.e. $x \in \Omega$. Hence, $u_\varepsilon(x) - u_\varepsilon^{(1)}(x) \to 0$ for a.e. $x \in \Omega$. On the other hand, by (2.4),

$$(u_\varepsilon(x) - u_\varepsilon^{(1)}(x))^2 \leq (u_\varepsilon(x))^2 \leq \frac{2}{c_W} W(u_\varepsilon(x)) + 2 \ ,$$

so that the conclusion follows from (4.2) and the (generalized) Lebesgue dominated convergence theorem. ■

In what follows, given a Borel set $E \subset \mathbb{R}^n$ and a function $u : E \to \mathbb{R}$, for every $\xi \in \mathbb{S}^{n-1}$ and for every $y \in \Pi^\xi$ (see (2.18)) we define the one-dimensional function

$$u_y^\xi(t) := u(y + t\xi) \ , \quad t \in E_y^\xi \ , \tag{4.4}$$

where E_y^ξ is defined in (2.19).

Lemma 4.2 *For every $A \subset \mathbb{R}^n$ open, $\varepsilon > 0$, and $u \in W^{1,2}_{\mathrm{loc}}(A) \cap L^2(A)$, we have*

$$\mathcal{F}_\varepsilon(u, A) \geq \int_{\mathbb{S}^{n-1}} \int_{\Pi^\xi} \mathcal{F}^\xi_\varepsilon(u^\xi_z, A^\xi_z)\ d\mathcal{H}^{n-1}(z) d\mathcal{H}^{n-1}(\xi)\ .$$

Proof. By Fubini's theorem, Proposition 2.2, (2.15), (2.23), and (2.24), we obtain

$$\begin{aligned}
&\mathcal{F}_\varepsilon(u, A)\\
&= \frac{1}{\sigma_{n-1}\varepsilon} \int_{\mathbb{S}^{n-1}} \int_{\Pi^\xi} \int_{A^\xi_z} W(u(z + t\xi))\ dt d\mathcal{H}^{n-1}(z) d\mathcal{H}^{n-1}(\xi)\\
&\quad + \frac{\varepsilon}{2} \int_{\mathbb{S}^{n-1}} \int_{\Pi^\xi} \int_{A^\xi_z} \int_{A^\xi_z} J^\xi_\varepsilon(t-s) |\nabla u(z+t\xi) - \nabla u(z+s\xi)|^2 dt ds d\mathcal{H}^{n-1}(z) d\mathcal{H}^{n-1}(\xi)\\
&\geq \frac{1}{\sigma_{n-1}\varepsilon} \int_{\mathbb{S}^{n-1}} \int_{\Pi^\xi} \int_{A^\xi_z} W(u^\xi_z(t))\ dt d\mathcal{H}^{n-1}(z) d\mathcal{H}^{n-1}(\xi)\\
&\quad + \frac{\varepsilon}{2} \int_{\mathbb{S}^{n-1}} \int_{\Pi^\xi} \int_{A^\xi_z} \int_{A^\xi_z} J^\xi_\varepsilon(t-s) ((u^\xi_z)'(t) - (u^\xi_z)'(s))^2 dt ds d\mathcal{H}^{n-1}(z) d\mathcal{H}^{n-1}(\xi)\\
&= \int_{\mathbb{S}^{n-1}} \int_{\Pi^\xi} \mathcal{F}^\xi_\varepsilon(u^\xi_z, A^\xi_z)\ d\mathcal{H}^{n-1}(z) d\mathcal{H}^{n-1}(\xi)\ .
\end{aligned}$$

■

Proof of Theorem 1.1. Let $\varepsilon_j \to 0^+$ and, for simplicity, write $u_j := u_{\varepsilon_j}$. By Lemma 4.2,

$$\int_{\mathbb{S}^{n-1}} \int_{\Pi^\xi} \mathcal{F}^\xi_{\varepsilon_j}((u_j)^\xi_z, \Omega^\xi_z)\ d\mathcal{H}^{n-1}(z) d\mathcal{H}^{n-1}(\xi) \leq M\ . \tag{4.5}$$

We claim that there exist a collection $\xi_1, \ldots, \xi_n \in \mathbb{S}^{n-1}$ of linearly independent vectors and a subsequence (not relabeled) such that

$$\lim_{j\to+\infty} \int_{\Pi^{\xi_i}} \mathcal{F}^{\xi_i}_{\varepsilon_j}((u_j)^{\xi_i}_z, \Omega^{\xi_i}_z)\ d\mathcal{H}^{n-1}(z) =: M_i < +\infty\ , \tag{4.6}$$

for every $i = 1, \ldots, n$.

Indeed, using Fatou's lemma by (4.5) we have that

$$\int_{\mathbb{S}^{n-1}} \liminf_{j\to+\infty} \int_{\Pi^\xi} \mathcal{F}^\xi_{\varepsilon_j}((u_j)^\xi_z, \Omega^\xi_z)\ d\mathcal{H}^{n-1}(z) d\mathcal{H}^{n-1}(\xi) \leq M\ . \tag{4.7}$$

Hence, there exists $\xi_1 \in \mathbb{S}^{n-1}$ such that

$$\liminf_{j\to+\infty} \int_{\Pi^{\xi_1}} \mathcal{F}^{\xi_1}_{\varepsilon_j}((u_j)^{\xi_1}_z, \Omega^{\xi_1}_z)\ d\mathcal{H}^{n-1}(z) =: M_1 < +\infty\ , \tag{4.8}$$

and we can extract a subsequence (not relabeled) such that (4.6) holds for $i = 1$.

We proceed by induction. Assume that we found a collection $\xi_1, \ldots, \xi_k \in \mathbb{S}^{n-1}$, $1 \leq k < n$, of linearly independent vectors and a subsequence (not relabeled) such that (4.6) holds for every $i = 1, \ldots, k$. Note that this subsequence still satisfies (4.5), and hence (4.7). Therefore we can find $\xi_{k+1} \in \mathbb{S}^{n-1}$, linearly independent of $\xi_1, \ldots, \xi_k$, such that

$$\liminf_{j\to+\infty} \int_{\Pi^{\xi_{k+1}}} \mathcal{F}_{\varepsilon_j}^{\xi_{k+1}}((u_j)_z^{\xi_{k+1}}, \Omega_z^{\xi_{k+1}})\, d\mathcal{H}^{n-1}(z) =: M_{k+1} < +\infty\ ,$$

and we can extract a subsequence (not relabeled) such that (4.6) holds also for $i = k+1$. After n steps we obtain that (4.6) is satisfied for every $i = 1, \ldots, n$.

Given $i = 1, \ldots, n$ and $\delta > 0$, for every j let

$$A_j^i := \left\{ z \in \Pi^{\xi_i} : \mathcal{F}_{\varepsilon_j}^{\xi_i}((u_j)_z^{\xi_i}, \Omega_z^{\xi_i}) > \frac{M_i}{\delta} \right\}\ , \tag{4.9}$$

and let $v_j^i \in L^2(\Omega)$ be defined by

$$\begin{cases} (v_j^i)_z^{\xi_i} := (u_j^{(1)})_z^{\xi_i} & \text{if } z \in \Pi^{\xi_i} \setminus A_j\ , \\ (v_j^i)_z^{\xi_i} := 0 & \text{if } z \in A_j\ , \end{cases} \tag{4.10}$$

where $u_j^{(1)}$ is the truncated function defined using (4.1). By (4.6) and (4.9) we have

$$\limsup_{j\to+\infty} \mathcal{H}^{n-1}(A_j^i) \le \delta\ ,$$

hence (4.10) yields

$$\limsup_{j\to+\infty} \|v_j^i - u_j^{(1)}\|_{L^2(\Omega)}^2 \le \delta\,\mathrm{diam}(\Omega)\ . \tag{4.11}$$

By Theorem 3.1 for every $z \in \Pi^{\xi_i}$ the set $\{(u_j)_z^{\xi_i}(1 - \chi_{A_j^i}(z)) : j \in \mathbb{N}\}$ is relatively compact in $L^2(\Omega_z^{\xi_i})$, where $\chi_{A_j^i}(z) = 1$ for $z \in A_j^i$ and $\chi_{A_j^i}(z) = 0$ for $z \notin A_j^i$. Therefore the same property holds for the set of truncated functions $\{(u_j^{(1)})_z^{\xi_i}(1 - \chi_{A_j^i}(z)) : j \in \mathbb{N}\}$. It follows that for every $z \in \Pi^{\xi_i}$ the set $\{(v_j^i)_z^{\xi_i} : j \in \mathbb{N}\}$ is relatively compact in $L^2(\Omega_z^{\xi_i})$. Since this property is valid for every $i = 1, \ldots, n$, we can apply the characterization by slicing of precompact sets of $L^2(\Omega)$ given by [5, Theorem 6.6] and we obtain that the set $\{u_j^{(1)} : j \in \mathbb{N}\}$ is relatively compact in $L^2(\Omega)$. In turn, by Lemma 4.1 the set $\{u_j : j \in \mathbb{N}\}$ is relatively compact in $L^2(\Omega)$, hence there exist a subsequence (not relabeled) , such that u_j converges in $L^2(\Omega)$ to some function u. By (1.9),

$$\lim_{j\to+\infty} \int_\Omega W(u_j(x))\, dx = 0\ ,$$

which, together with (2.3) and (2.4), implies that $u(x) \in \{-1, 1\}$ for a.e. $x \in \Omega$.

It remains to show that $u \in BV(\Omega)$. Using Fubini's theorem we find that there exists a subsequence (not relabeled) such that

$$(u_j)_z^{\xi_i} \to u_z^{\xi_i} \text{ in } L^2(\Omega_z^{\xi_i})\ . \tag{4.12}$$

Moreover, Fatou's lemma and (4.6) imply that

$$\int_{\Pi^{\xi_i}} \liminf_{j\to+\infty} \mathcal{F}^{\xi_i}_{\varepsilon_j}((u_j)^{\xi_i}_z, \Omega^{\xi_i}_z)\ d\mathcal{H}^{n-1}(z) \le M_i\ , \tag{4.13}$$

hence

$$\liminf_{j\to+\infty} \mathcal{F}^{\xi_i}_{\varepsilon_j}((u_j)^{\xi_i}_z, \Omega^{\xi_i}_z) < +\infty \tag{4.14}$$

for $\mathcal{H}^{n-1}$-a.e. $z \in \Pi^{\xi_i}$. Fix $z \in \Pi^{\xi_i}$ satisfying (4.12) and (4.14), and extract a subsequence $\{\hat{u}_j\}$, depending on z, such that

$$\lim_{j\to+\infty} \mathcal{F}^{\xi_i}_{\varepsilon_j}((\hat{u}_j)^{\xi_i}_z, \Omega^{\xi_i}_z) = \liminf_{j\to+\infty} \mathcal{F}^{\xi_i}_{\varepsilon_j}((u_j)^{\xi_i}_z, \Omega^{\xi_i}_z)\ . \tag{4.15}$$

By (3.3), (4.12), and (4.15) we have

$$\#S_{u^{\xi_i}_z} \le \frac{1}{c_{J,W}} \liminf_{j\to+\infty} \mathcal{F}^{\xi_i}_{\varepsilon_j}((u_j)^{\xi_i}_z, \Omega^{\xi_i}_z)\ .$$

Since $u^{\xi_i}_z(t) \in \{-1, 1\}$ for a.e. $t \in \Omega^{\xi_i}_z$, we deduce that

$$|Du^{\xi_i}_z|(\Omega^{\xi_i}_z) \le \frac{2}{c_{J,W}} \liminf_{j\to+\infty} \mathcal{F}^{\xi_i}_{\varepsilon_j}((u_j)^{\xi_i}_z, \Omega^{\xi_i}_z)$$

for $\mathcal{H}^{n-1}$-a.e. $z \in \Pi^{\xi_i}$. This property holds for every $i = 1, \ldots, n$. Therefore, we can apply the characterization by slicing of BV functions given by [7, Remark 3.104] and we obtain from (4.13) that $u \in BV(\Omega)$. ■

For $A \subset \mathbb{R}^n$ and $\eta > 0$ we recall the notation (3.1).

Lemma 4.3 (Interpolation inequality) *There exists a constant $c^{(n)}_{J,W}$ such that*

$$\varepsilon \int_A |\nabla u(x)|^2 dx \le c^{(n)}_{J,W} \mathcal{F}_\varepsilon(u, (A)^{2\varepsilon\gamma_J})\ . \tag{4.16}$$

for every $\varepsilon > 0$, for every open set $A \subset \mathbb{R}^n$, and for every $u \in W^{1,2}_{\mathrm{loc}}((A)_{2\varepsilon\gamma_J})$, where γ_J is the constant in (2.1).

Proof. Fix ε, A, and u as in the statement of the lemma, and define $B := (A)^{2\varepsilon\gamma_J}$. Given $\xi \in \mathbb{S}^{n-1}$, for $\mathcal{H}^{n-1}$ a.e. $z \in \Pi^\xi$ we have that $(A^\xi_z)^{2\varepsilon\gamma_J} \subset B^\xi_z$ and the sliced function u^ξ_z (see (4.4)) belongs to $W^{1,2}_{\mathrm{loc}}(B^\xi_z)$. Hence by Lemma 3.5 we have

$$\varepsilon \int_{A^\xi_z} ((u^\xi_z)'(t))^2 dt \le c^{(1)}_{J,W} \mathcal{F}^\xi_\varepsilon(u^\xi_z, B^\xi_z)\ .$$

Integrating this inequality in z over Π^ξ we obtain

$$\varepsilon \int_A (\nabla u(x) \cdot \xi)^2 dx \le c^{(1)}_{J,W} \int_{\Pi^\xi} \mathcal{F}^\xi_\varepsilon(u^\xi_z, B^\xi_z)\ d\mathcal{H}^{n-1}(z)\ .$$

Integrating this inequality in ξ over $\mathbb{S}^{n-1}$ and using Lemma 4.2, together with the identity $\int_{\mathbb{S}^{n-1}} |a\cdot\xi|^2 d\mathcal{H}^{n-1}(\xi) = \omega_n |a|^2$, we deduce

$$\omega_n \varepsilon \int_A |\nabla u(x)|^2 dx \le c^{(1)}_{J,W} \mathcal{F}_\varepsilon(u, B)\ .$$

This concludes the proof. ■

5 The modification theorem

In this section we prove that we can modify an admissible sequence to match a mollification of its limit in a neighborhood of the boundary, without increasing the limit energy.

Given $\nu \in \mathbb{S}^{n-1}$, let

$$w^\nu(x) := \begin{cases} 1 & \text{if } x \cdot \nu > 0 \ , \\ -1 & \text{if } x \cdot \nu < 0 \ . \end{cases} \tag{5.1}$$

When $\nu = e_n$, the superscript ν is omitted. Let $\theta \in C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp} \theta \subset B_1(0)$, $\int_{\mathbb{R}^n} \theta(x)\, dx = 1$, and for every $\sigma > 0$ define the mollifier

$$\theta_\sigma(x) := \frac{1}{\sigma^n} \theta\left(\frac{x}{\sigma}\right), \quad x \in \mathbb{R}^n \ . \tag{5.2}$$

Note that $\operatorname{supp} \theta_\sigma \subset B_\sigma(0)$. There exists a constant $C_\theta > 1$, independent of σ, such that

$$\sup_{\mathbb{R}^n} |(w^\nu * \theta_\sigma) - w^\nu| \leq 1 \ , \tag{5.3}$$

$$(w^\nu * \theta_\sigma)(x) = 1 \quad \text{if } x \cdot \nu > \sigma, \quad (w^\nu * \theta_\sigma)(x) = -1 \quad \text{if } x \cdot \nu < -\sigma \ , \tag{5.4}$$

$$\nabla (w^\nu * \theta_\sigma)(x) = 0 \quad \text{if } |x \cdot \nu| > \sigma \ , \tag{5.5}$$

$$\sup_{\mathbb{R}^n} |\nabla (w^\nu * \theta_\sigma)| \leq \frac{C_\theta}{\sigma} \quad \text{and} \quad \sup_{\mathbb{R}^n} |\nabla^2 (w^\nu * \theta_\sigma)| \leq \frac{C_\theta}{\sigma^2} \ . \tag{5.6}$$

Let P be a bounded polyhedron of dimension $n-1$ containing 0 and let $\nu \in \mathbb{S}^{n-1}$ be a normal to P. For every $\rho > 0$ we set

$$P_\rho := \{x + t\nu : \ x \in P \ , \ t \in (-\rho/2, \rho/2)\} \ . \tag{5.7}$$

Theorem 5.1 (Modification Theorem) *Let P be a bounded polyhedron of dimension $n-1$ containing 0, let $\rho > 0$, let $\varepsilon_j \to 0^+$, and let $\{u_j\}$ be a sequence in $W^{1,2}_{\text{loc}}(P_\rho) \cap L^2(P_\rho)$ such that $u_j \to w^\nu$ in $L^2(P_\rho)$. Then there exists a constant $\delta_{P_\rho} > 0$ depending only on P_ρ such that for every $0 < \delta < \delta_{P_\rho}$ there exists a sequence $\{v_j\} \subset W^{1,2}_{\text{loc}}(P_\rho) \cap L^2(P_\rho)$ such that $v_j \to w^\nu$ in $L^2(P_\rho)$, $v_j = u_j$ in $(P_\rho)_{2\delta}$, $v_j = w^\nu * \theta_{\varepsilon_j}$ on $P_\rho \setminus (P_\rho)_\delta$, and*

$$\limsup_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(v_j, P_\rho) \leq \limsup_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(u_j, P_\rho) + \kappa_1 \delta \ , \tag{5.8}$$

where $\kappa_1 > 0$ is a constant independent of j, δ, and P_ρ.

Remark 5.2 *By choosing a suitable subsequence, under the same assumptions of Theorem 5.1 we obtain that*

$$\liminf_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(v_j, P_\rho) \leq \liminf_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(u_j, P_\rho) + \kappa_1 \delta \ . \tag{5.9}$$

To prove Theorem 5.1 we use the estimate of the following lemma.

Lemma 5.3 *Let $\varepsilon > 0$, let $y \in \mathbb{R}^n$, let A be a measurable subset of $\mathbb{R}^n$, and let $g : A \to \mathbb{R}$ be a measurable function such that*

$$0 \le g(x) \le (a|x-y|)^2 \wedge b^2 \quad \text{for every } x \in A \ , \tag{5.10}$$

for some constants a and b. Then

$$\int_A J_\varepsilon(x-y) g(x) \ dx \le M_J \big((\varepsilon a) \vee b \big)^2 \ , \tag{5.11}$$

where M_J is the constant given in (1.6) and $\alpha \vee \beta := \max\{\alpha, \beta\}$.

Proof. Using (1.5) and the change of variables $z = (x-y)/\varepsilon$, we obtain

$$\begin{aligned}
\int_A J_\varepsilon(x-y) g(x) \ dx \le a^2 & \int_{A \cap B_\varepsilon(y)} J_\varepsilon(x-y)|x-y|^2 \ dx \\
& + b^2 \int_{A \setminus B_\varepsilon(y)} J_\varepsilon(x-y) \frac{|x-y|}{\varepsilon} \ dx \\
& \le \varepsilon^2 a^2 \int_{B_1(0)} J(z)|z|^2 \ dz + b^2 \int_{\mathbb{R}^n \setminus B_1(0)} J(z)|z| \ dz \ .
\end{aligned}$$

The conclusion follows from (1.6). ■

Lemma 5.4 *Let $0 < \varepsilon < \delta$, let A and B be open sets in $\mathbb{R}^n$, with $\operatorname{dist}(A, B) \ge \delta$, and let $u \in W^{1,2}_{\mathrm{loc}}(A \cup B)$. Then*

$$\mathcal{J}_\varepsilon(u, A, B) \le \varepsilon \omega_1\Big(\frac{\varepsilon}{\delta}\Big) \int_{A \cup B} |\nabla u(x)|^2 dx \ , \tag{5.12}$$

where

$$\omega_1(t) := 2 \int_{\mathbb{R}^n \setminus B_{1/t}(0)} J(z)|z| \ dz \to 0 \tag{5.13}$$

as $t \to 0^+$.

Proof. Using a change of variables we obtain

$$\begin{aligned}
\mathcal{J}_\varepsilon(u, A, B) &= \varepsilon \int_A \int_B J_\varepsilon(x-y)|\nabla u(x) - \nabla u(y)|^2 dx dy \\
&\le 2\varepsilon \int_B \Big(\int_A J_\varepsilon(x-y) \ dy \Big) |\nabla u(x)|^2 dx \\
&\quad + 2\varepsilon \int_A \Big(\int_B J_\varepsilon(x-y) \ dx \Big) |\nabla u(y)|^2 dy \\
&\le 2\varepsilon \int_B \Big(\int_{\mathbb{R}^n \setminus B_\delta(x)} J_\varepsilon(x-y) \ dy \Big) |\nabla u(x)|^2 dx \\
&\quad + 2\varepsilon \int_A \Big(\int_{\mathbb{R}^n \setminus B_\delta(y)} J_\varepsilon(x-y) \ dx \Big) |\nabla u(y)|^2 dy
\end{aligned}$$

$$\leq 2\varepsilon \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{\varepsilon}}(0)} J(z)\ dz \int_{A\cup B} |\nabla u(x)|^2 dx$$

$$\leq 2\varepsilon \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{\varepsilon}}(0)} J(z)|z|\ dz \int_{A\cup B} |\nabla u(x)|^2 dx\ .$$

This leads to (5.12). The fact that $\omega_1(t) \to 0^+$ as $t \to 0^+$ follows from (1.6). ■

Proof of Theorem 5.1. It is not restrictive to assume that $\delta < \frac{1}{4}$, $\varepsilon_j < \delta^2$, and $8\varepsilon_j \gamma_J < \delta$ for every j. To simplify the notation, set $\widetilde{u}_j := w^\nu * \theta_{\varepsilon_j}$. From (5.5) and (5.6) it follows that

$$\varepsilon_j \int_{P_\rho} |\nabla \widetilde{u}_j(x)|^2 dx \leq C_{\theta,P} \quad \text{for every } j\ , \tag{5.14}$$

for some constant $C_{\theta,P} > 0$ depending only on P and θ.

If the right-hand side of (5.8) is infinite, then there is nothing to prove. Thus, by extracting a subsequence (not relabeled), without loss of generality we may assume that

$$\mathcal{F}_{\varepsilon_j}(u_j, P_\rho) \leq M < +\infty \quad \text{for every } j\ , \tag{5.15}$$

for a suitable constant $M > 0$.

The functions v_j will be constructed as

$$v_j := \varphi_j u_j + (1 - \varphi_j)\widetilde{u}_j\ , \tag{5.16}$$

where $\varphi_j \in C_c^\infty(\mathbb{R}^n)$ are suitable cut-off functions satisfying $\varphi_j(x) = 1$ for $x \in (P_\rho)_\delta$ and $\varphi_j(x) = 0$ for $x \notin (P_\rho)_{\delta/2}$. Introduce the set

$$S := \Big\{x \in P_\rho :\ \frac{\delta}{2} < \operatorname{dist}\big(x, \partial P_\rho\big) \leq \delta\Big\}. \tag{5.17}$$

To construct the cut-off functions we divide S into m_j pairwise disjoint layers of width $\frac{\delta}{2m_j}$.

Consider the sequence $\{\eta_j\}$ defined by

$$\eta_j := \int_{P_\rho} (u_j(x) - \widetilde{u}_j(x))^2 dx + \int_{P_\rho} \int_{P_\rho \setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)(u_j(x) - \widetilde{u}_j(x))^2 dx dy\ . \tag{5.18}$$

By Fubini's theorem, a change of variables, (1.6), and (5.18), we obtain

$$\int_{P_\rho} \int_{P_\rho \setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)(u_j(x) - \widetilde{u}_j(x))^2 dx dy$$

$$= \int_{P_\rho} \bigg(\int_{P_\rho \setminus B_{\varepsilon_j}(x)} J_{\varepsilon_j}(x-y)\ dy \bigg) (u_j(x) - \widetilde{u}_j(x))^2 dx$$

$$\leq \int_{P_\rho} (u_j(x) - \widetilde{u}_j(x))^2 dx \int_{\mathbb{R}^n \setminus B_1(0)} J(z)\ dz \leq M_J \int_{P_\rho} (u_j(x) - \widetilde{u}_j(x))^2 dx\ .$$

Hence, $\eta_j \to 0^+$ as $j \to +\infty$, because $\{u_j\}$ and $\{\widetilde{u}_j\}$ converge to w^ν in $L^2(P_\rho)$. Without loss of generality, we assume that $\eta_j < \frac{1}{4}$ for every j. Let m_j be the unique integer such that

$$\frac{\sqrt{\varepsilon_j} + \sqrt{\eta_j}}{\varepsilon_j} < m_j \le \frac{\sqrt{\varepsilon_j} + \sqrt{\eta_j}}{\varepsilon_j} + 1 \; . \tag{5.19}$$

Since $\varepsilon_j < 1$ we have

$$\frac{1}{m_j} < \sqrt{\varepsilon_j} \quad \text{and} \quad m_j < 2\frac{\sqrt{\varepsilon_j} + \sqrt{\eta_j}}{\varepsilon_j} \tag{5.20}$$

and

$$\frac{\eta_j}{m_j \varepsilon_j} \le \sqrt{\varepsilon_j} + \sqrt{\eta_j} \quad \text{and} \quad m_j \varepsilon_j \le 2(\sqrt{\varepsilon_j} + \sqrt{\eta_j}) \; . \tag{5.21}$$

Divide S into m_j pairwise disjoint layers of width $\frac{\delta}{2m_j}$,

$$S_j^i := \left\{ x \in P_\rho : \ \frac{\delta}{2} + \frac{(i-1)\delta}{2m_j} < \operatorname{dist}\big(x, \partial P_\rho\big) < \frac{\delta}{2} + \frac{i\delta}{2m_j} \right\} , \tag{5.22}$$

$i = 1, \dots, m_j$.

For every open set $A \subset \mathbb{R}^d$ define

$$\begin{aligned} \mathcal{G}_j(A) := {} & \mathcal{J}_{\varepsilon_j}(u_j, A, P_\rho) + \mathcal{W}_{\varepsilon_j}(u_j, A) \\ & + \varepsilon_j \int_A |\nabla u_j(x)|^2 dx + \frac{1}{\varepsilon_j} \int_A (u_j(x) - \widetilde{u}_j(x))^2 dx \\ & + \frac{1}{\varepsilon_j} \int_A \int_{P_\rho \setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)(u_j(x) - \widetilde{u}_j(x))^2 dx dy \; . \end{aligned} \tag{5.23}$$

Hence, using (5.15), (5.18), and Lemma 4.3, we obtain

$$\sum_{i=1}^{m_j} \mathcal{G}_j(S_j^i) \le \mathcal{G}_j(S) \le K - 1 + \frac{\eta_j}{\varepsilon_j} \; ,$$

where $K := M + c_{J,W}^{(n)} M + 1$, and so there exists $i_j \in \{1, \dots, m_j\}$ such that, setting

$$S_j := S_j^{i_j} \; ,$$

we have

$$\mathcal{G}_j(S_j) \le \frac{K-1}{m_j} + \frac{\eta_j}{m_j \varepsilon_j} \le K\sqrt{\varepsilon_j} + \sqrt{\eta_j} \le K \; , \tag{5.24}$$

where in the last inequalities we used (5.20), (5.21), and the fact that $\varepsilon_j < \frac{1}{4}$, $\eta_j < \frac{1}{4}$, and $K \ge 1$. Define

$$\begin{aligned} A_j &:= \left\{ x \in P_\rho : \ \operatorname{dist}(x, \partial P_\rho) > \frac{\delta}{2} + \frac{i_j \delta}{2m_j} \right\} , \\ A_j^* &:= \left\{ x \in P_\rho : \ \operatorname{dist}(x, \partial P_\rho) > \frac{\delta}{2} + \frac{i_j \delta}{2m_j} - \frac{\delta}{4m_j} \right\} , \\ B_j &:= \left\{ x \in P_\rho : \ \operatorname{dist}(x, \partial P_\rho) < \frac{\delta}{2} + \frac{(i_j - 1)\delta}{2m_j} \right\} , \end{aligned} \tag{5.25}$$

and let

$$\varphi_j(x) := \int_{A_j^*} \theta_{\frac{\delta}{4m_j}}(x-y)\,dy \ .$$

Then $\varphi_j \in C_c^\infty(\mathbb{R}^n)$ and the following properties hold, thanks to (5.6) and (5.20):

$$\varphi_j = 1 \text{ in } A_j, \quad 0 \le \varphi_j \le 1 \text{ in } S_j, \quad \varphi_j = 0 \text{ in } B_j \ , \tag{5.26}$$

$$\sup|\nabla\varphi_j| \le 8\frac{C_\theta}{\delta}\frac{\sqrt{\varepsilon_j}+\sqrt{\eta_j}}{\varepsilon_j} \le \frac{8C_\theta}{\delta\varepsilon_j} \ , \ \sup|\nabla^2\varphi_j| \le 2^7\frac{C_\theta}{\delta^2}\frac{\varepsilon_j+\eta_j}{\varepsilon_j^2} \ , \tag{5.27}$$

where C_θ is the constant given in (5.6).

Let v_j be the function defined by (5.16). Since $(P_\rho)_\delta \subset A_j$ and $P_\rho \setminus (P_\rho)_{\delta/2} \subset B_j$, we have that $v_j = u_j$ in $(P_\rho)_\delta$ and $v_j = \widetilde{u}_j$ on $P_\rho \setminus (P_\rho)_{\delta/2}$. Moreover, since u_j and $\widetilde{u}_j$ converge to w^ν in $L^2(P_\rho)$, we have that $v_j \to w^\nu$ in $L^2(P_\rho)$. Note that

$$\nabla v_j := \varphi_j \nabla u_j + (1-\varphi_j)\nabla\widetilde{u}_j + (u_j - \widetilde{u}_j)\nabla\varphi_j \ . \tag{5.28}$$

Fix $0 < \eta < \frac{1}{2}$. Using the inequality $|a+b|^2 \le \frac{|a|^2}{1-\eta} + \frac{|b|^2}{\eta}$, we obtain

$$\begin{aligned}
|\nabla v_j(x) - \nabla v_j(y)|^2 \le \frac{1}{1-\eta}\big|&\varphi_j(x)\nabla u_j(x) - \varphi_j(y)\nabla u_j(y) \\
&+ (1-\varphi_j(x))\nabla\widetilde{u}_j(x) - (1-\varphi_j(y))\nabla\widetilde{u}_j(y)\big|^2 \\
+ \frac{1}{\eta}\big|&(u_j(x) - \widetilde{u}_j(x))\nabla\varphi_j(x) - (u_j(y) - \widetilde{u}_j(y))\nabla\varphi_j(y)\big|^2 \ .
\end{aligned} \tag{5.29}$$

In view of the same inequality and the convexity of $|\cdot|^2$, we get

$$\begin{aligned}
\big|\varphi_j(x)&\nabla u_j(x) - \varphi_j(y)\nabla u_j(y) + (1-\varphi_j(x))\nabla\widetilde{u}_j(x) - (1-\varphi_j(y))\nabla\widetilde{u}_j(y)\big|^2 \\
&= \big|\varphi_j(x)(\nabla u_j(x) - \nabla u_j(y)) + (\varphi_j(x) - \varphi_j(y))\nabla u_j(y) \\
&\quad + (1-\varphi_j(x))(\nabla\widetilde{u}_j(x) - \nabla\widetilde{u}_j(y)) - (\varphi_j(x) - \varphi_j(y))\nabla\widetilde{u}_j(y)\big|^2 \\
&\le \frac{1}{1-\eta}\big|\varphi_j(x)(\nabla u_j(x) - \nabla u_j(y)) + (1-\varphi_j(x))(\nabla\widetilde{u}_j(x) - \nabla\widetilde{u}_j(y))\big|^2 \\
&\quad + \frac{1}{\eta}\big|(\varphi_j(x) - \varphi_j(y))(\nabla u_j(y) - \nabla\widetilde{u}_j(y))\big|^2 \\
&\le \frac{\varphi_j(x)}{1-\eta}\big|\nabla u_j(x) - \nabla u_j(y)\big|^2 + \frac{1-\varphi_j(x)}{1-\eta}\big|\nabla\widetilde{u}_j(x) - \nabla\widetilde{u}_j(y)\big|^2 \\
&\quad + \frac{1}{\eta}(\varphi_j(x) - \varphi_j(y))^2\big|\nabla u_j(y) - \nabla\widetilde{u}_j(y)\big|^2 \ .
\end{aligned}$$

This inequality and (5.29) yield

$$
\begin{aligned}
|\nabla v_j(x) - \nabla v_j(y)|^2 \leq{}& \frac{\varphi_j(x)}{(1-\eta)^2}\big|\nabla u_j(x) - \nabla u_j(y)\big|^2 \\
&+ \frac{1-\varphi_j(x)}{(1-\eta)^2}\big|\nabla \widetilde{u}_j(x) - \nabla \widetilde{u}_j(y)\big|^2 \\
&+ \frac{2}{\eta}(\varphi_j(x) - \varphi_j(y))^2\big|\nabla u_j(y) - \nabla \widetilde{u}_j(y)\big|^2 \\
&+ \frac{1}{\eta}\big|(u_j(x) - \widetilde{u}_j(x))\nabla\varphi_j(x) - (u_j(y) - \widetilde{u}_j(y))\nabla\varphi_j(y)\big|^2 ,
\end{aligned}
$$

hence for every pair of open sets A, $B \subset P_\rho$ we obtain by (2.14)

$$
\begin{aligned}
&\mathcal{J}_{\varepsilon_j}(v_j, A, B) \leq \frac{\mathcal{J}_{\varepsilon_j}(u_j, A, B\cap(A_j \cup S_j))}{(1-\eta)^2} + \frac{\mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, A, B\cap(S_j \cup B_j))}{(1-\eta)^2} \\
&+ \frac{2\varepsilon_j}{\eta}\int_A \Big(\int_B J_{\varepsilon_j}(x-y)(\varphi_j(x)-\varphi_j(y))^2 dx\Big)|\nabla u_j(y) - \nabla\widetilde{u}_j(y)|^2 dy \qquad (5.30) \\
&+ \frac{\varepsilon_j}{\eta}\int_A \Big(\int_B J_{\varepsilon_j}(x-y)\big|(u_j(x)-\widetilde{u}_j(x))\nabla\varphi_j(x) - (u_j(y)-\widetilde{u}_j(y))\nabla\varphi_j(y)\big|^2 dxdy.
\end{aligned}
$$

By (2.17) we have

$$
\begin{aligned}
\mathcal{J}_{\varepsilon_j}(v_j, P_\rho) = {}& \mathcal{J}_{\varepsilon_j}(u_j, A_j) + \mathcal{J}_{\varepsilon_j}(v_j, S_j) + \mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, B_j) \\
&+ 2\mathcal{J}_{\varepsilon_j}(v_j, S_j, A_j \cup B_j) + 2\mathcal{J}_{\varepsilon_j}(v_j, A_j, B_j) \ . \qquad (5.31)
\end{aligned}
$$

We now estimate all the terms but the first on the right-hand side of (5.31).

By (5.30),

$$
\begin{aligned}
&\mathcal{J}_{\varepsilon_j}(v_j, S_j) \leq \frac{\mathcal{J}_{\varepsilon_j}(u_j, S_j)}{(1-\eta)^2} + \frac{\mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, S_j)}{(1-\eta)^2} \qquad (5.32) \\
&+ \frac{2\varepsilon_j}{\eta}\int_{S_j}\Big(\int_{S_j} J_{\varepsilon_j}(x-y)(\varphi_j(x)-\varphi_j(y))^2 dx\Big)|\nabla u_j(y) - \nabla\widetilde{u}_j(y)|^2 dy \\
&+ \frac{\varepsilon_j}{\eta}\int_{S_j}\Big(\int_{S_j} J_{\varepsilon_j}(x-y)\big|(u_j(x)-\widetilde{u}_j(x))\nabla\varphi_j(x) - (u_j(y)-\widetilde{u}_j(y))\nabla\varphi_j(y)\big|^2 dxdy.
\end{aligned}
$$

From (2.17) and (5.5) it follows that

$$
\begin{aligned}
\mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, S_j \cup B_j) = {}& \mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, (S_j \cup B_j)\cap P_{2\varepsilon_j}) \\
&+ 2\mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, (S_j \cup B_j)\cap P_{2\varepsilon_j}, (S_j \cup B_j)\setminus P_{2\varepsilon_j}) \ . \qquad (5.33)
\end{aligned}
$$

By the mean value theorem and by (5.6), for every $y \in P_\rho$ the function $g(x) := |\nabla\widetilde{u}_j(x) - \nabla\widetilde{u}_j(y)|^2$ satisfies (5.10) with $a = \frac{C_\theta}{\varepsilon_j^2}$ and $b = \frac{2C_\theta}{\varepsilon_j}$, hence by Lemma 5.10 we obtain

$$
\int_{P_\rho} J_{\varepsilon_j}(x-y)|\nabla\widetilde{u}_j(x) - \nabla\widetilde{u}_j(y)|^2 dx \leq 4C_\theta^2 M_J \frac{1}{\varepsilon_j^2} \ .
$$

Therefore by (2.14) and (5.33) we have

$$\begin{aligned}&\mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, S_j, S_j \cup B_j) + \mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, B_j) \leq \mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, S_j \cup B_j)\\ &\leq \mathcal{L}^n((S_j \cup B_j) \cap P_{2\varepsilon_j})\, 4C_\theta^2 M_J \frac{1}{\varepsilon_j}.\end{aligned}$$

We now use the fact that there exist two constants $C_{P_\rho} > 0$ and $\delta_{P_\rho} > 0$, depending only on P_ρ, such that

$$\mathcal{L}^n(((P_\rho)_{\delta_1} \setminus (P_\rho)_{\delta_2}) \cap P_\varepsilon) \leq C_{P_\rho}\varepsilon(\delta_2 - \delta_1) \tag{5.34}$$

for every $0 < \varepsilon < \delta_1 < \delta_2 < \delta_{P_\rho}$. Therefore

$$\mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, S_j, S_j \cup B_j) + \mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, B_j) \leq 4C_{P_\rho} C_\theta^2 M_J \delta\ . \tag{5.35}$$

By the mean value theorem, (5.20), and (5.27), for every $y \in S_j$ the function $g(x) = (\varphi_j(x) - \varphi_j(y))^2$ satisfies (5.10) with $a = \frac{8C_\theta}{\delta\varepsilon_j}$ and $b = 1 \leq \frac{8C_\theta}{\delta}$, where we used the inequalities $C_\theta \geq 1$ and $\delta \leq 1$. Hence, by Lemma 5.3 we have

$$\int_{P_\rho} J_{\varepsilon_j}(x-y)(\varphi_j(x) - \varphi_j(y))^2 dx \leq 2^6 \frac{C_\theta^2}{\delta^2} M_J\ .$$

In turn, by (5.5), (5.6), (5.23), and (5.24),

$$\begin{aligned}&\frac{2\varepsilon_j}{\eta}\int_{S_j}\Big(\int_{P_\rho} J_{\varepsilon_j}(x-y)(\varphi_j(x) - \varphi_j(y))^2 dx\Big)|\nabla u_j(y) - \nabla\widetilde{u}_j(y)|^2 dy\\ &\leq 2^8\frac{C_\theta^2 M_J}{\eta\delta^2}\varepsilon_j\int_{S_j}|\nabla u_j(y)|^2 dy + 2^8\frac{C_\theta^4 M_J}{\eta\delta^2}\frac{1}{\varepsilon_j}\mathcal{L}^n(S_j \cap P_{2\varepsilon_j})\\ &\leq 2^8\frac{C_\theta^2 M_J}{\eta\delta^2}\big(K\sqrt{\varepsilon_j} + \sqrt{\eta_j}\big) + 2^8 C_{P_\rho}\frac{C_\theta^4 M_J}{\eta\delta}\sqrt{\varepsilon_j}\ ,\end{aligned} \tag{5.36}$$

where in the last inequality we used the estimate

$$\mathcal{L}^n(S_j \cap P_{\varepsilon_j}) \leq C_{P_\rho}\delta\frac{\varepsilon_j}{m_j} \leq C_{P_\rho}\delta\varepsilon_j\sqrt{\varepsilon_j}\ , \tag{5.37}$$

which follows fron (5.20) and (5.34).

To treat the last term on the right-hand side of (5.32) we observe that

$$\begin{aligned}&\big|(u_j(x) - \widetilde{u}_j(x))\nabla\varphi_j(x) - (u_j(y) - \widetilde{u}_j(y))\nabla\varphi_j(y)\big|^2\\ &\qquad = \big|(u_j(x) - \widetilde{u}_j(x))(\nabla\varphi_j(x) - \nabla\varphi_j(y)) +\\ &\qquad\quad + (u_j(x) - \widetilde{u}_j(x) - u_j(y) + \widetilde{u}_j(y))\nabla\varphi_j(y)\big|^2\\ &\qquad \leq 2(u_j(x) - \widetilde{u}_j(x))^2\big|\nabla\varphi_j(x) - \nabla\varphi_j(y)\big|^2\\ &\qquad\ + 2(u_j(x) - \widetilde{u}_j(x) - u_j(y) + \widetilde{u}_j(y))^2\big|\nabla\varphi_j(y)\big|^2\ .\end{aligned}$$

Integrating and using the symmetry of J, we obtain

$$\begin{aligned} &\frac{\varepsilon_j}{\eta}\int_{S_j}\Big(\int_{S_j} J_{\varepsilon_j}(x-y)\big|(u_j(x)-\widetilde{u}_j(x))\nabla\varphi_j(x)-(u_j(y)-\widetilde{u}_j(y))\nabla\varphi_j(y)\big|^2dxdy \\ &\le \frac{2\varepsilon_j}{\eta}\int_{S_j}\Big(\int_{S_j} J_{\varepsilon_j}(x-y)|\nabla\varphi_j(x)-\nabla\varphi_j(y)|^2dx\Big)(u_j(y)-\widetilde{u}_j(y))^2dy \quad (5.38)\\ &+\frac{2\varepsilon_j}{\eta}\int_{S_j}\Big(\int_{S_j} J_{\varepsilon_j}(x-y)(u_j(x)-\widetilde{u}_j(x)-u_j(y)+\widetilde{u}_j(y))^2dx\Big)|\nabla\varphi_j(y)|^2dy\ . \end{aligned}$$

By the mean value theorem and (5.27), for every $y\in S_j$ the function $g(x)=|\nabla\varphi_j(x)-\nabla\varphi_j(y)|^2$ satisfies (5.10) for every $x\in\mathbb{R}^n$, with $a=\frac{2^7C_\theta}{\delta^2}\frac{\varepsilon_j+\eta_j}{\varepsilon_j^2}\le\frac{2^6C_\theta}{\delta^2}\frac{\sqrt{\varepsilon_j}+\sqrt{\eta_j}}{\varepsilon_j^2}$ and $b=\frac{2^4C_\theta}{\delta}\frac{\sqrt{\varepsilon_j}+\sqrt{\eta_j}}{\varepsilon_j}\le\frac{2^6C_\theta}{\delta^2}\frac{\sqrt{\varepsilon_j}+\sqrt{\eta_j}}{\varepsilon_j}$, where we used the inequalities $\delta\le 1$, $\varepsilon_j\le\frac14$, and $\eta_j\le\frac14$. Hence, by Lemma 5.3 we have

$$\int_{P_\rho} J_{\varepsilon_j}(x-y)|\nabla\varphi_j(x)-\nabla\varphi_j(y)|^2dx\le 2^{13}\frac{C_\theta^2M_J}{\delta^4}\frac{\varepsilon_j+\eta_j}{\varepsilon_j^2}\ .$$

In turn, by (5.23) and (5.24),

$$\begin{aligned} &\frac{2\varepsilon_j}{\eta}\int_{S_j}\Big(\int_{P_\rho} J_{\varepsilon_j}(x-y)|\nabla\varphi_j(x)-\nabla\varphi_j(y)|^2dx\Big)(u_j(y)-\widetilde{u}_j(y))^2dy\\ &\le 2^{14}\frac{C_\theta^2M_J}{\eta\delta^4}(\varepsilon_j+\eta_j)\frac{1}{\varepsilon_j}\int_{S_j}(u_j(y)-\widetilde{u}_j(y))^2dy \qquad (5.39)\\ &\le 2^{14}\frac{C_\theta^2M_JK}{\eta\delta^4}(\varepsilon_j+\eta_j)\ . \end{aligned}$$

Since J is even, by Fubini's theorem, a change of variables, and (5.27),

$$\begin{aligned} &\frac{2\varepsilon_j}{\eta}\int_{S_j}\Big(\int_{P_\rho} J_{\varepsilon_j}(x-y)(u_j(x)-\widetilde{u}_j(x)-u_j(y)+\widetilde{u}_j(y))^2dx\Big)|\nabla\varphi_j(y)|^2dy\\ &\le\frac{2^8C_\theta^2}{\eta\delta^2}\frac{\varepsilon_j+\eta_j}{\varepsilon_j}\int_{S_j}\Big(\int_{P_\rho\cap B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)(u_j(x)-\widetilde{u}_j(x)-u_j(y)+\widetilde{u}_j(y))^2dx\Big)dy\\ &+\frac{2^8C_\theta^2}{\eta\delta^2}\frac{\varepsilon_j+\eta_j}{\varepsilon_j}\int_{S_j}\Big(\int_{P_\rho\setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)(u_j(x)-\widetilde{u}_j(x)-u_j(y)+\widetilde{u}_j(y))^2dx\Big)dy\\ &\le\frac{2^8C_\theta^2}{\eta\delta^2}\frac{\varepsilon_j+\eta_j}{\varepsilon_j}\int_{B_{\varepsilon_j}(0)} J_{\varepsilon_j}(z)\Big(\int_{S_j}(u_j(y+z)-\widetilde{u}_j(y+z)-u_j(y)+\widetilde{u}_j(y))^2dy\Big)dz\\ &+\frac{2^9C_\theta^2}{\eta\delta^2}\frac{\varepsilon_j+\eta_j}{\varepsilon_j}\int_{S_j}\Big(\int_{P_\rho\setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)(u_j(x)-\widetilde{u}_j(x))^2dx\Big)dy \qquad (5.40)\\ &+\frac{2^9C_\theta^2}{\eta\delta^2}\frac{\varepsilon_j+\eta_j}{\varepsilon_j}\int_{S_j}\Big(\int_{P_\rho\setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)dx\Big)(u_j(y)-\widetilde{u}_j(y))^2dy\ . \end{aligned}$$

Since $\varepsilon_j < \delta/4$, by (5.20) and (5.22) for $y \in S_j$ and $|z| \le \varepsilon_j$ the segment joning y and $y+z$ is contained in $(P_\rho)_{\delta/4}$, and so by the mean value theorem for $|z| \le \varepsilon_j$,

$$\int_{S_j} (u_j(y+z) - \widetilde{u}_j(y+z) - u_j(y) + \widetilde{u}_j(y))^2 dy \le |z|^2 \int_{(P_\rho)_{\delta/4}} |\nabla u_j(y) - \nabla \widetilde{u}_j(y)|^2 dy \ .$$

Therefore, recalling that $2\varepsilon_j \gamma_J < \delta/4$, it follows from (1.5), (1.6), (5.14), and Lemma 4.3, that

$$\begin{aligned}
&\frac{2^8 C_\theta^2}{\eta\delta^2} \frac{\varepsilon_j + \eta_j}{\varepsilon_j} \int_{B_{\varepsilon_j}(0)} J_{\varepsilon_j}(z) \Big(\int_{S_j} (u_j(y+z) - \widetilde{u}_j(y+z) - u_j(y) + \widetilde{u}_j(y))^2 dy \Big) dz \\
&\le \frac{2^8 C_\theta^2}{\eta\delta^2} \frac{\varepsilon_j + \eta_j}{\varepsilon_j} \int_{B_{\varepsilon_j}(0)} J_{\varepsilon_j}(z) |z|^2 dz \int_{(P_\rho)_{\delta/4}} |\nabla u_j(y) - \nabla \widetilde{u}_j(y)|^2 dy \\
&\le \frac{2^9 C_\theta^2}{\eta\delta^2} (\varepsilon_j + \eta_j) \varepsilon_j \int_{B_1(0)} J(z)|z|^2 dz \int_{(P_\rho)_{\delta/4}} |\nabla u_j(y)|^2 dy \qquad (5.41) \\
&\quad + \frac{2^9 C_\theta^2}{\eta\delta^2} (\varepsilon_j + \eta_j) \varepsilon_j \int_{B_1(0)} J(z)|z|^2 dz \int_{(P_\rho)_{\delta/4}} |\nabla \widetilde{u}_j(y)|^2 dy \\
&\le \frac{2^9 C_\theta^2 M_J c_{J,W}^{(n)} M}{\eta\delta^2} (\varepsilon_j + \eta_j) + \frac{2^9 C_\theta^2 C_{\theta,P} M_J}{\eta\delta^2} (\varepsilon_j + \eta_j) \ .
\end{aligned}$$

By (5.23) and (5.24)

$$\begin{aligned}
&\frac{2^9 C_\theta^2}{\eta\delta^2} \frac{\varepsilon_j + \eta_j}{\varepsilon_j} \int_{S_j} \Big(\int_{P_\rho \setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)(u_j(x) - \widetilde{u}_j(x))^2 dx \Big) dy \qquad (5.42) \\
&\le \frac{2^9 C_\theta^2 K}{\eta\delta^2} (\varepsilon_j + \eta_j) \ .
\end{aligned}$$

Using (1.6), (5.23), and (5.24) we obtain

$$\begin{aligned}
&\frac{2^9 C_\theta^2}{\eta\delta^2} \frac{\varepsilon_j + \eta_j}{\varepsilon_j} \int_{S_j} \Big(\int_{P_\rho \setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y) dx \Big) (u_j(y) - \widetilde{u}_j(y))^2 dy \\
&\le \frac{2^9 C_\theta^2 M_J}{\eta\delta^2} \frac{\varepsilon_j + \eta_j}{\varepsilon_j} \int_{S_j} (u_j(y) - \widetilde{u}_j(y))^2 dy \le \frac{2^9 C_\theta^2 M_J K}{\eta\delta^2} (\varepsilon_j + \eta_j) \ . \qquad (5.43)
\end{aligned}$$

Combining (5.32), (5.35), (5.36), (5.38), (5.39), (5.40), (5.41), (5.42), and (5.43), we have

$$\mathcal{J}_{\varepsilon_j}(v_j, S_j) + \mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, B_j) \le \frac{\mathcal{J}_{\varepsilon_j}(u_j, S_j)}{(1-\eta)^2} + \frac{4 C_{P_\rho} C_\theta^2 M_J}{(1-\eta)^2} \delta + \sigma_j^{(1)} \ , \qquad (5.44)$$

where $\sigma_j^{(1)} \to 0^+$ as $j \to +\infty$.

Next we consider the term $\mathcal{J}_{\varepsilon_j}(v_j, S_j, A_j \cup B_j)$ in (5.31) . By (5.30), using (5.26),

$$
\begin{aligned}
\mathcal{J}_{\varepsilon_j}(v_j, S_j, A_j \cup B_j) \leq & \frac{\mathcal{J}_{\varepsilon_j}(u_j, S_j, A_j)}{(1-\eta)^2} + \frac{\mathcal{J}_{\varepsilon_j}(\widetilde{u}_j, S_j, B_j)}{(1-\eta)^2} \\
& + \frac{2\varepsilon_j}{\eta} \int_{S_j} \Big(\int_{A_j \cup B_j} J_{\varepsilon_j}(x-y)(\varphi_j(x) - \varphi_j(y))^2 dx \Big) |\nabla u_j(y) - \nabla \widetilde{u}_j(y)|^2 dy \\
& + \frac{\varepsilon_j}{\eta} \int_{S_j} \int_{A_j \cup B_j} J_{\varepsilon_j}(x-y)(u_j(y) - \widetilde{u}_j(y))^2 |\nabla \varphi_j(y)|^2 dx dy \ . \qquad (5.45)
\end{aligned}
$$

Since $\eta < 1/2$, by (5.23) and (5.24) we have

$$
\frac{\mathcal{J}_{\varepsilon_j}(u_j, S_j, A_j)}{(1-\eta)^2} \leq 4\mathcal{J}_{\varepsilon_j}(u_j, S_j, A_j) \leq 4K\sqrt{\varepsilon_j} + 4\sqrt{\eta_j} \ . \qquad (5.46)
$$

The second and third terms on the right-hand side of (5.45) can be estimated using (5.35) and (5.36). For the last term, we use the fact that $\nabla \varphi_j(x) = 0$ if $x \in A_j \cup B_j$. Hence, by a change of variables, from (1.6), (5.23), (5.24), (5.27) and from the inequalities $\delta \leq 1$, $\varepsilon_j \leq 1$, and $\eta_j \leq 1$, we obtain

$$
\begin{aligned}
& \frac{\varepsilon_j}{\eta} \int_{S_j} \int_{A_j \cup B_j} J_{\varepsilon_j}(x-y)(u_j(y) - \widetilde{u}_j(y))^2 |\nabla \varphi_j(y)|^2 dx dy \\
& \leq \frac{\varepsilon_j}{\eta} \int_{S_j} \int_{B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)(u_j(y) - \widetilde{u}_j(y))^2 |\nabla \varphi_j(y) - \nabla \varphi_j(x)|^2 dx dy \\
& + \frac{\varepsilon_j}{\eta} \int_{S_j} \int_{P_\rho \setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)(u_j(y) - \widetilde{u}_j(y))^2 |\nabla \varphi_j(y)|^2 dx dy \\
& \leq 2^{14} \frac{C_\theta^2}{\eta \delta^4} \frac{(\varepsilon_j + \eta_j)^2}{\varepsilon_j^3} \int_{S_j} \Big(\int_{B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y)|x-y|^2 dx \Big) (u_j(y) - \widetilde{u}_j(y))^2 dy \\
& + \frac{2^7 C_\theta^2}{\eta \delta^2} \frac{\varepsilon_j + \eta_j}{\varepsilon_j} \int_{S_j} \Big(\int_{P_\rho \setminus B_{\varepsilon_j}(y)} J_{\varepsilon_j}(x-y) \ dx \Big) (u_j(y) - \widetilde{u}_j(y))^2 dx dy \qquad (5.47) \\
& \leq 2^{14} \frac{C_\theta^2 M_J}{\eta \delta^4} \frac{\varepsilon_j + \eta_j}{\varepsilon_j} \int_{S_j} (u_j(y) - \widetilde{u}_j(y))^2 dy \leq 2^{14} \frac{C_\theta^2 M_J K}{\eta \delta^4} (\varepsilon_j + \eta_j) \ .
\end{aligned}
$$

Therefore, by (5.35), (5.36), (5.45), (5.46), and (5.47) we get

$$
\mathcal{J}_{\varepsilon_j}(v_j, S_j, A_j \cup B_j) \leq \frac{4C_{P_\rho} C_\theta^2 M_J}{(1-\eta)^2} \delta + \sigma_j^{(2)} \ , \qquad (5.48)
$$

where $\sigma_j^{(2)} \to 0^+$ as $j \to +\infty$.

We now estimate the term $\mathcal{J}_{\varepsilon_j}(v_j, A_j, B_j)$ in (5.31). Since $v_j = u_j$ in A_j, $v_j = \widetilde{u}_j = 1$ in B_j, and $\mathrm{dist}(A_j, B_j) = \frac{\delta}{2m_j}$, by a change of variables and in view of (5.14), (5.21), and Lemmas 4.3 and 5.4, for j large enough we obtain

$$\begin{aligned}\mathcal{J}_{\varepsilon_j}(v_j, A_j, B_j) &\leq 2\omega_1\big(2\frac{m_j\varepsilon_j}{\delta}\big)\Big(\varepsilon_j\int_{B_j}|\nabla\widetilde{u}_j(x)|^2dx+\varepsilon_j\int_{A_j}|\nabla u_j(y)|^2dy\Big)\\ &\leq 2\omega_1\Big(4\frac{\sqrt{\varepsilon_j}+\sqrt{\eta_j}}{\delta}\Big)(C_{\theta,P}+c^{(n)}_{J,W}M)\ . \qquad (5.49)\end{aligned}$$

Combining (5.31), (5.35), (5.44), (5.48), and (5.49) we deduce

$$\mathcal{J}_{\varepsilon_j}(v_j, P_\rho) \leq \frac{\mathcal{J}_{\varepsilon_j}(u_j, P_\rho)}{(1-\eta)^2}+\frac{12C_{P_\rho}C_\theta^2M_J}{(1-\eta)^2}\delta+\sigma_j^{(3)}\ , \qquad (5.50)$$

where $\sigma_j^{(3)}\to 0^+$ as $j\to+\infty$.

Next we consider the term $\mathcal{W}_{\varepsilon_j}(v_j, P_\rho)$. Fix $x\in S_j$ with $x\cdot\nu>\varepsilon_j$, so that $\widetilde{u}_j(x)=1$. By (2.5) and (2.6) we have $W(v_j(x))\leq W(u_j(x))$ if $u_j(x)\geq 1-a_W$. Let $s_0<-1$ be such that

$$W(s_0)=\max_{[-1,1]}W=:M_W\ . \qquad (5.51)$$

If $u_j(x)\leq s_0$, then either $u_j(x)\leq v_j(x)\leq -1$ or $-1\leq v_j(x)\leq 1$. In both cases we get $W(v_j(x))\leq W(u_j(x))$, either by (2.6) or by (5.51). If $s_0<u_j(x)<1-a_W$, then $s_0<v_j(x)<1$ and we have

$$W(v_j(x))\leq W(s_0)=M_W$$

by (2.6) and (5.51). We conclude that

$$W(v_j(x))\leq W(u_j(x))+M_W$$

for every $x\in S_j$ with $x\cdot\nu>\varepsilon_j$. Integrating we obtain

$$\begin{aligned}\frac{1}{\varepsilon_j}\int_{S_j\cap\{x\cdot\nu>\varepsilon_j\}}W(v_j(x))\,dx &\leq \frac{1}{\varepsilon_j}\int_{S_j\cap\{x\cdot\nu>\sigma_j\}}W(u_j(x))\,dx\\ &+\frac{M_W}{\varepsilon_j}\mathcal{L}^n(S_j\cap\{|u_j-1|>a_W\}\cap\{x\cdot\nu>\varepsilon_j\})\\ &\leq\frac{1}{\varepsilon_j}\int_{S_j\cap\{x\cdot\nu>\varepsilon_j\}}W(u_j(x))\,dx+\frac{M_W}{\varepsilon_j a_W^2}\int_{S_j\cap\{x\cdot\nu>\varepsilon_j\}}(u_j(x)-1)^2dx\end{aligned}$$

A similar inequality can be obtained for $S_j\cap\{x\cdot\nu<-\varepsilon_j\}$, and adding these two inequalities we conclude that

$$\begin{aligned}\frac{1}{\varepsilon_j}\int_{S_j\setminus P_{\varepsilon_j}}W(v_j(x))\,dx &\leq\frac{1}{\varepsilon_j}\int_{S_j\setminus P_{\varepsilon_j}}W(u_j(x))\,dx\\ &+\frac{M_W}{a_W^2}\frac{1}{\varepsilon_j}\int_{S_j\setminus P_{\varepsilon_j}}(u_j(x)-\widetilde{u}_j(x))^2dx\ , \qquad (5.52)\end{aligned}$$

where in the last inequality we used the fact that $\widetilde{u}_j = w^\nu$ on $P_\rho \setminus P_{\varepsilon_j}$.

On the other hand, since $W(v_j(x)) \le W(u_j(x)) + M_W$ for every $x \in P_\rho$, integrating over $S_j \cap P_{\varepsilon_j}$ and using (5.37), we obtain

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{S_j \cap P_{\varepsilon_j}} W(v_j(x))\, dx &\le \frac{1}{\varepsilon_j} \int_{S_j \cap P_{\varepsilon_j}} W(u_j(x))\, dx + \frac{M_W}{\varepsilon_j} \mathcal{L}^n(S_j \cap P_{\varepsilon_j}) \\ &\le \frac{1}{\varepsilon_j} \int_{S_j \cap P_{\varepsilon_j}} W(u_j(x))\, dx + C_{P_\rho} M_W \delta \sqrt{\varepsilon_j}\ . \end{aligned} \tag{5.53}$$

Adding (5.52) and (5.53) gives

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{S_j} W(v_j(x))\, dx &\le \frac{1}{\varepsilon_j} \int_{S_j} W(u_j(x))\, dx \\ &\quad + \frac{M_W}{a_W^2} \frac{1}{\varepsilon_j} \int_{S_j} (u_j(x) - \widetilde{u}_j(x))^2 dx + C_{P_\rho} M_W \delta \sqrt{\varepsilon_j}\ , \end{aligned}$$

hence by (5.23) and (5.24) we have

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{S_j} W(v_j(x))\, dx &\le \frac{1}{\varepsilon_j} \int_{S_j} W(u_j(x))\, dx \\ &\quad + \frac{M_W}{a_W^2} (K\sqrt{\varepsilon_j} + \sqrt{\eta_j}) + C_{P_\rho} M_W \delta \sqrt{\varepsilon_j}\ . \end{aligned} \tag{5.54}$$

By (5.3), (5.4), (5.34), and (5.51) we get

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{B_j} W(v_j(x))\, dx &= \frac{1}{\varepsilon_j} \int_{B_j} W(\widetilde{u}_j(x))\, dx \\ &\le \frac{M_W}{\varepsilon_j} \mathcal{L}^n(B_j \cap P_{\varepsilon_j}) \le C_{P_\rho} M_W \delta\ . \end{aligned} \tag{5.55}$$

From (5.54) and (5.55) it follows that

$$\frac{1}{\varepsilon_j} \int_{P_\rho} W(v_j(x))\, dx \le \frac{1}{\varepsilon_j} \int_{P_\rho} W(u(x))\, dx + C_{P_\rho} M_W \delta + \sigma_j^{(4)}\ , \tag{5.56}$$

where $\sigma_j^{(4)} \to 0^+$ as $j \to +\infty$.

Adding (5.50) and (5.56) we obtain

$$\mathcal{F}_{\varepsilon_j}(v_j, P_\rho) \le \frac{\mathcal{F}_{\varepsilon_j}(u_j, P_\rho)}{(1-\eta)^2} + C_{P_\rho}(48\, C_\theta^2 M_J + M_W)\delta + \sigma_j^{(5)}$$

where $\sigma_j^{(5)} \to 0^+$ as $j \to +\infty$. This implies that

$$\limsup_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(v_j, P_\rho) \le \frac{1}{(1-\eta)^2} \limsup_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(u_j, P_\rho) + \kappa_1 \delta\ ,$$

where κ_1 is a constant independent of j, δ, and P_ρ. Passing to the limit as $\eta \to 0^+$ we obtain (5.8). ■

6 Gamma Liminf Inequality

In this section we prove the Γ-liminf inequality.

Theorem 6.1 (Γ-Liminf) *Let $\varepsilon_j \to 0^+$ and let $\{u_j\}$ be a sequence in $W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^2(\Omega)$ such that $u_j \to u$ in $L^2(\Omega)$ and*

$$\liminf_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(u_j, \Omega) < +\infty \ . \tag{6.1}$$

Then $u \in BV(\Omega; \{-1, 1\})$ and

$$\liminf_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(u_j, \Omega) \geq \int_{S_u} \psi(\nu_u)\ d\mathcal{H}^{n-1} \ , \tag{6.2}$$

where ψ is defined by (1.13).

Given $\nu \in \mathbb{S}^{n-1}$, let $\nu_1, \ldots, \nu_n$ be an orthonormal basis in $\mathbb{R}^n$ with $\nu_n = \nu$, let

$$Q^\nu_\rho := \{x \in \mathbb{R}^n : \ |x \cdot \nu_i| < \rho/2 \ , \ i = 1, \ldots, n\} \ , \quad \hat{Q}^\nu_\rho := \mathbb{R}^n \setminus Q^\nu_\rho, \tag{6.3}$$

and let

$$S^\nu_\rho := \{x \in \mathbb{R}^n : \ |x \cdot \nu| < \rho/2\} \ , \quad \hat{S}^\nu_\rho := \mathbb{R}^n \setminus S^\nu_\rho \ .$$

When $\nu_1, \ldots, \nu_n$ is the canonical basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$ we omit the superscript ν in the above notation.

We recall the definition of the sets V^ν and X^ν in (1.10) and in (1.12), respectively. We will use these sets in what follows. Further, as in Section 5, θ_ε is the standard mollifier (see (5.2)), and we set

$$\tilde{u}_\varepsilon := w^\nu * \theta_\varepsilon \ , \tag{6.4}$$

where w^ν is the function defined in (5.1), with $\nu \in \mathbb{S}^{n-1}$.

Lemma 6.2 *Let $0 < \varepsilon < \delta < 1/3$, let $C_\delta := Q_{1+\delta} \setminus Q_{1-\delta}$, and let $\tilde{u}_\varepsilon$ be the function in (6.4), with $\nu = e_n$. Then*

$$\mathcal{J}_\varepsilon(\tilde{u}_\varepsilon, C_\delta) \leq \kappa_2 \delta$$

for some constant $\kappa_2 > 0$ independent of ε and δ.

Proof. For every $\sigma > 0$ define $C^\sigma_\delta := C_\delta \cap \{|x_n| < \sigma\}$, $\hat{C}^\sigma_\delta := C_\delta \cap \{|x_n| \geq \sigma\}$, and write

$$C_\delta \times C_\delta = (C^{2\varepsilon}_\delta \times C^{2\varepsilon}_\delta) \cup (C^\varepsilon_\delta \times \hat{C}^{2\varepsilon}_\delta) \cup (\hat{C}^{2\varepsilon}_\delta \times C^\varepsilon_\delta) \cup (\hat{C}^\varepsilon_\delta \times \hat{C}^\varepsilon_\delta) \ .$$

Since J is even, we have

$$\mathcal{J}_\varepsilon(\tilde{u}_\varepsilon, C_\delta) \leq \mathcal{J}_\varepsilon(\tilde{u}_\varepsilon, C^{2\varepsilon}_\delta) + 2\mathcal{J}_\varepsilon(\tilde{u}_\varepsilon, C^\varepsilon_\delta, \hat{C}^{2\varepsilon}_\delta) + \mathcal{J}_\varepsilon(\tilde{u}_\varepsilon, \hat{C}^\varepsilon_\delta) \ . \tag{6.5}$$

By (5.2) we have that $\nabla\tilde{u}_\varepsilon = 0$ on $\hat{C}^\varepsilon_\delta$ and so

$$\mathcal{J}_\varepsilon(\tilde{u}_\varepsilon, \hat{C}^\varepsilon_\delta) = 0 \ . \tag{6.6}$$

We now estimate the first term on the right-hand side of (6.5). Since $\varepsilon\nabla\tilde{u}_\varepsilon$ and $\varepsilon^2\nabla^2\tilde{u}_\varepsilon$ are bounded in L^∞ uniformly with respect to ε, there exists a constant $c > 0$ such that

$$|\nabla\tilde{u}_\varepsilon(x) - \nabla\tilde{u}_\varepsilon(y)|^2 \leq \frac{c}{\varepsilon^2}\Big(\Big|\frac{x-y}{\varepsilon}\Big| \wedge \Big|\frac{x-y}{\varepsilon}\Big|^2\Big)$$

for every x, $y \in \mathbb{R}^n$. Therefore, by the change of variables $z = (x-y)/\varepsilon$ and (1.6) we get

$$\begin{aligned} \mathcal{J}_\varepsilon(\tilde{u}_\varepsilon, C^{2\varepsilon}_\delta) &\leq \frac{c}{\varepsilon}\int_{C^{2\varepsilon}_\delta}\int_{C^{2\varepsilon}_\delta} J_\varepsilon(x-y)\Big(\Big|\frac{x-y}{\varepsilon}\Big| \wedge \Big|\frac{x-y}{\varepsilon}\Big|^2\Big)\,dxdy \\ &\leq \frac{cM_J}{\varepsilon}\mathcal{L}^n(C^{2\varepsilon}_\delta) \leq 2^{n+1}cM_J\delta \ . \end{aligned} \tag{6.7}$$

Next we study the second term on the right-hand side of (6.5). Since $\nabla\tilde{u}_\varepsilon = 0$ on $\hat{C}^{2\varepsilon}_\delta$ and $\varepsilon\nabla\tilde{u}_\varepsilon$ is bounded in L^∞ uniformly with respect to ε, there exists a constant $c > 0$ such that

$$\begin{aligned} \mathcal{J}_\varepsilon(\tilde{u}_\varepsilon, C^\varepsilon_\delta, \hat{C}^{2\varepsilon}_\delta) &= \varepsilon\int_{C^\varepsilon_\delta}\Big(\int_{\hat{C}^{2\varepsilon}_\delta} J_\varepsilon(x-y)dx\Big)|\nabla\tilde{u}_\varepsilon(y)|^2dy \\ &\leq \frac{c}{\varepsilon}\mathcal{L}^n(C^\varepsilon_\delta)\int_{\mathbb{R}^n\setminus B_1(0)} J(z)\,dz \leq 2^n cM_J\delta \ , \end{aligned} \tag{6.8}$$

where we used again the change of variables $z = (x-y)/\varepsilon$ and (1.6). The conclusion follows by combining (6.5)–(6.8). ■

The following result will be crucial in the proof of the Γ-liminf inequality.

Lemma 6.3 *Let $0 < \varepsilon < \delta < 1/3$, let $u \in X^\nu$ be such $u = \tilde{u}_\varepsilon$ in $Q^\nu_1 \setminus Q^\nu_{1-\delta}$, where $\tilde{u}_\varepsilon$ is the function defined in (6.4). Then there exist two constants κ_3 and κ_4, depending only on the dimension n of the space, such that*

$$\mathcal{J}_\varepsilon(u, V^\nu, \mathbb{R}^n) - \mathcal{J}_\varepsilon(u, Q^\nu_1) \leq \kappa_2\delta + \Big(\kappa_3\omega_1\Big(\frac{\varepsilon}{\delta}\Big) + \kappa_4\omega_1(\varepsilon)\Big)\varepsilon\int_{Q^\nu_1}|\nabla u(x)|^2dx \ ,$$

where κ_2 is the constant in Lemma 6.2, and ω_1 is the function defined in (5.13).

Proof. Without loss of generality, we may assume that $\nu = e_n$, the n-th vector of the canonical basis. For simplicity we omit the superscript ν in the notation for Q^ν_ρ, $\hat{Q}^\nu_\rho$, S^ν_ρ, $\hat{S}^\nu_\rho$, V^ν, X^ν, w^ν, and the subscript ρ when $\rho = 1$. Write

$$\begin{aligned} V \times \mathbb{R}^n &= ((V\setminus Q)\times Q) \cup ((V\setminus Q)\times\hat{Q}) \cup (Q\times Q) \cup (Q\times\hat{Q}) \\ &\subset (\hat{S}\times Q) \cup ((V\setminus Q)\times S) \cup (\hat{S}\times\hat{S}) \cup (Q\times Q) \cup (Q\times(S\setminus Q)) \cup (Q\times\hat{S}) \ . \end{aligned} \tag{6.9}$$

Since J is even we have

$$\begin{aligned}
\mathcal{J}_\varepsilon(u,V,\mathbb{R}^n) - \mathcal{J}_\varepsilon(u,Q) \le 2\varepsilon \int_{\hat S}\Big(\int_{Q_{1-\delta}} & J_\varepsilon(x-y)|\nabla u(x)|^2 dx\Big) dy \\
&+ \varepsilon \int_{V\setminus Q}\Big(\int_{S_{1-\delta}} J_\varepsilon(x-y)|\nabla u(x)|^2 dx\Big) dy \qquad (6.10)\\
&+ \varepsilon \int_{Q}\Big(\int_{S\setminus Q} J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2 dx\Big) dy \ ,
\end{aligned}$$

where we have used the equalities $u = \pm 1$ and $\nabla u = 0$ in $\hat S_{1-\delta}$, which follow from the facts that $u \in X$ and $u = \tilde u_\varepsilon$ on $Q_1 \setminus Q_{1-\delta}$ (see (5.4), (5.5), and the inequalities $0 < \varepsilon < \delta < 1/3$).

We now estimate the first term on the right-hand side of (6.10). By Lemma 5.4 and because $\nabla u = 0$ in $\hat S$, we have

$$\varepsilon \int_{\hat S}\Big(\int_{Q_{1-\delta}} J_\varepsilon(x-y)|\nabla u(x)|^2 dx\Big) dy \le \varepsilon\omega_1\Big(\frac{\varepsilon}{\delta}\Big) \int_{Q_{1-\delta}} |\nabla u(x)|^2 dx \ . \qquad (6.11)$$

To estimate the second term on the right-hand side of (6.10), we identify $\mathbb{Z}^n$ with $\mathbb{Z}^{n-1}\times\mathbb{Z}$ so that for $\alpha = (\alpha_1,\ldots,\alpha_{n-1}) \in \mathbb{Z}^{n-1}$ and $\beta \in \mathbb{Z}$ we have $(\alpha,\beta) = (\alpha_1,\ldots,\alpha_{n-1},\beta) \in \mathbb{Z}^n$. Write

$$S \setminus Q_3 = \bigcup_{\alpha\in\mathbb{Z}^{n-1},\ |\alpha|_\infty \ge 2} ((\alpha,0)+Q) \ , \quad V = \bigcup_{\beta\in\mathbb{Z}} ((0,\beta)+Q) \ ,$$

where $|\alpha|_\infty := \max\{|\alpha_1|,\ldots,|\alpha_{n-1}|\}$. Then

$$\begin{aligned}
\varepsilon \int_{V\setminus Q} & \Big(\int_{S_{1-\delta}} J_\varepsilon(x-y)|\nabla u(x)|^2 dx\Big) dy \\
&\le \varepsilon \int_{V\setminus Q}\Big(\int_{S_{1-\delta}\cap Q_3} J_\varepsilon(x-y)|\nabla u(x)|^2 dx\Big) dy \qquad (6.12)\\
&\quad + \sum_{\alpha\in\mathbb{Z}^{n-1},\ |\alpha|_\infty\ge 2}\ \sum_{\beta\in\mathbb{Z}} \varepsilon \int_{(0,\beta)+Q}\Big(\int_{(\alpha,0)+Q} J_\varepsilon(x-y)|\nabla u(x)|^2 dx\Big) dy \ .
\end{aligned}$$

By Lemma 5.4 and because $\nabla u = 0$ in $V \setminus Q$, we have

$$\varepsilon \int_{V\setminus Q}\Big(\int_{S_{1-\delta}\cap Q_3} J_\varepsilon(x-y)|\nabla u(x)|^2 dx\Big) dy \le \varepsilon\omega_1\Big(\frac{\varepsilon}{\delta}\Big) \int_{S_{1-\delta}\cap Q_3} |\nabla u(x)|^2 dx \ .$$

To estimate the second term on the right-hand side of (6.12), we use the change of variables $\zeta = x - y$ and observe that for $x \in (\alpha,0)+Q$ and $y \in (0,\beta)+Q$ we have $\zeta \in (\alpha, -\beta) + Q_2$. Therefore, we obtain

$$\begin{aligned}
&\int_{(0,\beta)+Q} \Big(\int_{(\alpha,0)+Q} J_\varepsilon(x-y) |\nabla u(x)|^2 dx \Big) dy \\
&\qquad = \int_{(\alpha,0)+Q} |\nabla u(x)|^2 \Big(\int_{(0,\beta)+Q} J_\varepsilon(x-y)\ dy \Big) dx \\
&\qquad \leq \int_{(\alpha,0)+Q} |\nabla u(x)|^2 dx \int_{(\alpha,-\beta)+Q_2} J_\varepsilon(\zeta)\ d\zeta \\
&\qquad = \int_{Q} |\nabla u(x)|^2 dx \int_{(\alpha,-\beta)+Q_2} J_\varepsilon(\zeta)\ d\zeta\ ,
\end{aligned}$$

where in the last equality we used the periodicity of $u \in X$. Hence

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}^{n-1},\ |\alpha|_\infty\geq 2} \sum_{\beta\in\mathbb{Z}} \varepsilon \int_{(0,\beta)+Q} \Big(\int_{(\alpha,0)+Q} J_\varepsilon(x-y) |\nabla u(x)|^2 dx \Big) dy \\
&\qquad \leq \varepsilon \int_{Q} |\nabla u(x)|^2 dx \sum_{\alpha\in\mathbb{Z}^{n-1},\ |\alpha|_\infty\geq 2} \sum_{\beta\in\mathbb{Z}} \int_{(\alpha,-\beta)+Q_2} J_\varepsilon(\zeta)\ d\zeta \\
&\qquad \leq 2^n \varepsilon \int_{Q} |\nabla u(x)|^2 dx \int_{\hat{Q}_2} J_\varepsilon(\zeta)\ d\zeta\ .
\end{aligned}$$

In the last inequality we used the fact that each point of $\hat{Q}_2$ belongs to at most 2^n cubes of the form $(\alpha, -\beta) + Q_2$ for $\alpha \in \mathbb{Z}^{n-1}$, with $|\alpha|_\infty \geq 2$, and $\beta \in \mathbb{Z}$. After the change of variables $z = \zeta/\varepsilon$ we obtain (see (5.13))

$$\int_{\hat{Q}_2} J_\varepsilon(\zeta)\ d\zeta \leq \int_{\mathbb{R}^n \setminus B_{1/\varepsilon}(0)} J(z)\ dz \leq \omega_1(\varepsilon)\ .$$

Combining the last five inequalities and using the periodicity of u, from (6.12) we obtain

$$\begin{aligned}
&\varepsilon \int_{V\setminus Q} \Big(\int_{S_{1-\delta}} J_\varepsilon(x-y) |\nabla u(x)|^2 dx \Big) dy \qquad (6.13) \\
&\qquad \leq \Big(\omega_1\Big(\frac{\varepsilon}{\delta}\Big) + 2^n \omega_1(\varepsilon) \Big) \varepsilon \int_{S\cap Q_3} |\nabla u(x)|^2 dx \\
&\qquad = 3^{n-1} \Big(\omega_1\Big(\frac{\varepsilon}{\delta}\Big) + 2^n \omega_1(\varepsilon) \Big) \varepsilon \int_{Q} |\nabla u(x)|^2 dx\ .
\end{aligned}$$

Finally, to estimate the last term on the right-hand side of (6.10), we use the inclusion

$$\begin{aligned}
&Q \times (S \setminus Q) \subset \big(Q \times (S \setminus Q_3)\big) \cup \big(Q_{1-\delta} \times (S \cap (Q_3 \setminus Q_1)\big) \\
&\quad \cup \big((Q_1 \setminus Q_{1-\delta}) \times (Q_{1+\delta} \setminus Q_1)\big) \cup \big((Q_1 \setminus Q_{1-\delta}) \times (S \cap (Q_3 \setminus Q_{1+\delta}))\big)
\end{aligned}$$

and we write

$$
\begin{aligned}
\varepsilon\int_Q\Big(\int_{S\setminus Q}&J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2dx\Big)dy\\
&\le \varepsilon\int_Q\Big(\int_{S\setminus Q_3}J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2dx\Big)dy\\
&\quad+\varepsilon\int_{Q_{1-\delta}}\Big(\int_{S\cap(Q_3\setminus Q_1)}J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2dx\Big)dy \qquad (6.14)\\
&\quad+\varepsilon\int_{Q_1\setminus Q_{1-\delta}}\Big(\int_{Q_{1+\delta}\setminus Q_1}J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2dx\Big)dy\\
&\quad+\varepsilon\int_{Q_1\setminus Q_{1-\delta}}\Big(\int_{S\cap(Q_3\setminus Q_{1+\delta})}J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2dx\Big)dy\ .
\end{aligned}
$$

By Lemma 5.4,

$$
\begin{aligned}
\varepsilon\int_{Q_{1-\delta}}\Big(\int_{S\cap(Q_3\setminus Q_1)}&J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2dx\Big)dy\\
&+\varepsilon\int_{Q_1\setminus Q_{1-\delta}}\Big(\int_{S\cap(Q_3\setminus Q_{1+\delta})}J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2dx\Big)dy \qquad (6.15)\\
&\le 2\varepsilon\omega_1\Big(\frac{\varepsilon}{\delta}\Big)\int_{S\cap Q_3}|\nabla u(x)|^2dx = 2{\cdot}3^{n-1}\varepsilon\omega_1\Big(\frac{\varepsilon}{\delta}\Big)\int_Q|\nabla u(x)|^2dx\ ,
\end{aligned}
$$

where in the last equality we used the periodicity of u. On the other hand, by Lemma 6.2

$$
\varepsilon\int_{Q_1\setminus Q_{1-\delta}}\Big(\int_{(Q_{1+\delta}\setminus Q_1)}J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2dx\Big)dy\le\kappa_2\delta\ . \qquad (6.16)
$$

It remains to study the first term on the right-hand side of (6.14). We have

$$
\begin{aligned}
\varepsilon\int_Q\Big(\int_{S\setminus Q_3}&J_\varepsilon(x-y)|\nabla u(x)-\nabla u(y)|^2dx\Big)dy\\
&\le 2\varepsilon\int_Q\Big(\int_{S\setminus Q_3}J_\varepsilon(x-y)|\nabla u(x)|^2dx\Big)dy \qquad (6.17)\\
&\quad+2\varepsilon\int_Q\Big(\int_{S\setminus Q_3}J_\varepsilon(x-y)\ dx\Big)|\nabla u(y)|^2dy\ .
\end{aligned}
$$

To estimate the first term on the right-hand side of (6.17) we write

$$
\begin{aligned}
2\varepsilon\int_Q\Big(\int_{S\setminus Q_3}&J_\varepsilon(x-y)|\nabla u(x)|^2dx\Big)dy\\
&=2\varepsilon\sum_{\alpha\in\mathbb{Z}^n\cap(S\setminus Q_3)}\int_Q\Big(\int_{\alpha+Q}J_\varepsilon(x-y)|\nabla u(x)|^2dx\Big)dy\ .
\end{aligned}
$$

By Fubini's theorem and the change of variables $\zeta = x - y$, we get

$$\begin{aligned}\int_Q \Big(\int_{\alpha+Q} J_\varepsilon(x-y)|\nabla u(x)|^2 dx \Big) dy &= \int_{\alpha+Q} \Big(\int_Q J_\varepsilon(x-y)\ dy \Big) |\nabla u(x)|^2 dx \\ \leq \int_{\alpha+Q} \Big(\int_{x-Q} J_\varepsilon(\zeta)\ d\zeta \Big) |\nabla u(x)|^2 dx &\leq \int_Q |\nabla u(x)|^2 dx \int_{\alpha - Q_2} J_\varepsilon(\zeta)\ d\zeta\ ,\end{aligned}$$

where in the last inequality we have used the periodicity of u and the inclusion $x - Q \subset \alpha - Q_2$ for $x \in \alpha + Q$. Hence,

$$\begin{aligned}2\varepsilon \sum_{\alpha \in \mathbb{Z}^n \cap (S \setminus Q_3)} &\int_Q \Big(\int_{\alpha+Q} J_\varepsilon(x-y)|\nabla u(x)|^2 dx \Big) dy \\ &\leq 2\varepsilon \int_Q |\nabla u(x)|^2 dx \sum_{\alpha \in \mathbb{Z}^n \cap (S \setminus Q_3)} \int_{\alpha - Q_2} J_\varepsilon(\zeta)\ d\zeta \\ &\leq 2^n \varepsilon \int_Q |\nabla u(x)|^2 dx \int_{\hat{Q}_2} J_\varepsilon(\zeta)\ d\zeta\ ,\end{aligned}$$

where in the last inequality we used the fact that each point of $\hat{Q}_2$ belongs to at most 2^{n-1} cubes of the form $\alpha - Q_2$ for $\alpha \in \mathbb{Z}^n \cap (S \setminus Q_3)$. After the change of variables $z = \zeta/\varepsilon$, we obtain

$$2\varepsilon \int_Q \Big(\int_{S \setminus Q_3} J_\varepsilon(x-y)|\nabla u(x)|^2 dx \Big) dy \leq 2^n \varepsilon \int_Q |\nabla u(x)|^2 dx \int_{\mathbb{R}^n \setminus B_{1/\varepsilon}(0)} J(z)|z|\ dz\ . \quad (6.18)$$

We now estimate the second term on the right-hand side of (6.17). With the change of variables $z = (x-y)/\varepsilon$, we have

$$2\varepsilon \int_Q \Big(\int_{S \setminus Q_3} J_\varepsilon(x-y)\ dx \Big) |\nabla u(y)|^2 dy \leq 2\varepsilon \int_{\mathbb{R}^n \setminus B_{1/\varepsilon}(0)} J(z)|z|\ dz \int_Q |\nabla u(y)|^2 dy\ . \quad (6.19)$$

Combining the inequalities (6.17)–(6.19), we obtain

$$2\varepsilon \int_Q \Big(\int_{S \setminus Q_3} J_\varepsilon(x-y)|\nabla u(x)|^2 dx \Big) dy \leq 2^n \varepsilon \omega_1(\varepsilon) \int_Q |\nabla u(x)|^2 dx\ . \quad (6.20)$$

The conclusion follows from (6.11), (6.13), (6.14), (6.15), (6.16), and (6.20). ■

Proof of Theorem 6.1. By Theorem 1.1 we deduce that $u \in BV(\Omega; \{-1, 1\})$. Let μ_j be the nonnegative Radon measure on Ω defined by

$$\mu_j(B) := \frac{1}{\varepsilon} \int_B W(u_j(x))\, dx + \varepsilon \int_B \int_\Omega J_\varepsilon(x-y)|\nabla u_j(x) - \nabla u_j(y)|^2 dx dy \quad (6.21)$$

for every Borel set $B \subset \Omega$. Since $\mu_j(\Omega) = \mathcal{F}_{\varepsilon_j}(u_j, \Omega)$, by (6.1) $\mu_j(\Omega)$ is bounded uniformly with respect to j. Extracting a subsequence (not relabeled), we may assume that the liminf in (6.2) is a limit and that $\mu_j \overset{*}{\rightharpoonup} \mu$ weakly* in the space $\mathcal{M}_b(\Omega)$ of bounded Radon measures on Ω, considered, as usual, as the dual of the space $C_0(\Omega)$ of continuous functions on $\overline{\Omega}$ vanishing on $\partial\Omega$. Let g be the density of the absolutely continuous part of μ with respect to $\mathcal{H}^{n-1}$ restricted to S_u. Then the inequality (6.2) will follow from

$$g(x_0) \geq \psi(\nu_u(x_0)) \text{ for } \mathcal{H}^{n-1} \text{ a.e. } x_0 \in S_u \; . \tag{6.22}$$

To prove this inequality, fix $x_0 \in S_u$ such that, setting $\nu := \nu_u(x_0)$, we have

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n} \int_{Q^\nu_\rho} |u(x+x_0) - w^\nu(x+x_0)| \, dx = 0 \; , \tag{6.23}$$

$$g(x_0) = \lim_{\rho\to 0^+} \frac{\mu(x_0 + \overline{Q^\nu_\rho})}{\rho^{n-1}} < +\infty \; . \tag{6.24}$$

It is well-known (see [21, Theorem 3 in Section 5.9]) that (6.23) and (6.24) hold for $\mathcal{H}^{n-1}$ a.e. $x_0 \in S_u$. Since $\mu_j \overset{*}{\rightharpoonup} \mu$ weakly* in $\mathcal{M}_b(\Omega)$, by (2.15) and (6.21), using a change of variables, we get

$$\begin{aligned} g(x_0) = \lim_{\rho\to 0^+} \frac{\mu(x_0 + \overline{Q^\nu_\rho})}{\rho^{n-1}} &\geq \limsup_{\rho\to 0^+} \limsup_{j\to+\infty} \frac{\mu_j(x_0 + Q^\nu_\rho)}{\rho^{n-1}} \\ &\geq \limsup_{\rho\to 0^+} \limsup_{j\to+\infty} \frac{\mathcal{F}_{\varepsilon_j}(u_j, x_0 + Q^\nu_\rho)}{\rho^{n-1}} = \limsup_{\rho\to 0^+} \limsup_{j\to+\infty} \mathcal{F}_{\eta_{j,\rho}}(v_{j,\rho}, Q^\nu_1) \; , \end{aligned}$$

where $\eta_{j,\rho} := \varepsilon_j/\rho$ and $v_{j,\rho}(y) := u_j(x_0 + \rho y)$. On the other hand, since $u_j \to u$ in $L^2(\Omega)$, by (6.23) we obtain

$$\begin{aligned} 0 &= \lim_{\rho\to 0^+} \lim_{j\to+\infty} \frac{1}{\rho^n} \int_{Q^\nu_\rho} |u_j(x+x_0) - w^\nu(x+x_0)| \, dx \\ &= \lim_{\rho\to 0^+} \lim_{j\to+\infty} \int_{Q^\nu_1} |v_{j,\rho}(x) - w^\nu(x)| \, dx \; . \end{aligned}$$

Since for every $\rho > 0$

$$\lim_{j\to+\infty} \eta_{j,\rho} = 0 \; ,$$

by a diagonal argument we can choose $\rho_j \to 0^+$ such that, setting $\eta_j := \eta_{j,\rho_j}$ and $v_j := v_{j,\rho_j}$, we have $\eta_j \to 0^+$, $v_j \to w^\nu$ in $L^1(Q^\nu_1)$, and

$$g(x_0) \geq \limsup_{j\to+\infty} \mathcal{F}_{\eta_j}(v_j, Q^\nu_1) \; . \tag{6.25}$$

The finiteness of $g(x_0)$ and Theorem 1.1 yield that $v_j \to w^\nu$ in $L^2(Q^\nu_1)$. We can now apply the modification Theorem 5.1: there exists $\delta_\nu > 0$ such that for every $0 < \delta < \delta_\nu$ we obtain a sequence $\{w_j\} \subset W^{1,2}_{\mathrm{loc}}(Q^\nu_1) \cap L^2(Q^\nu_1)$ with $w_j \to w^\nu$ in $L^2(Q^\nu_1)$, $w_j = w^\nu * \theta_{\varepsilon_j}$ in $Q^\nu_1 \setminus Q^\nu_{1-\delta}$, and

$$\limsup_{j\to+\infty} \mathcal{F}_{\eta_j}(v_j, Q^\nu_1) \geq \limsup_{j\to+\infty} \mathcal{F}_{\eta_j}(w_j, Q^\nu_1) - \kappa_1 \delta \; , \tag{6.26}$$

where, we recall, the constant κ_1 is independent of δ. Extend w_j to $\mathbb{R}^n$ in such a way that $w_j(x) = \pm 1$ for $\pm x \cdot \nu \geq \frac{1}{2}$ and $w(x + \nu_i) = w(x)$ for all $x \in \mathbb{R}^n$ and for all $i = 1, \ldots, n-1$, where ν_i are the vectors in (1.11). Then $w_j \in X^\nu$ and so we can apply Lemma 6.3 to obtain

$$\limsup_{j\to+\infty} \mathcal{F}_{\eta_j}(w_j, Q_1^\nu) \geq \limsup_{j\to+\infty} (\mathcal{W}_{\eta_j}(w_j, Q_1^\nu) + \mathcal{J}_{\eta_j}(w_j, V^\nu, \mathbb{R}^n)) + \\ - \kappa_2 \delta - \limsup_{j\to+\infty} \Big(\kappa_3 \omega_1\Big(\frac{\eta_j}{\delta}\Big) + \kappa_4 \omega_1(\eta_j)\Big) \eta_j \int_{Q_1^\nu} |\nabla w_j(x)|^2 dx \ , \tag{6.27}$$

where we recall that $\mathcal{W}_{\eta_j}$ is defined in (2.13). By (1.13),

$$\mathcal{W}_{\eta_j}(w_j, Q_1^\nu) + \mathcal{J}_{\eta_j}(w_j, V^\nu, \mathbb{R}^n) \geq \psi(\nu) \tag{6.28}$$

for every j with $\eta_j < 1$. By (6.25) and (6.25) the finiteness of $g(x_0)$ implies that $\mathcal{F}_{\eta_j}(w_j, Q_1^\nu)$ is bounded uniformly with respect to j. Therefore Lemma 4.3, together with the periodicity of w_j, proves that the same property holds for $\eta_j \int_{Q_1^\nu} |\nabla w_j(x)|^2 dx$. Together with (5.13), (6.25), (6.26), (6.27), and (6.28), this shows that $g(x_0) \geq \psi(\nu) - \kappa_1 \delta - \kappa_2 \delta$ for every $0 < \delta < \delta_\nu$. Taking the limit as $\delta \to 0^+$ we obtain (6.22). This concludes the proof of the theorem. ■

7 Gamma Limsup Inequality

In this section we prove the Γ-limsup inequality. Fix $\varepsilon_j \to 0^+$. For every $u \in BV(\Omega; \{-1, 1\})$ we define

$$\mathcal{F}''(u, \Omega) := \inf\big\{\limsup_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(u_j, \Omega) : \ u_j \to u \text{ in } L^2(\Omega)\big\} \ . \tag{7.1}$$

Theorem 7.1 (Γ-Limsup) *For every $u \in BV(\Omega; \{-1, 1\})$ we have*

$$\mathcal{F}''(u, \Omega) \leq \int_{S_u} \psi(\nu_u) \ d\mathcal{H}^{n-1} \ . \tag{7.2}$$

To prove the Γ-limsup inequality we need the results proved in the following lemmas.

Lemma 7.2 *Let $u \in BV_{\mathrm{loc}}(\mathbb{R}^n; \{-1, 1\})$ and, for every $\varepsilon > 0$, let $\tilde{u}_\varepsilon$ be as in (6.4). Assume that there exists a bounded polyhedral set Σ of dimension $n-1$ such that $S_u = \Sigma$, let Σ^{n-2} the union of all its $n-2$ dimensional faces, and let $(\Sigma^{n-2})^\delta$ be defined as in (3.1). Then there exists $\delta_\Sigma > 0$ such that for $0 < \varepsilon < \delta < \delta_\Sigma$ we have*

$$\mathcal{J}_\varepsilon(\tilde{u}_\varepsilon, (\Sigma^{n-2})^\delta) \leq c_1 \delta \mathcal{H}^{n-2}(\Sigma^{n-2})$$

for some constant $c_1 > 0$ independent of ε, δ, and Σ.

Proof. It is enough to repeat the proof of Lemma 6.2 with C^σ_δ and $\hat{C}^\sigma_\delta$ replaced by $\{x \in (\Sigma^{n-2})^\delta : \ \mathrm{dist}(x,\Sigma) < \varepsilon\}$ and $\{x \in (\Sigma^{n-2})^\delta : \ \mathrm{dist}(x,\Sigma) \geq \varepsilon\}$. ■

Lemma 7.3 *Let P be a bounded polyhedron of dimension $n-1$ containing 0 with normal ν, let $\rho > 0$, and let P_ρ be the n-dimensional prism defined in (5.7). Then for every $\eta > 0$ there exists a sequence $\{u_\varepsilon\} \subset W^{1,2}(P_\rho)$ such that $u_\varepsilon \to w^\nu$ in $L^2(P_\rho)$ and*

$$\limsup_{\varepsilon \to 0^+} \big(\mathcal{W}_\varepsilon(u_\varepsilon, P_\rho) + \mathcal{J}_\varepsilon(u_\varepsilon, P_\rho, \mathbb{R}^n)\big) \leq (\psi(\nu) + \eta)\mathcal{H}^{n-1}(P) \ .$$

Proof. Without loss of generality, we assume that $\nu = e_n$. For simplicity, we omit the superscript ν in the notation for w^ν, X^ν, V^ν, Q^ν_1, and the subscript ρ when $\rho = 1$. By the definition of ψ (see (1.13)), given $\eta > 0$ there exist $\varepsilon_* \in (0,1)$ and $u_* \in X$ such that

$$\mathcal{W}_{\varepsilon_*}(u_*, Q) + \mathcal{J}_{\varepsilon_*}(u_*, V, \mathbb{R}^n) \leq \psi(e_n) + \eta \ . \tag{7.3}$$

Define $u_\varepsilon(x) := u_*(\frac{\varepsilon_*}{\varepsilon}x)$ for $x \in \mathbb{R}^n$. Since $u_*(x) = \pm 1$ for $\pm x_n \geq 1/2$, the sequence $\{u_\varepsilon\}$ converges to w in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

To estimate $\mathcal{W}_\varepsilon(u_\varepsilon, P_\rho)$ and $\mathcal{J}_\varepsilon(u_\varepsilon, P_\rho, \mathbb{R}^n)$, we consider the $(n-1)$-dimensional cube $Q^{(n-1)} := Q \cap \{x_n = 0\}$ and we set

$$Z_\varepsilon := \Big\{\{\alpha \in \mathbb{Z}^n : \ \alpha_n = 0 \ , \ (\alpha + Q^{(n-1)}) \cap \Big(\frac{\varepsilon_*}{\varepsilon}P\Big) \neq \emptyset\Big\} \ .$$

Observe that

$$\Big(\frac{\varepsilon}{\varepsilon_*}\Big)^{n-1} \# Z_\varepsilon \to \mathcal{H}^{n-1}(P) \quad \text{as } \varepsilon \to 0^+ \ , \tag{7.4}$$

where $\# Z_\varepsilon$ is the number of elements of Z_ε.

Let $S := \{x \in \mathbb{R}^n : \ |x_n| < 1/2\}$. Since $u_*(x) = \pm 1$ for $\pm x_n \geq 1/2$, by (2.3) we have $W(u_*(x)) = 0$ for $x \in \mathbb{R}^n \setminus S$. Therefore a change of variables and the periodicity of u_* give

$$\begin{aligned} \mathcal{W}_\varepsilon(u_\varepsilon, P_\rho) &= \Big(\frac{\varepsilon}{\varepsilon_*}\Big)^{n-1} \mathcal{W}_{\varepsilon_*}\Big(u_*, \frac{\varepsilon_*}{\varepsilon}P_\rho\Big) = \Big(\frac{\varepsilon}{\varepsilon_*}\Big)^{n-1} \mathcal{W}_{\varepsilon_*}\Big(u_*, \Big(\frac{\varepsilon_*}{\varepsilon}P_\rho\Big) \cap S\Big) \\ &\leq \Big(\frac{\varepsilon}{\varepsilon_*}\Big)^{n-1} \sum_{\alpha \in Z_\varepsilon} \mathcal{W}_{\varepsilon_*}(u_*, \alpha + Q) = \Big(\frac{\varepsilon}{\varepsilon_*}\Big)^{n-1} \# Z_\varepsilon \mathcal{W}_{\varepsilon_*}(u_*, Q) \ . \end{aligned} \tag{7.5}$$

Similarly,

$$\begin{aligned} \mathcal{J}_\varepsilon(u_\varepsilon, P_\rho, \mathbb{R}^n) &= \Big(\frac{\varepsilon}{\varepsilon_*}\Big)^{n-1} \mathcal{J}_{\varepsilon_*}\Big(u_*, \frac{\varepsilon_*}{\varepsilon}P_\rho, \mathbb{R}^n\Big) \\ &\leq \Big(\frac{\varepsilon}{\varepsilon_*}\Big)^{n-1} \sum_{\alpha \in Z_\varepsilon} \mathcal{J}_{\varepsilon_*}(u_*, \alpha + V, \mathbb{R}^n) = \Big(\frac{\varepsilon}{\varepsilon_*}\Big)^{n-1} \# Z_\varepsilon \mathcal{J}_{\varepsilon_*}(u_*, V, \mathbb{R}^n) \ . \end{aligned} \tag{7.6}$$

The result now follows from (7.3)–(7.6). ■

Lemma 7.4 *Let $u \in BV_{\mathrm{loc}}(\mathbb{R}^n; \{-1,1\})$. Assume that there exists a bounded polyhedral set Σ of dimension $n-1$ such that $S_u = \Sigma$. For every $\rho > 0$ let $\Sigma_\rho := \{x \in \mathbb{R}^n : \ \mathrm{dist}(x, \Sigma) < \rho/2\}$. Then for every $\sigma > 0$ there exist $\rho > 0$ and $\delta \in (0, \rho)$ with the following property: for every $\varepsilon_j \to 0^+$ there exists $v_j \in W^{1,2}(\Sigma_\rho)$ such that $v_j = u$ on $\Sigma_\rho \setminus \Sigma_{\rho-\delta}$ and*

$$\limsup_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(v_j, \Sigma_\rho) \le \int_\Sigma \psi(\nu_u) \, d\mathcal{H}^{n-1} + \sigma \ .$$

Proof. Let $\delta_\Sigma > 0$ be as in Lemma 7.2. Fix σ and $\hat\sigma$ with $\hat\sigma \in (0, \min\{\sigma, \delta_\Sigma\})$. There exist $\rho \in (0, \hat\sigma)$ and a finite number of bounded polyhedra $P^1, \ldots, P^k$ of dimension $n-1$ and contained in the $n-1$ dimensional faces of Σ such that $\overline{P}^i_\rho \cap \overline{P}^j_\rho = \emptyset$ for $i \neq j$ and

$$\Sigma_\rho \setminus \bigcup_{i=1}^k P^i_\rho \subset (\Sigma^{n-2})^{\hat\sigma}, \tag{7.7}$$

where P^i_ρ and $(\Sigma^{n-2})^{\hat\sigma}$ are defined as in (5.7) and Lemma 7.2, respectively. Find $R^1, \ldots, R^k$, bounded polyhedra of dimension $n-1$ contained in the $n-1$ dimensional faces of Σ, such that $P^i \Subset R^i$ and $\overline{R}^i_\rho \cap \overline{R}^j_\rho = \emptyset$ for $i \neq j$.

Fix $\eta > 0$ such that $\eta \mathcal{H}^{n-1}(\Sigma) < \sigma/2$. By Lemma 7.3 for every $i = 1, \ldots, k$, there exists a sequence $\{u^i_j\} \subset W^{1,2}(R^i_\rho)$ such that $u^i_j \to u$ in $L^2(R^i_\rho)$, and

$$\limsup_{j \to +\infty} \big(\mathcal{W}_{\varepsilon_j}(u^i_j, R^i_\rho) + \mathcal{J}_{\varepsilon_j}(u^i_j, R^i_\rho, \mathbb{R}^n)\big) \le (\psi(\nu^i) + \eta)\mathcal{H}^{n-1}(R^i) \ . \tag{7.8}$$

By Theorem 5.1 there exist $\delta \in (0, \min\{\hat\sigma, \rho/2\})$ and $\{v^i_j\} \subset W^{1,2}(R^i_\rho)$ such that $v^i_j \to u$ in $L^2(R^i_\rho)$ as $j \to +\infty$, $v^i_j = u * \theta_{\varepsilon_j}$ on $R^i_\rho \setminus (R^i_\rho)_\delta$, and

$$\begin{aligned} \limsup_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(v^i_j, R^i_\rho) &\le \limsup_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(u^i_j, R^i_\rho) + \kappa_1 \delta \\ &\le (\psi(\nu^i) + \eta)\mathcal{H}^{n-1}(R^i) + \kappa_1 \hat\sigma \ , \end{aligned} \tag{7.9}$$

where, we recall, the costant $\kappa_1 > 0$ is independent of j, $\hat\sigma$, and R^i_ρ. Define $v_j := v^i_j$ on R^i_ρ and $v_j := u * \theta_{\varepsilon_j}$ on $A_\rho := \Sigma_\rho \setminus \bigcup_{i=1}^k R^i_\rho$. Then $v_j \in W^{1,2}(\Sigma_\rho)$ and $v_j \to u$ in $L^2(\Sigma_\rho)$. Moreover $v_j = u$ on $\Sigma_\rho \setminus \Sigma_{\rho-\delta}$ for all j sufficiently large.

By additivity we obtain

$$\mathcal{W}_{\varepsilon_j}(v_j, \Sigma_\rho) \le \sum_{i=1}^k \mathcal{W}_{\varepsilon_j}(v_j, R^i_\rho) + \mathcal{W}_{\varepsilon_j}(v_j, A_\rho) \ . \tag{7.10}$$

Since $(u * \theta_{\varepsilon_j})(x) = \pm 1$ for $x \notin \Sigma_{2\varepsilon_j}$ and $-1 \le (u * \theta_{\varepsilon_j})(x) \le 1$, by (2.3) and (7.7) we have

$$\begin{aligned} \mathcal{W}_{\varepsilon_j}(v_j, A_\rho) &\le \mathcal{W}_{\varepsilon_j}(u * \theta_{\varepsilon_j}, (\Sigma^{n-2})^{\hat\sigma} \cap \Sigma_{2\varepsilon_j}) \\ &\le \frac{1}{\varepsilon_j} M_W \mathcal{L}^n((\Sigma^{n-2})^{\hat\sigma} \cap \Sigma_{2\varepsilon_j}) \le M_W c_\Sigma \hat\sigma \mathcal{H}^{n-2}(\Sigma^{n-2}) \ , \end{aligned}$$

where M_W is the constant in (5.51) and $c_\Sigma > 0$ is a constant depending only on the geometry of Σ. The previous inequality together with (7.10) gives

$$\mathcal{W}_{\varepsilon_j}(v_j, \Sigma_\rho) \leq \sum_{i=1}^{k} \mathcal{W}_{\varepsilon_j}(v_j, R_\rho^i) + M_W c_\Sigma \hat{\sigma} \mathcal{H}^{n-2}(\Sigma^{n-2}) \ . \tag{7.11}$$

To estimate $\mathcal{J}_{\varepsilon_j}(v_j, \Sigma_\rho)$ we use the inclusion

$$\begin{aligned}\Sigma_\rho \times \Sigma_\rho \subset \bigcup_{i=1}^{k} (R_\rho^i \times R_\rho^i) \cup \bigcup_{i=1}^{k} (P_\rho^i \times (\Sigma_\rho \setminus R_\rho^i)) \cup \bigcup_{i=1}^{k} ((\Sigma_\rho \setminus R_\rho^i) \times P_\rho^i) \\ \cup \left(\Big(\Sigma_\rho \setminus \bigcup_{i=1}^{k} P_\rho^i\Big) \times \Big(\Sigma_\rho \setminus \bigcup_{i=1}^{k} P_\rho^i\Big) \right) \cup \bigcup_{i \neq j} (R_\rho^i \times R_\rho^j) \ ,\end{aligned}$$

which, together with (7.7), gives

$$\begin{aligned}\mathcal{J}_{\varepsilon_j}(v_j, \Sigma_\rho) \leq \sum_{i=1}^{k} \mathcal{J}_{\varepsilon_j}(v_j, R_\rho^i) + \sum_{i=1}^{k} \mathcal{J}_{\varepsilon_j}(v_j, P_\rho^i, \Sigma_\rho \setminus R_\rho^i) \\ + \sum_{i=1}^{k} \mathcal{J}_{\varepsilon_j}(v_j, \Sigma_\rho \setminus R_\rho^i, P_\rho^i) + \mathcal{J}_{\varepsilon_j}(v_j, (\Sigma^{n-2})^{\hat{\sigma}}) + \sum_{i \neq j} \mathcal{J}_{\varepsilon_j}(v_j, R_\rho^i, R_\rho^j) \ .\end{aligned} \tag{7.12}$$

By Lemma 4.3 and (7.9) the sequence $\{\varepsilon_j \int_{R_\rho^i} |\nabla v_j^i|^2 dx\}$ is uniformly bounded with respect to j. Taking into account (5.5) and (5.6) we see that the same property holds for $\{\varepsilon_j \int_{\Sigma_\rho} |\nabla v_j|^2 dx\}$. Hence, by Lemma 5.4, the second, third, and fifth terms on the right-hand side of (7.12) tend to zero as $j \to +\infty$. By Lemma 7.2,

$$\mathcal{J}_{\varepsilon_j}(v_j, (\Sigma^{n-2})^{\hat{\sigma}}) \leq c_1 \hat{\sigma} \mathcal{H}^{n-2}(\Sigma^{n-2}) \ . \tag{7.13}$$

Combining (7.9), (7.11), (7.12), and (7.13) we get

$$\begin{aligned}\limsup_{j \to +\infty} \mathcal{F}_{\varepsilon_j}(v_j, \Sigma_\rho) \leq \int_\Sigma \psi(\nu_u) \ d\mathcal{H}^{n-1} + \eta \mathcal{H}^{n-1}(\Sigma) \\ + \kappa_1 \hat{\sigma} + M_W c_\Sigma \hat{\sigma} \mathcal{H}^{n-2}(\Sigma^{n-2}) + c_1 \hat{\sigma} \mathcal{H}^{n-2}(\Sigma^{n-2}) \ .\end{aligned}$$

Since $\eta \mathcal{H}^{n-1}(\Sigma) < \sigma/2$, the conclusion can be obtained by taking $\hat{\sigma}$ sufficiently small. ■

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. By [8, Lemma 3.1] for every $u \in BV(\Omega; \{-1, 1\})$ there exists a sequence $\{z_k\}$ in $BV(\Omega; \{-1, 1\})$ converging to u in $L^2(\Omega)$ such that S_{z_k} is given by the intersection with Ω with a bounded polyhedral set Σ_k of dimension $n-1$ and $\mathcal{H}^{n-1}(S_{z_k}) \to \mathcal{H}^{n-1}(S_u)$. By Reshetnyak's convergence theorem (see, e.g., [42]) this implies that

$$\lim_{k \to +\infty} \int_{S_{z_k}} \psi(\nu_{z_k}) \ d\mathcal{H}^{n-1} = \int_{S_u} \psi(\nu_u) \ d\mathcal{H}^{n-1} \ .$$

Hence, using the lower semicontinuity of $\mathcal{F}^{''}(\cdot,\Omega)$ with respect to convergence in $L^2(\Omega)$ it suffices to prove (7.2) for $u \in BV(\Omega;\{-1,1\})$ such that $S_u = \Omega\cap\Sigma$ with Σ a bounded polyhedral set of dimension $n-1$.

In this case, for every $\sigma > 0$ let $0 < \delta < \rho$ and $v_j \in W^{1,2}(\Sigma_\rho)$ be as in Lemma 7.4. Define $u_j := v_j$ on Σ_ρ and $u_j := u$ on $\Omega\setminus\Sigma_\rho$. The properties of v_j imply that $u_j := u$ on $\Omega\setminus\Sigma_{\rho-\delta}$ for all j sufficiently large. Hence, by (2.3) we have

$$\mathcal{W}_{\varepsilon_j}(u_j,\Omega) \le \mathcal{W}_{\varepsilon_j}(u_j,\Sigma_\rho)\ . \tag{7.14}$$

To estimate $\mathcal{J}_{\varepsilon_j}(u_j,\Omega)$ we consider the inclusion

$$\begin{aligned}\Omega\times\Omega \subset &(\Sigma_\rho\times\Sigma_\rho)\cup(\Sigma_{\rho-\delta}\times(\Omega\setminus\Sigma_\rho))\cup((\Omega\setminus\Sigma_\rho)\times\Sigma_{\rho-\delta})\\ &\cup((\Omega\setminus\Sigma_{\rho-\delta})\times(\Omega\setminus\Sigma_{\rho-\delta}))\ .\end{aligned} \tag{7.15}$$

Since $\nabla u_j = \nabla u = 0$ on $\Omega\setminus\Sigma_{\rho-\delta}$, in view of (7.15) we obtain

$$\mathcal{J}_{\varepsilon_j}(u_j,\Omega) \le \mathcal{J}_{\varepsilon_j}(u_j,\Sigma_\rho)+\mathcal{J}_{\varepsilon_j}(u_j,\Sigma_{\rho-\delta},\Omega\setminus\Sigma_\rho)+\mathcal{J}_{\varepsilon_j}(u_j,\Omega\setminus\Sigma_\rho,\Sigma_{\rho-\delta})\ . \tag{7.16}$$

By Lemmas 4.3 and 5.4 the last two terms tend to zero as $j\to\infty$, and by Lemma 7.4 we deduce

$$\limsup_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j,\Sigma_\rho) \le \int_\Sigma \psi(\nu_u)\ d\mathcal{H}^{n-1}+\sigma\ .$$

Together with (7.14) and (7.16) this shows that

$$\mathcal{F}^{''}(u,\Omega) \le \limsup_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j,\Omega) \le \int_\Sigma \psi(\nu_u)\ d\mathcal{H}^{n-1}+\sigma\ .$$

Letting σ tend to 0 we obtain (7.2). ■

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