Elasticity with Gradient-Disarrangements: a Multiscale Perspective for Strain-Gradient Theories of Elasticity and of Plasticity

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Abstract

Standard measures of local deformation such as deformation gradient, strain, elastic deformation, and plastic deformation are dimensionless. However, many macroscopic and submacroscopic geometrical changes observed in continuous bodies result in the formation of zones across whose boundaries significant changes in geometry can occur. In order to predict the sizes of such zones and their influence on material response, theories of elasticity and plasticity have been employed in which second gradients of deformation, gradients of strain, as well as gradients of elastic or of plastic deformation are taken into account. The theory of structured deformations provides additive decompositions of first deformation gradient and of second deformation gradient, valid for large deformations of any material, in which each term has a multiscale geometrical interpretation corresponding to the presence or absence of submacroscopic disarrangements (non-smooth geometrical changes such as slips and void formation). This article provides a field theory that broadens the earlier field theory, elasticity with disarrangements, by including energetic contributions from submacroscopic "gradient-disarrangements" (limits of averages of jumps in gradients of approximating deformations) and by treating particular kinematical conditions as internal constraints. An explicit formula is obtained showing the manner in which submacroscopic gradient-disarrangements determine a defectiveness density analogous to the dislocation density in theories of plasticity. A version of the new field theory incorporates this defectiveness density to obtain a counterpart of strain-gradient plasticity, while another instance of elasticity with gradient-disarrangements recovers an instance of strain-gradient elasticity with symmetric Cauchy stress. All versions of the new theory included here are compatible with the Second Law of Thermodynamics.

Keywords: strain-gradient elasticity; strain-gradient plasticity; elasticity with disarrangements; structured deformations; multiscale geometry; field theory

Contents

1 Introduction

The widely used continuum theories of elasticity and plasticity in their earliest formulations employ local geometrical measures of changes in shape that are dimensionless: deformation gradient, strain, elastic deformation, and plastic deformation. However, there are significant phenomena in continuous bodies at macroscopic or at submacroscopic levels in which any number of zones may form that carry significantly different geometrical changes from one zone to another, including strain-localization in metals, in polymers, and in granular materials [12] - [15] as well as fine mixtures of phases in a variety of solids

[16] - [19]. Moreover, some materials exhibit responses to changes in shape that depend upon the size of zones under consideration [20] - [23],[30].

When models of elasticity or plasticity employ dimensionless measures of changes in shape, there is no apparent length-scale available to capture the sizes of such deformation zones, and this absence of an inherent length-scale has led to the introduction of spatial gradients of the dimensionless local measures of changes in shape: second deformation gradient, strain-gradient, gradient of elastic deformation, and gradient of plastic deformation [24] - [28]. In the case of plasticity, the incorporation of the curl of plastic deformation as a measure of dislocation density provides not only a length-scale but also a connection at the continuum level to the submacroscopic presence of defects (see [29], Part XV, for a variety of approaches).

The goal in this paper is to approach the introduction of a length scale and its connections to submacroscopic geometrical changes through the use of the multiscale geometry of structured deformations and through a broadening of an associated field theory [1],[3]. That theory, elasticity with disarrangements [3], itself broadens classical, finite elasticity by incorporating into the energetic response both the contributions at the macrolevel of smooth, submacroscopic changes and the contributions of non-smooth submacroscopic changes (disarrangements) such as void formation and slips. Moreover, it allows for the dissipation of energy in smooth motions of the body and reduces to finite elasticity when restricted to motions that are free from disarrangements. Elasticity with disarrangements [3] rests on the theory of first-order structured deformations [1] that provides and justifies an additive decomposition $\nabla g = G + M$ of macroscopic deformation gradient ∇g into a part G without disarrangements and a part M due to disarrangements. This additive decomposition is available without restriction on the size of deformations or on the particular material undergoing the structured deformations. Moreover, this decomposition incorporates disarrangements in the form of submacroscopic jumps in approximating piecewise smooth deformations, but it does not incorporate disarrangements in the form of submacroscopic jumps in the gradients of approximating deformations. In that respect, the theory [3] resembles the theories of elasticity and plasticity in their earliest formulations, because the refined local measures of multiscale deformation G and M arising in the theory of (first-order) structured deformations are dimensionless.

The remedy used here for incorporating submacroscopic jumps in gradients of approximating deformations is to recast elasticity with disarrangements in the broader context of second-order structured deformations [11]. That theory goes beyond the first-order context by providing and justifying an additive decomposition $\nabla G = \mathbb{G} + \mathbb{M}$ of the gradient ∇G of the deformation without disarrangements into a part $\mathbb{G}$ without disarrangements and a part $\mathbb{M}$ due to gradient-disarrangements. As in the case of first-order structured deformations, this decomposition is valid for large deformations and for any material. However, both terms in this additive decomposition carry the dimension of reciprocal length and so provide a variety of possible additional fields with precise multiscale geometrical interpretations and with the desired dimension of length. An appropriate choice of one or more such fields then can be incorporated into the consititutive assumptions underlying elasticity with disarrangements to arrive eventually at an enriched version of that field theory.

Our main goal in this paper is to provide the detailed steps that lead from elasticity with disarrangements [3] to the enriched field theory "elasticity with gradient-disarrangements". In Section 2 we review the multiscale geometry of structured deformations and the ingredients required for the derivation of the field relations of elasticity with disarrangements [3]. The principal ingredient that permits one eventually to close the system of field equations is a tensorial consistency relation for stresses. The consistency relation follows from a unique, additive and universal decomposition of continuum fluxes that is induced by submacroscopic disarrangements [10]. In this manner, one obtains an additional tensorial relation valid for any continuous body undergoing structured deformations in the presence of a system of contact forces, without regard to the particular material comprising the body. One is then in a position to provide constitutive relations that relate both the free energy density and the stresses to the fields G and M whose sum is the macroscopic deformation gradient ∇g. Our discussion of elasticity with disarrangements in Section 2 not only reviews the derivation [3] of the field equations (49) - (53) but also includes a brief indication of specific applications to granular materials that arise when disarrangements as measured by the field M are "purely dissipative", i.e., do not contribute to the free energy of the body. We also provide for the first time in Subsection 2.4 a treatment of the accommodation inequality $\det G \leq \det \nabla g$ as an internal constraint. (Parts of this treatment are based on unpublished joint research with Luca Deseri during the period 2003-2006.) This internal constraint in the context of structured deformations is tantamount to the requirement that matter is impenetrable.

The desired broadening of elasticity with disarrangements is carried out in detail in Section 3. The main results from the theory of second-order structured deformations [11] are presented in Section 3.1 where the decomposition

$$\nabla G = \mathbb{G} + \mathbb{M} \tag{1}$$

and the interpretation of each term on the right-hand side is justified in the case of a general, second-order structured deformation $(g, G, \mathbb{G})$. Identification relations for the third-order tensor fields $\mathbb{G}$ and $\mathbb{M}$ justify our calling $\mathbb{G}$ the deformation without gradient-disarrangements and $\mathbb{M}$ the deformation due to gradient disarrangements. Subsection 3.2 provides a direct connection between the skew part $\tilde{\mathbb{M}}$ of the deformaion due to gradient disarrangements and the second-order tensor field $\operatorname{curl} G$, already known to be an area density of defects [1]. That analysis yields an identification relation for $\operatorname{curl} G$ in terms of the gradient-disarrangements associated with approximating injective and piecewise-smooth deformations.

Subsections 3.3 - 3.9 contain a derivation of the field relations for an elastic body undergoing gradient-disarrangements that parallels the corresponding derivation in Subsections 2.3 - 2.8. (The material on third-order tensors prerequisite for understanding the derivation is provided in the Appendix.) A central new element in the derivation is the assumption that the continuum under consideration is subject to both contact forces and contact moments. The presence of these two contact interactions in the formula for external power expended leads not only to the usual second-order tensor of stress but also to a third-order tensor of hyperstress $\mathbb{S}$ that expends power against changes in ∇G. The material in Subsection 3.3 provides a separate consistency relation for the hyperstresses with and without disarrangements, $\mathbb{S}_d$ and $\mathbb{S}_\backslash$, similar in form to the one treated in Subsection 2.3 that remains available for the stresses with and without disarrangements, S_d and $S_\backslash$. The consistency relation for the hyperstresses arises by applying the unique, additive and universal decomposition of continuum fluxes, discussed in Subsection 2.3, to the flux associated with the hyperstress. Subsection 3.4 contains a treatment of not only the internal constraint $\det G \leq \det \nabla g$ but also of the constraint "$\mathbb{G}$ is symmetric" in terms of reactive stresses and hyperstresses that arise in any material undergoing second-order structured deformations.

The additive decompositions of ∇g, of ∇G, of S, and of $\mathbb{S}$ imply an additive decomposition of the volume density of internal power into two groups of terms: one group of four terms accounts exclusively for power expended by forces proximate to the corresponding submacroscopic geometrical changes, and the other accounts exclusively for power expended by forces remote from the corresponding changes. These matters are treated in Subsection 3.6 and motivate the manner in which the internal dissipation is made explicit using the refined geometry available in the present theory.

The class of materials that is covered by the present field theory is specified in Subsection 3.7 by means of the free energy response function $(G, M, \mathbb{G}, \mathbb{M}) \longmapsto \Psi(G, M, \mathbb{G}, \mathbb{M})$. The response functions for constitutively determined parts of $S_\backslash$ and S_d are specified in terms of Ψ in terms of the single response function for the constitutively determined part of the stress S, itself. As a result, the consistency relation for stresses is satisfied as an identity and, *unlike* the case in Section 2 for elasticity with disarrangements, places no restriction on the geometrical fields g, G, and $\mathbb{G}$ that define a second-order structured deformation. In contrast, the response functions for the constitutively determined parts of $\mathbb{S}_d$ and $\mathbb{S}_\backslash$ are specified individually, again in terms of Ψ, so that the consistency relation for hyperstresses does restrict the fields g, G, and $\mathbb{G}$. As in Section 2, the choice of constitutive relations in Subsection 3.7 provides an explicit formula for the internal dissipation, and the requirement that the dissipation be nonnegative is imposed as a restriction on the structured motions experienced by the body.

The requirements of frame-indifference are treated in Subsection 3.8. The field relations (109) - (114) for elasticity with gradient-disarrangements are recorded and discussed in Subsection 3.9. They amount to 42 scalar equations and two inequalitites that restrict the unknown fields g, G, $\mathbb{G}$, and and a reactive hyperstress $\mathbb{R}_\backslash$ that together amount to 39 scalar unknowns. Three of the scalar equations arise from the requirement that the internal dissipation be frame-indifferent and, therefore, may be satisfied through appropriate choices of the response function Ψ.

The final two subsections of Section 3 treat two special cases that provide connections with strain-gradient theories of elasticity and of plasticity. In Subsection 3.10 the geometrical setting is restricted to coherently submacroscopically affine motions, i.e., second-order structured motions in which $M = 0$ and in which $\mathbb{G} = 0$. In this setting, there is only one unknown geometrical field, the macroscopic motion g itself. Its gradient $\nabla g = G$ provides the deformation without disarrangements, while its second gradient $\nabla^2 g = \nabla G = \mathbb{G} + \mathbb{M} = \mathbb{M}$ provides the deformation due to gradient-disarrangements. The geometrical restriction on M and $\mathbb{G}$ are treated as internal constriants, and the field equations reduce to a pair of partial differential equations (129) for g: the balance of linear momentum and the consistency relation. The latter amounts to the assertion that, given the deformation gradient $\nabla g(X)$ at a point X of the body, the second gradient $\nabla^2 g(X)$ must render the free energy stationary with respect to changes in the deformation due to gradient-disarrangements : $D_{\mathbb{M}}\Psi(\nabla g(X), 0, 0, \mathbb{M}) \mid_{\mathbb{M}=\nabla^2 g(X)} = 0$. This stationarity condition results from the consistency relation and the special geometrical setting of coherent submacroscopically affine motions. The setting of coherent, submacroscopically affine motions is appropriate for the description of fine mixtures of phases in which $\nabla g(X)$ provides the average contribution of smooth deformation from the preferred phases of the body, while $\nabla^2 g(X)$ provides the average contribution of jumps in gradients across phase boundaries. This instance of elasticity with gradient-disarrangments is compatible with symmetric Cauchy stress and with zero internal dissipation.

Subsection 3.11 treats the case of defect-dominant gradient energetics in which the free energy is assumed to depend upon G, M, and $\tilde{\mathbb{M}}$, the skew part of $\mathbb{M}$. The analysis in Subsection 3.2 shows that this special dependence on $\tilde{\mathbb{M}}$ amounts to a dependence of the energy upon G, $M = \nabla g - G$, and $\operatorname{curl} G$, the defectiveness density. In order to point out the connection of this case to theories of strain-gradient plasticity, we use the additive decomposition $\nabla g = G + M$ as in [1] to obtain the multiplicative decomposition $\nabla g = F_e F_p$, with $F_e = G$ and $F_p = I + G^{-1}M$. In this manner the field relations (109) - (114) reduce to a system of partial differential equations for the fields g and F_e in which both F_e and $\operatorname{curl} F_e$ appear explicitly. The consistency relation reduces to the partial differential equation $D_{\tilde{\mathbb{M}}}\tilde{\Psi}(\nabla g(X), F_e(X), 0, \tilde{\mathbb{M}}) \mid_{\tilde{\mathbb{M}}=\operatorname{curl} F_e(X)} = 0$ that amounts to a stationarity condition for the free energy with respect to defectiveness density. This instance of elasticity with gradient-disarrangements also is compatible with symmetric Cauchy stress, but the internal dissipation need not be zero, as was the case for the instance of coherent, submacroscopically affine motions that obey (129).

The examples and applications for elasticity with disarrangements briefly mentioned in Subsection 2.9 can set the stage for future research in the context of elasticity with gradient-disarrangements. In particular, the problems of determining the portfolio of disarrangement phases and of studying the propagation of moving interfaces described in Subsection 2.9 are meaningful and of equal interest when gradient-disarrangements play an explicit role. For the case of coherent, submacroscopically affine deformations in Subsection 3.10, it is of particluar interest for materials that exhibit fine phase-mixtures to study the propagation of waves, and, for the defect-dominant gradient energetics outlined in Subsection 3.11, the study of the evolution of interfaces separating slipped and unslipped phases of single crystals promises to bring the current development closer to main-stream treatments in crystal plasticity.

2 Elasticity with disarrangements: a comprehensive summary

2.1 First-order structured deformations and the additive decomposition of ∇g

In this subsection we select crucial elements of the geometry of structured deformations [1], [2] needed to describe elasticity with disarrangements as presented in [3]. Let $\mathcal{A}$ be a suitably regular subset of Euclidean space $\mathcal{E}$ and let $\mathcal{V}$ denote the translation space of $\mathcal{E}$. A structured deformation from $\mathcal{A}$ is a pair (g, G) of mappings $g : \mathcal{A} \longrightarrow \mathcal{E}$ and $G : \mathcal{A} \longrightarrow Lin\mathcal{V}$ such that g is smooth, injective, with smooth inverse, G is continuous, and the pair (g, G) of mappings satisfy the accommodation inequality

$$0 < c < \det G(X) \leq \det \nabla g(X), \tag{2}$$

for every $X \in \mathcal{A}$. We use the term "disarrangements" to describe non-smooth, submacroscopic geometrical changes such as microscopic slips and void formation. The additional field G is then called the *deformation without disarrangements*, a designation that is justified by means of the following approximation theorem:

Theorem 1 *[1] If (g, G) is a structured deformation from $\mathcal{A}$, then there exists a sequence $n \longmapsto f_n$ of injective, piecewise continuously differentiable mappings on $\mathcal{A}$ such that*

$$\lim_{n\to\infty} f_n = g \quad \textit{and} \quad \lim_{n\to\infty} \nabla f_n = G \tag{3}$$

with convergence in the sense of $L^\infty(\mathcal{A}, \mathcal{E})$.

Indeed, (3) shows that G as a limit of gradients is not influenced by any discontinuities associated with the piecewise smooth mappings f_n. In a complementary way, the tensor-valued mapping

$$M = \nabla g - G \tag{4}$$

is called the *deformation due to disarrangements*, based on the following result:

Theorem 2 *[2] If (g, G) is a structured deformation, then for each sequence $n \longmapsto f_n$ as in the Approximation Theorem and for each $X \in \mathcal{A}$ there holds*

$$M(X) = \lim_{r\to 0} \lim_{n\to\infty} \frac{\int_{J(f_n)\cap B(X,r)} [f_n](Y) \otimes \nu_{f_n}(Y) dA_Y}{|B(X,r)|} \tag{5}$$

where $J(f_n)$ is the jump set of f_n, $[f_n](Y)$ is its jump at the point Y, $\nu_{f_n}(Y)$ is the normal to $J(f_n)$ at Y, $B(X,r)$ is the sphere centered at X of radius r, and $|B(X,r)|$ is its volume.

In fact, (5) shows that only the non-smooth part of the approximation f_n affects the values of M, thereby justifying application of the attribute "deformation due to disarrangements" to the tensor field M. With these specific identifications of the fields G and M at hand, we may rewrite (4) as an additive decomposition of macroscopic deformation ∇g into a part G without disarrangements and a part M due to disarrangements:

$$\nabla g = G + M. \tag{6}$$

We emphasize that the additive decomposition (6) is

- purely kinematical, i.e., involves only geometrical notions,
- valid without restriction on the size of deformations
- universal, i.e., involves no restrictions as to the type of material under consideration.

The only assumption required in order to invoke (6) is that (g, G) be a structured deformation. It is useful in subsequent considerations to record the following formula for G that follows from the Approximation Theorem and the specific type of convergence guaranteed in the formula $(3)_2$:

$$G(X) = \lim_{r\to 0}\lim_{n\to\infty} \frac{\int_{B(X,r)} \nabla f_n(Y) dV_Y}{|B(X,r)|}. \tag{7}$$

Suppose now that (g, G) is a structured deformation from $\mathcal{A}$ and (g', G') is a structured deformation from $g(\mathcal{A})$. We define as in [1] the composition $(g', G') \diamond (g, G)$ to be the structured deformation from $\mathcal{A}$ given by

$$(g', G') \diamond (g, G) = (g' \circ g,\ (G' \circ g)G) \tag{8}$$

Here, $\circ$ denotes the usual composition of mappings. We are justified in calling this pair a structured deformation from $\mathcal{A}$ because the entry $g' \circ g$ IS smooth and has a smooth inverse, the entry $(G' \circ g)G$ is continuous, and because the accommodation inequality is satisfied for the pair $(g' \circ g,\ (G' \circ g)G)$:

$$0 < cc' < \det(G' \circ g) \det G = \det((G' \circ g)G) \leq \det((\nabla g') \circ g) \det \nabla g = \det(\nabla(g' \circ g)).$$

The composition $\diamond$ yields the following factorization of an arbitrary structured deformation from $\mathcal{A}$:

$$(g, G) = (g, \nabla g) \diamond (i_{\mathcal{A}}, K) \tag{9}$$

in which $i_{\mathcal{A}}$ Is the identity mapping, $i_{\mathcal{A}}(X) = X$ for all $X \in \mathcal{A}$, and

$$K = (\nabla g)^{-1} G. \tag{10}$$

The "classical" (structured) deformation $(g, \nabla g)$ on the right-hand side of (9) introduces no disarrangements, since the gradient of the first entry in the pair equals the second entry. Because $i_{\mathcal{A}}$ is the identity mapping, the structured deformation $(i_{\mathcal{A}}, K)$ has no macroscopic effect on points in $\mathcal{A}$ and, hence, $(i_{\mathcal{A}}, K)$ is called a purely submacroscopic deformation from $\mathcal{A}$. We note from the accommodation inequality (2) for (g, G) that $\det K \leq 1$, which implies that the accommodation inequality for $(i_{\mathcal{A}}, K)$ is satisfied.

The structured deformation (g, G) from $\mathcal{A}$ is said to be invertible if there is a structured deformation (g', G') from $g(\mathcal{A})$ satisfying

$$(g', G') \diamond (g, G) = (i_{\mathcal{A}}, I) \quad \text{AND} \quad (g, G) \diamond (g', G') = (i_{\mathcal{A}'}, I). \tag{11}$$

This definition permits one to show that (g, G) is invertible if and only if $\det G = \det \nabla g$, so that no volume change occurs due to disarrangements. Equivalently, (g, G) is invertible if and only if $\det K = 1$. In this case, the inverse structured deformation $(g, G)^{-1}$ is given by

$$(g, G)^{-1} = (g^{-1}, G^{-1} \circ g^{-1}).$$

In particular, the purely submacroscopic deformation $(i_{\mathcal{A}}, K)$ from $\mathcal{A}$ is invertible if and only if $\det K = 1$, and we then have

$$(i_{\mathcal{A}}, K)^{-1} = (i_{\mathcal{A}}, K^{-1}).$$

2.2 Material bodies and their structured configurations

In the context of classical deformations, there is a widely accepted notion of a material body as a smooth, differentiable manifold $\mathcal{B}$ whose globally defined charts $\kappa : \mathcal{B} \longrightarrow \mathcal{E}$ are called configurations of the body. This concept is tied to classical deformations through the requirement that, whenever κ and κ' are configurations of the body, the mapping $g = \kappa' \circ \kappa^{-1}$ is required to be a diffeomorphism of $\kappa(\mathcal{B})$. It is common to choose a particular configuration κ_r in advance and to identify the material points X of the body with their corresponding positions $\kappa_r(X)$ in space. The distinguished configuration κ_r is called a (classical) reference configuration for the body, and one frequently identifies the body, itself, with the range $\kappa_r(\mathcal{B}) \subset \mathcal{E}$. This identification permits us to identify each configuration κ of the body with the classical deformation $\kappa \circ \kappa_r^{-1}$ from $\kappa_r(\mathcal{B})$.

For structured deformations, a corresponding mathematical structure for a material body based on differential geometry has not yet been provided, although the article [36] provides hints that may help in that regard. Consequently, we start by choosing a material body $\mathcal{B}$ with respect to classical deformations, as described above, along with a specified classical reference configuration κ_r. We again identify material points of the body with their positions in $\kappa_r(\mathcal{B}) \subset \mathcal{E}$ and consider as deformations of the body the structured deformations (g, G) from $\kappa_r(\mathcal{B})$. In order to provide structured counterparts of the classical configurations κ for $\mathcal{B}$, we note that each structured deformation (g, G)

from $\kappa_r(\mathcal{B})$ determines for each material point $X \in \kappa_r(\mathcal{B})$ both the position $g(X)$ of X as well as the deformation without disarrangements $G(X)$. The assignment $\kappa_{(g,G)}$ of pairs $(g(X), G(X))$ to points $X \in \kappa_r(\mathcal{B})$ will be called the structured configuration corresponding to (g, G). In this way, once the classical reference configuration κ_r has been assigned, each structured deformation (g, G) from $\kappa_r(\mathcal{B})$ determines a structured configuration $\kappa_{(g,G)}$ of the body. In particular, the classical configurations of the body are those of the form $\kappa_{(g,\nabla g)}$ with g a diffeomorphism of $\kappa_r(\mathcal{B})$, and the configuration κ_r is the configuration determined by the structured deformation $(i_{\kappa_r(\mathcal{B})}, I)$. It is helpful to refer to the preassigned classical reference configuration κ_r as the (preassigned) virgin configuration, because it is a configuration determined by the classical deformation $(i_{\kappa_r(\mathcal{B})}, I)$ and, hence, not only is free of disarrangements but also is free of any change in position of points.

With these concepts at hand, we may use the factorization (9)

$$(g, G) = (g, \nabla g) \diamond (i_{\kappa_r(\mathcal{B})}, K)$$

to identify for each structured deformation (g, G) from $\kappa_r(\mathcal{B})$ two structured configurations of the body: the (non-classical) deformed configuration $\kappa_{(g,G)}$ of $\mathcal{B}$, and the submacroscopically disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K)}$ of $\mathcal{B}$ with, as before, $K = (\nabla g)^{-1}G$. The submacroscopically disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K)}$ is macroscopically indistinguishable from the virgin configuration $\kappa_r = \kappa_{(i_{\kappa_r(\mathcal{B})},I)}$, because it is obtained from the virgin configuration through the purely submacroscopic structured deformation $(i_{\kappa_r(\mathcal{B})}, K)$ and because $i_{\kappa_r(\mathcal{B})}(X) = X$ for all $X \in \kappa_r(\mathcal{B})$. The deformed configuration $\kappa_{(g,G)}$ is obtained from the submacroscopically disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K)}$ through the classical deformation $(g, \nabla g)$, so that no additional disarrangements are introduced in attaining the deformed configuration from the submacroscopically disarranged configuration. While the deformed configuration and the submacroscopically disarranged configuration depend upon the given structured deformation (g, G), the (preassigned) virgin configuration does not.

It is important in these considerations to keep in mind that each material point X of the body has the same position $\kappa_r(X)$ in space in both the virgin configuration and in the disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K)}$, despite the fact that, in general, these two structured configurations are different. We note also that in previous descriptions [3], [6] - [9] of the configurations available to bodies undergoing structured deformations, the disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K)}$ was called the (macroscopic) reference configuration, because it corresponds to the configuration from which the classical deformation $(g, \nabla g)$ procedes. With that usage, the reference configuration would vary as the tensor $K = (\nabla g)^{-1}G$ for the structured deformation varies. In the present article, the term reference configuration refers to the fixed, preassigned classical configuration $\kappa_r = \kappa_{(i_{\kappa_r(\mathcal{B})},I)}$; the term virgin configuration also is used here to denote the preassigned classical configuration $\kappa_r = \kappa_{(i_{\kappa_r(\mathcal{B})},I)}$ because the structured deformation $(i_{\kappa_r(\mathcal{B})}, I)$ introduces no disarrangements. Because we identify material points of the body with their positions $X \in \kappa_r(\mathcal{B})$, we call $g(X)$ the location of

X in the final configuration (g, G) of the body, we call $G(X)$ the deformation without disarrangements at X, and $M(X) = \nabla g(X) - G(X)$ the deformation due to disarrangements at X.

2.3 The universal decomposition of continuum fluxes

It is standard in continuum mechanics that a spatial continuum flux $w_s : g(\kappa_r(\mathcal{B})) \to \mathcal{V}$, for example, the traction field or the heat-flux vector field acting on the deformed configuration $g(\kappa_r(\mathcal{B}))$ of a body, induces through a smooth deformation g the following flux $w : \kappa_r(\mathcal{B}) \to \mathcal{V}$ on the configuration $\kappa_r(\mathcal{B})$:

$$w := \det(\nabla g)(\nabla g)^{-1}(w_s \circ g). \tag{12}$$

In the context of structured configurations of the body as described above, the structured configuration $\kappa_{(g,G)}$ is obtained from the disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K)}$ by means of the classical deformation $(g, \nabla g)$, and we therefore may regard the spatial field w_s as acting on the structured configuration $\kappa_{(g,G)}$ and the flux w as acting on the disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K)}$. In [10] the fact that the disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K)}$ is obtained from the virgin configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},I)}$ by means of the purely submacroscopic deformation $(i_{\kappa_r(\mathcal{B})}, K)$ was used to define the flux $w_\backslash : \kappa_r(\mathcal{B}) \to \mathcal{V}$ by means of a formula analogous to (12):

$$w_\backslash := (\det K)K^{-1}w \tag{13}$$

as well as to define the flux $w_d : \kappa_r(\mathcal{B}) \to \mathcal{V}$

$$w_d := (\det K)w - w_\backslash. \tag{14}$$

Just as we may regard the flux w as acting on the disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K)}$, we may regard through (13) and (14) the fluxes $w_\backslash$, w_d, and $(\det K)w$ as acting on the virgin configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},I)}$. It was demonstrated in [10] that the resulting formula

$$(\det K)w = w_\backslash + w_d \tag{15}$$

represents a unique, additive and universal decomposition of the flux $(\det K)w$ into the part $w_\backslash$ without disarrangements and the part w_d due to disarrangements. The adjective "universal" is used because the fields appearing in (15), as well as the validity of that relation, itself, depend only upon the given flux w and the structured deformation (g, G) from $\kappa_r(\mathcal{B})$, and not upon the material comprising the body. The terminology above is justified by the fact established in [10] and references cited therein, that $w_\backslash$ is not affected by fluxes across surfaces on which approximations f_n of the purely submacroscopic deformation (i, K) have jumps, while w_d is not affected by fluxes across preassigned surfaces on which f_n does not jump.

It is immediate that the relations (13) and (15) imply the consistency relation

$$Kw_\backslash = w_\backslash + w_d \tag{16}$$

that shows that, for a given structured deformation (g, G), the fields $w_\backslash$ and w_d are related in a specific way, independent of the particular material comprising the body. The presence of the refined versions $w_\backslash$ and w_d of the flux w permits one to provide sharper versions of constitutive equations. In fact, the short-range interactions associated with $w_\backslash$ and with w_d have different physical mechanisms and, therefore, in a given material, the dependence of the fields $w_\backslash$ and w_d upon the deformation may differ significantly. For example, the mechanism for transmitting forces across the interface separating adjacent grains of a granular body may differ significantly from that for transmitting forces across a preassigned surface within a single grain. Nevertheless, whatever the mechanisms are, the two resulting mechanisms must be consistent with the given structured deformation in the sense that (16) is satisfied.

2.3.1 Stresses with and without disarrangements

We follow the argument in [3] and suppose now that a system of contact forces on the body is prescribed that gives rise to a stress field $T : g(\kappa_r(\mathcal{B})) \rightarrow Lin\mathcal{V}$, the Cauchy stress, on the deformed configuration $\kappa_{(g,G)}$ of the body. For a fixed vector $a \in \mathcal{V}$, we put $w_s^a := T^T a$ and note that the fields w^a, $w_\backslash^a$, w_d^a associated with w_s^a through (12)-(14) must satisfy the consistency relation (16). Using the arbitrariness of a and the definition

$$S = \det(\nabla g)(T \circ g)(\nabla g)^{-T}$$

of the Piola-Kirchhoff stress S, the consistency relation becomes

$$S_\backslash K^T = S_\backslash + S_d \tag{17}$$

or, equivalently,

$$S_\backslash M^T + S_d(\nabla g)^T = 0 \tag{18}$$

where

$$S_\backslash := (\det K) S K^{-T} \quad \text{and} \quad S_d := (\det K) S - S_\backslash \tag{19}$$

are the stresses with and without disarrangements, respectively. These definitions immediately yield the additive decomposition of stress:

$$(\det K) S = S_\backslash + S_d. \tag{20}$$

When two constitutive assumptions are made, each relating one of the two stress fields $S_\backslash$ and S_d to the structured deformation (g, G) that the body undergoes, the consistency relation (18) becomes a tensorial restriction on the fields g and G that describe the multiscale deformation of the body. (Because the restrictions imposed by the two constitutive equations and the consistency relation depend upon the material that comprises the body, the restrictions differ from the material-independent internal constraints discussed in the next subsection.) Alternatively, when a single constitutive relation is made that gives the stress field S in terms of g and G, then the constitutive relations for $S_\backslash$ and

S_d are determined by g and G through (19) and the constitutive relation for S. In this case, the consistency relation (18) is satisfied as an identity and places no additional restriction on g and G. Each of the two alternatives describes a distinct material, and the former alternative, in which separate constitutive relations are made for $S_\backslash$ and S_d, takes advantage of the more refined measures of stress provided by the multiscale geometry of structured deformations. In this section, that former alternative is used, while in Section 3 dealing with "gradient-disarrangements" there is an additional continuum flux that appears, and both alternatives will play a role in the formulation of constitutive equations.

2.4 $\det(G + M) - \det G \geq 0$ as an internal constraint

The accommodation inequality (2) contains the inequality $\det G \leq \det F$ with $F = \nabla g$ the macroscopic deformation gradient; using (6) we may write that inequality in the form

$$\det(G + M) - \det G \geq 0. \tag{21}$$

This inequality is part of the definition of a structured deformation and, therefore, may be viewed as an internal, kinematical constraint imposed on the dynamical processes experienced by every body undergoing structured deformations, whatever may be the material from which the body is formed.

In the case of classical deformations and internal constraints of the form $f(F) \geq 0$, with f a smooth function, there is only a single reaction stress S^r that depends upon F in a manner which is restricted by the following "dissipation axiom" [38] that we adapt here to the case of unilateral constraints: for each F_0 satisfying the constraint $f(F_0) \geq 0$ and for each smooth curve $\tau \longmapsto F(\tau)$ that satisfies (i) $F(0) = F_0$ and (ii) $f(F(\tau)) \geq 0$ for all τ, there holds $S^r(F_0) \cdot F'(0) \geq 0$. Here, F' denotes the derivative of $\tau \longmapsto F(\tau)$. It is straightforward to show (see [38] and references therein for a proof of $(22)_1$) that this principle is equivalent to the following condition: for every F_0 satisfying the constraint $f(F_0) \geq 0$ there exists $\lambda^r \geq 0$ such that

$$S^r(F_0) = \lambda^r Df(F_0) \text{ and } \lambda^r f(F_0) = 0. \tag{22}$$

Note that if $f(F_0) = 0$, then the inequality $\lambda^r \geq 0$ is the only restriction placed on the number λ^r. The dissipation axiom as employed in [38] contains an additional term there associated with reactive entropy production that we here set equal to zero without affecting the conclusions drawn. Moreover, we have not required that the set of possible reaction stresses at F_0 is closed under scalar multiplication.

In the context of structured deformations the inequality (21) is an internal constraint of the form $f(G, M) \geq 0$, and it is appropriate to introduce reaction stresses $R_\backslash$ and R_d that depend upon G and M according to the following broadening of the "dissipation axiom": for each (G_0, M_0) satisfying the constraint $f(G_0, M_0) \geq 0$ and for each smooth curve $\tau \longmapsto (G(\tau), M(\tau))$ that satisfies (i) $(G(0), M(0)) = (G_0, M_0)$ and (ii) $f(G(\tau), M(\tau)) \geq 0$ for all τ, there holds $(R_\backslash(G_0, M_0), R_d(G_0, M_0)) \cdot (G'(0), M'(0)) \geq 0$. Here, "$\cdot$" denotes the inner product on the product space $Lin\mathcal{V} \times Lin\mathcal{V}$. The same argument underlying the classical case above yields: the "dissipation axiom" is equivalent to the condition that, for every (G_0, M_0) satisfying the constraint $f(G_0, M_0) \geq 0$, there exists $\lambda^r \geq 0$ such that

$$\begin{aligned} (R_\backslash, R_d) \quad &| \quad _{(G_0,M_0)} = \lambda^r (D_G f, D_M f) \,|_{(G_0,M_0)} \\ \lambda^r f(G_0, M_0) \quad &= \quad 0. \end{aligned} \tag{23}$$

Again, if $f(G_0, M_0) = 0$, then $\lambda^r \geq 0$ is the only restriction placed on the number λ^r.

The additive decomposition of the stress (20) and the formula (23) lead us to define the reaction stress S^r through the formula

$$\begin{aligned} (\det K) S^r &= R_\backslash + R_d \\ &= \lambda^r (D_G f(G_0, M_0) + D_M f(G_0, M_0)). \end{aligned} \tag{24}$$

For the particular internal constraint (21) induced by the accommodation inequality, the formula $(23)_1$ for the reaction stresses become

$$R_\backslash(G, M) = \lambda^r (F^{-T} - G^{-T}), \; R_d(G, M) = \lambda^r F^{-T} \tag{25}$$

$$\lambda^r (\det(G + M) - \det G) = 0 \tag{26}$$

with $F = G + M$, and the reaction stress S^r in (24) becomes

$$S^r(G, M) = \tilde{\lambda}^r (2F^{-T} - G^{-T}). \tag{27}$$

with $\tilde{\lambda}^r = \lambda^r \backslash (\det K) \geq 0$. In particular, when the structured deformation is a classical one, we have $M = F - G = 0$, $K = I$, and the formulas above reduce to

$$R_\backslash(F, 0) = 0, \; R_d(F, 0) = \lambda^r F^{-T} = S^r(F, 0), \tag{28}$$

so that the reaction stress in the current configuration reduces to a hydrostatic tension in the case of classical deformations.

It is convenient and customary to call the differences $S - S^r$, $S_\backslash - R_\backslash$, and $S_d - R_d$ the *constitutively determined part* of the corresponding stresses to emphasize that the specific material that occupies the body will be defined through functions that provide relations between each field $S - S^r$, $S_\backslash - R_\backslash$, and $S_d - R_d$ and the fields G and M that describe the local geometrical changes occurring in the body. We note that in [3] the accommodation inequality was not treated as an internal constraint, so that no reaction stresses where introduced. In the remainder of the present section, one need only set $\lambda^r = 0$ to recover corresponding formulas and relations appearing in [3].

2.5 Power and balance laws

We consider now structured motions of a body, so that at each time t there is given a structured deformation $(g(\cdot, t), G(\cdot, t))$ from $\kappa_r(\mathcal{B})$. The dependence of the two fields $g(\cdot,t), G(\cdot,t)$ upon t is assumed to be smooth. In anticipation of our treatment of the more complex case of gradient-disarrangements, we here derive the field relations for the body by requiring that two measures of power expended agree in all virtual perturbations of a given structured motion. The external power expended at time t in each part $\mathcal{P} \subset \kappa_r(\mathcal{B})$ of the body is defined to be the number

$$P_{ext}(\mathcal{P},t,g) := \int_{\partial\mathcal{P}} S(X,t)n(X)\cdot\dot{g}(X,t)dA_X + \int_{\mathcal{P}} b_{tot}(X,t)\cdot\dot{g}(X,t)dV_X, \quad (29)$$

where $S(\cdot,t)$ is the stress field at time t and $b_{tot}(\cdot,t)$ is the body force field on $\kappa_r(\mathcal{B})$ at time t,

$$b_{tot}(\cdot,t) = b_r(X,t) - \rho_r(X)\ddot{g}(X,t),$$

including inertial forces $\rho_r\ddot{g}$ per unit volume, with ρ_r the mass density in the virgin configuration. The vector $n(X)$ is the outward normal to $\partial\mathcal{P}$ at the point X, with superposed dots denoting time derivatives. The internal power is defined to be

$$P_{int}(\mathcal{P},t,g) := \int_{\mathcal{P}} S(X,t)\cdot\nabla\dot{g}(X,t)dV_X, \quad (30)$$

and we note that the definitions (30) and (29) require only that classical systems of contact and body forces be imposed during the structured motion of the body. Moreover, neither the internal power nor the external power depends upon the field G of deformation without disarrangements. (For the case of gradient-disarrangements, both will depend upon the field G.)

Given a structured motion $t \longmapsto (g(\cdot,t), G(\cdot,t))$ of $\mathcal{B}$ and a smooth velocity field $v : \kappa_r(\mathcal{B}) \to \mathcal{V}$, we define the virtual motion g_v of $\mathcal{B}$ by

$$g_v(X,t) = g(X,t) + tv(X), \quad (31)$$

and require that, for the given fields S, b_r, ρ_r, g, and G, there holds

$$P_{int}(\mathcal{P},t,g_v) = P_{ext}(\mathcal{P},t,g_v) \quad \text{for all } v,\ t, \text{ and } \mathcal{P}, \quad (32)$$

We require as well that the internal power be frame-indifferent, i.e.,

$$P_{int}(\mathcal{P},t,r\circ_1 g) = P_{int}(\mathcal{P},t,g) \quad (33)$$

for every rigid motion r:

$$r(y,t) = Q(t)(y - y_o) + w(t).$$

Here, the composition $r\circ_1 g$ denotes the motion defined by:

$$\begin{aligned}(r\circ_1 g)(X,t) &= r\left(g(X,t),t\right) \\ &= Q(t)(g(X,t) - y_o) + w(t).\end{aligned}$$

Standard arguments show that (32) and (33) imply that the balance of linear momentum and angular momentum in the local forms

$$\rho_r\ddot{g} = \operatorname{div} S + b_r \quad (34)$$

$$S(\nabla g)^T = (\nabla g)S^T. \quad (35)$$

2.6 The additive decomposition of stress-power

The stress-power $S \cdot \dot{F}$, i.e., the power expended per unit volume in the virgin configuration by internal forces, plays a key role in formulating the field relations that govern an elastic body undergoing disarrangements. Here, $F = \nabla g$ denotes the macroscopic deformation gradient. The additive decomposition (6) of F may be differentiated, and the additive decomposition (20) of stress then yields the *additive decomposition of the stress-power* $(\det K) S \cdot \dot{F}$:

$$(\det K) S \cdot \dot{F} = S_{\backslash} \cdot \dot{G} + S_d \cdot \dot{M} + S_{\backslash} \cdot \dot{M} + S_d \cdot \dot{G}. \tag{36}$$

The first term on the right-hand side of (36), $S_{\backslash} \cdot \dot{G}$, may be described as a "pure term" in the sense that both factors $S_{\backslash}$ and $\dot{G}$ of the tensor inner product $S_{\backslash} \cdot \dot{G}$ carry the same attribute "without disarrangements". The applied contact forces are in proximity to the geometrical changes that enter into the pure term $S_{\backslash} \cdot \dot{G}$, because both are associated with actions away from disarrangement sites. The same is the case for the "pure term" $S_d \cdot \dot{M}$, although here the common attribute for this term is "due to disarrangments", and both factors are associated with actions at disarrangement sites. The point of view taken here follows that taken in [3] : the two pure terms comprise the part of the stress-power that will contribute to the energy *stored* by the body as it undergoes structured deformations.

The third and fourth terms on the right-hand side of (36) may be described as "mixed terms". In each mixed term, the applied contact forces are remotely located with respect to where the geometrical changes occur, in the sense that one is associated with disarrangement sites and the other is not. The point of view taken here again follows that taken in [3] : the two mixed terms comprise the part of the stress-power that will contribute to the energy *dissipated* by the body as it undergoes structured deformations. These considerations on storage and dissipation of energy will be realized concretely through the particular constitutive assumptions laid down in the next subsection.

2.7 Constitutive assumptions

The local form (34) of the balance of linear momentum amounts to three scalar equations that restrict the fields g and G that, together, have twelve scalar components. By laying down separate and independent consitutive equations for $S_{\backslash} - R_{\backslash}$ and $S_d - R_d$, we can use the consistency relation for stresses (18) to achieve an additional nine scalar equations that further restrict g and G. This approach exploits the presence of the refined measures of stress $S_{\backslash}$ and S_d. The additive decomposition of stress (20) and the corresponding decomposition of reaction stress $(24)_1$ then provide together a constitutive equation for the constitutively determined stress $S - S^r$, itself. The resulting class of materials is the one considered in [3] and is reviewed in detail below. We also indicate in this subsection an alternative approach that makes a constitutive assumption directly on $S - S^r$ and so defines a different class of elastic materials that will be relevant in Section 3 in the context of gradient-disarrangements.

We let ψ denote the Helmholtz free energy density (measured per unit volume in the virgin configuration), and we identify a particular elastic body undergoing structured motions through the specification of a scalar-valued free energy response function $(G, M) \longmapsto \Psi(G, M)$ that determines the field ψ through the constitutive relation [3]

$$\psi(X,t) = \Psi(G(X,t), M(X,t)). \tag{37}$$

This relation determines the manner in which the Helmholtz free energy varies during every (isothermal) structured motion $t \longmapsto (g(\cdot,t), G(\cdot,t))$. In addition, we require that the response function Ψ also determine the fields $S_\backslash - R_\backslash$ and $S_d - R_d$ through the constitutive relations [3]

$$\begin{aligned} S_\backslash(X,t)) - R_\backslash(X,t) &= (\det K(X,t))\, D_G\Psi(G(X,t), M(X,t)) \\ S_d(X,t) - R_d(X,t) &= (\det K(X,t)))\, D_M\Psi(G(X,t), M(X,t)). \end{aligned} \tag{38}$$

Here, $D_G\Psi$ and $D_M\Psi$ denote the partial derivatives of Ψ, each of which is a tensor-valued function of the pair (G, M). The additive decomposition of stress (20), the corresponding decomposition of reaction stress $(24)_1$, and the constitutive relations (38) then yield the following constitutive relation for the stress $S - S^r$:

$$S - S^r = D_G\Psi + D_M\Psi \tag{39}$$

where for the sake of brevity we have omitted the arguments involving X and t. We refer to this formula for $S - S^r$ as the stress relation.

The specific form (38) chosen here is motivated by two considerations: that choice guarantees that the pure terms $S_\backslash \cdot \dot{G} + S_d \cdot \dot{M}$ in the stress power account for all of the energy stored (see (41) below) and, therefore, that the mixed terms $S_\backslash \cdot \dot{M} + S_d \cdot \dot{G}$ can contribute only to the power dissipated. That choice also permits one to interpret $S_\backslash - R_\backslash$ and $S_d - R_d$ as instances of the "driving forces" corresponding to the kinematical variables G and M, familiar in the physics literature. The constitutive assumptions (38), the formulas (25), and $M = \nabla g - G = F - G$ when substituted into the consistency relation (18) yield

$$\begin{aligned} 0 \;=\; & \{D_G\Psi(G, \nabla g - G) + \tilde{\lambda}^r((\nabla g)^{-T} - G^{-T})\}(\nabla g - G)^T + \\ & +\{D_M\Psi(G, \nabla g - G) + \tilde{\lambda}^r(\nabla g)^{-T}\}(\nabla g)^T, \end{aligned} \tag{40}$$

where $\tilde{\lambda}^r = \lambda^r/(\det K)$. We conclude that (40) is a tensorial relation that restricts the structured deformation (g, G) and the non-negative scalar field $\tilde{\lambda}^r$.

Alternatively, one could make a constitive assumption directly on the stress S of the form $S - S^r = \hat{S}(G, M)$ and deduce from the definitions of $S_\backslash$ and S_d the dependence of these refined measures on G and M. This alternative would describe a different material from the one described in (38) and (37). Moreover, this alternative would result in the satisfaction of the consistency as an identity, thus removing nine scalar equations restricting the pair of fields g and G. An instance of the use of such an alternative appears in our treatment of gradient disarrangements in Section 3 where an analogous consistency relation for hyperstress provides additional equations that restrict the unknown fields.

The constitutive relations (38) and the chain-rule yield the formula

$$\begin{aligned}(\det K)\dot{\psi} &= (\det K)(D_G\Psi \cdot \dot{G} + D_M\Psi \cdot \dot{M}) \\ &= S_\backslash \cdot \dot{G} + S_d \cdot \dot{M} - (R_\backslash \cdot \dot{G} + R_d \cdot \dot{M}). \end{aligned} \tag{41}$$

We note by the "dissipation axiom" and its consequences (25), (26), that $R_\backslash \cdot \dot{G} + R_d \cdot \dot{M}$ must be non-negative and, provided that the mapping $t \longmapsto f(G(X,t), M(X,t))$ is piecewise monotone, for a given material point X, the number $R_\backslash \cdot \dot{G} + R_d \cdot \dot{M}$ can be positive only at isolated times, . Consequently, except at isolated times, the constitutive assumptions imply that for each material point the pure terms $S_\backslash \cdot \dot{G} + S_d \cdot \dot{M}$ in the decomposition (36) of the stress-power account for the rate of change of free energy, as proposed in Section 2.6. Similarly, our constitutive assumptions show that $S \cdot \dot{F} - \dot{\psi}$, the rate of dissipation of energy per unit volume in the virgin configuration, satisfies the relation

$$(\det K)(S \cdot \dot{F} - \dot{\psi}) = S_\backslash \cdot \dot{M} + S_d \cdot \dot{G} + R_\backslash \cdot \dot{G} + R_d \cdot \dot{M} \tag{42}$$

which, except at isolated times for a given material point, reduces to the expression $S_\backslash \cdot \dot{M} + S_d \cdot \dot{G}$. Consequently, in order that the Second Law of Thermodynamics holds in the present, isothermal context, we impose the dissipation inequality

$$S_\backslash \cdot \dot{M} + S_d \cdot \dot{G} \geq 0. \tag{43}$$

Through the relation (43), we impose the Second Law as a restriction on dynamical processes [3]. The constitutive assumptions (38) and the formulas (25), when substituted into the dissipation inequality (43), yield

$$\begin{aligned} 0 \ \leq \ & \{D_G\Psi(G,M) + \tilde{\lambda}^r(F^{-T} - G^{-T})\} \cdot \dot{M} + \\ & +\{D_M\Psi(G,M) + \tilde{\lambda}^r F^{-T}\} \cdot \dot{G}. \end{aligned} \tag{44}$$

We require as in [3] that the free-energy response $(G,M) \longmapsto \Psi(G,M)$ be frame indifferent, i.e., for all orthogonal tensors Q and for all tensors G and M there holds

$$\Psi(QG, QM) = \Psi(G,M). \tag{45}$$

Because the rate of dissipation of energy also should be frame-indifferent, we also impose here the requirement made in [3] that the mixed power $S_\backslash \cdot \dot{M} + S_d \cdot \dot{G}$ be frame-indifferent or equivalently, by (38) and the formulas (25), that the scalar

$$\{D_G\Psi(G,M) + \tilde{\lambda}^r(F^{-T} - G^{-T})\} \cdot \dot{M} + \{D_M\Psi(G,M) + \tilde{\lambda}^r F^{-T}\} \cdot \dot{G} \tag{46}$$

be frame-indifferent. Given (45), frame-indifference of the mixed power is shown in [3] to be equivalent to the assertion

$$sk\{(D_G\Psi(G,M) + \tilde{\lambda}^r(F^{-T} - G^{-T}))M^T + (D_M\Psi(G,M) + \tilde{\lambda}^r F^{-T})G^T\} = 0 \tag{47}$$

where $skA := (A - A^T)/2$ for all tensors A. In that article we showed that (47) and (45) imply that the Cauchy stress T is symmetric, or, equivalently:

$$sk\{SF^T\} = 0. \tag{48}$$

2.8 Field relations for elasticity with disarrangements

Given a free energy response function $(G, M) \longmapsto \Psi(G, M)$ that is frame indifferent in the sense of (45) and (47), along with the constitutive relations (38), the formulas for the reaction stresses (25) and the inequality (43), we may record the following field relations for an elastic body undergoing disarrangements (see [3], where only the case $\tilde{\lambda}^r = 0$, i.e., where the reaction stresses are identically zero, was considered).

- balance of linear momentum:

$$\rho_r \ddot{g} = \mathrm{div}(D_G\Psi + \tilde{\lambda}^r(F^{-T} - G^{-T}) + D_M\Psi + \tilde{\lambda}^r F^{-T}) + \rho_r b_r \tag{49}$$

  with ρ_r and b_r the mass density and the body force per unit volume in the virgin configuration, and with $F = \nabla g = G + M$,

- consistency relation :

$$(D_G\Psi + \tilde{\lambda}^r(F^{-T} - G^{-T}))\, M^T + (D_M\Psi + \tilde{\lambda}^r F^{-T})\, F^T = 0 \tag{50}$$

- frame-indifference of the mixed power:

$$sk\{(D_G\Psi + \tilde{\lambda}^r(F^{-T} - G^{-T}))\, M^T + (D_M\Psi + \tilde{\lambda}^r F^{-T})\, G^T\} = 0, \tag{51}$$

- dissipation inequality:

$$(D_G\Psi + \tilde{\lambda}^r(F^{-T} - G^{-T})) \cdot \dot{M} + (D_M\Psi + \tilde{\lambda}^r F^{-T}) \cdot \dot{G} \geq 0, \tag{52}$$

- (weakened) accommodation inequality:

$$0 < \det G \leq \det F. \tag{53}$$

We call (49) - (53) the field relations for elasticity with disarrangements. The reaction scalar field $\tilde{\lambda}^r$ is non-negative, and, for the case $\tilde{\lambda}^r = 0$, the article [3] provides equivalent forms of these field relations with the alternative choice of variables (F, G) or the choice (F, K) in specifying the free energy response. The particular choice of variables $(F, \Delta) = (G + M, G - M)$ is shown [6] in the case of statics to provide a variational formulation of the field relations (49) - (53) when $\tilde{\lambda}^r = 0$.

The pair of fields g and G and the non-negative single scalar field $\tilde{\lambda}^r$ amount to thirteen unknown scalar fields to be determined through the twelve scalar relations (49) and (50), along with the three scalar relations (51) and the two inequalities (52), (53).

Because frame indifference (45) of the response Ψ and frame-indifference of the mixed power (51) imply the symmetry of the Cauchy stress T, the balance of angular momentum is satisfied and need not be imposed separately. Moreover, if Ψ satisfies for all tensors G

$$D_M \Psi(G, 0) = 0, \tag{54}$$

then the field relations for elasticity with disarrangements, when restricted to classical structured motions $(g, \nabla g)$, imply that the reactive hydrostatic tension $\tilde{\lambda}^r I$ vanishes and that the field relations reduce to those for classical, finite elasticity [3]. Therefore, elasticity with disarrangements provides a broadening of finite elasticity that systematically incorporates geometrical changes at submacroscopic levels.

2.9 Examples and applications

A variety of examples of response functions Ψ and applications of elasticity with disarrangements (with $\tilde{\lambda}^r = 0$) have been studied in recent years [3],[5] - [9]. Some of these have centered on applications to cohesionless, granular media through the restriction to "purely dissipative disarrangements": for all tensors G, M there holds

$$D_M \Psi(G, M) = 0. \tag{55}$$

For such materials, submacroscopic slips and separations do not contribute to energy stored in the body. When the continuum is viewed as an aggregate of tiny elastic bodies or grains, G captures through the formula (7) the average deformation of grains and M caputures through (5) the average separation and slip among the grains. The restriction (55) then amounts to the assumption that short-range forces between grains are weak enough to make negligible any energy stored through relative motions among grains. In this situation, the only mechanism for storage of energy is through deformation of individual grains as reflected in the constitutive relation that follows from (55):

$$\psi(X, t) = \Psi(G(X, t)), \tag{56}$$

and the field relations (49) - (53) then reduce to

$$\begin{array}{rcl} \rho_r \ddot{g} & = & \operatorname{div}(D\Psi(G)) + \rho_r b_r \\ D\Psi(G)\ (\nabla g - G)^T & = & 0 \\ D\Psi(G) \cdot (\nabla \dot{g} - \dot{G}) & \geq & 0 \\ 0 & < & \det G \leq \det \nabla g. \end{array} \tag{57}$$

The articles [7] - [9] have highlighted the determination of the portfolio of "disarrangement phases", i.e., pairs $(\nabla g, G)$ satisfying the consistency relation $(57)_2$ and accommodation inequality $(57)_4$ available to an aggregate of elastic bodies described by the field relations (57). They also provide an analysis of the propagation of moving interfaces separating loose and compact phases of granular media, the appearance of which can reveal information about the frictional properties of the continuum; they provide in addition connections between elasticity with purely dissipative disarrangements and materials that cannot support tensile stresses, the so-called "no-tension materials.".

3 Elasticity with gradient-disarrangements

The field theory elasticity with disarrangements described in Section 2 broadens classical, finite elasticty by taking into account both smooth and non-smooth geometrical changes at submacroscopic levels. The additive decomposition (6) of ∇g,

$$\nabla g = G + M, \tag{58}$$

isolates the smooth submacroscopic changes by means of the term G and the disarrangements through the term M. Moreover, the identification relation (5) for M shows that the disarrangements that M measures are precisely those associated with the jumps $[f_n]$ in the approximating deformations f_n that appear in (5).

Notably absent in the considerations leading to the field equations (49) - (53) are any effects of the jumps $[\nabla f_n]$ in gradients of approximating functions f_n. Consequently, elasticity with disarrangements does not explicitly take into account such "gradient-disarrangements" and, therefore, only partially incorporates submacroscopic geometrical changes associated with structured deformations. One physical context in which gradient-disarrangements arise is that of fine phase mixtures in which ∇f_n, itself, jumps across phase boundaries, while f_n does not jump across these boundaries [18]. A second such context is that of large deformations of metals in which domains form at the microlevel across whose walls both f_n as well as ∇f_n can jump [30]. The present section is devoted to broadening elasticity with disarrangements, itself, so that these contexts can be addressed through an analogous field theory. The additional field introduced in order to broaden elasticity with disarrangements then provides the field theory with a natural length scale and so fits into the broader category of strain-gradient theories formulated in order to address size effects observed in continua.

3.1 Second-order structured deformations and the additive decompositon of ∇G

A helpful guide toward capturing the effects of gradient-disarrangements is the definition (4) of M , rewritten by means of the Approximation Theorem in the form

$$M = \nabla \lim_{n\to\infty} f_n - \lim_{n\to\infty} \nabla f_n, \tag{59}$$

that reveals M as a measure of the lack of commutativity of the operations ∇ and $\lim_{n\to\infty}$. Moreover, M is revealed through (5) as a measure of the jumps $[f_n]$ in the approximating deformations f_n, and we may conclude that $\nabla \lim_{n\to\infty} f_n - \lim_{n\to\infty} \nabla f_n$ is a measure of the jumps in the approximating deformations f_n. This conclusion suggests that the expression $\nabla \lim_{n\to\infty} \nabla f_n - \lim_{n\to\infty} \nabla^2 f_n$ may serve as a measure of the jumps $[\nabla f_n]$ in ∇f_n, i.e., as a measure of gradient-disarrangements, and indicates the need to make sense of the expression $\lim_{n\to\infty} \nabla^2 f_n$ in order to procede further along these lines. We do so through a notion of second-order structured deformation [11].

We consider a triple $(g, G, \mathbb{G})$ of fields such that, for each time t, the pair $(g(\cdot, t,), G(\cdot, t))$ is a (first-order) structured deformation from $\kappa_r(\mathcal{B})$ and $\mathbb{G}(\cdot, t)$ is a continuous, third-order tensor field on $\kappa_r(\mathcal{B})$ with the symmetries of a second gradient, i.e., for every $X \in \kappa_r(\mathcal{B})$ there holds for all vectors u, v:

$$(\mathbb{G}(X, t)u)v = (\mathbb{G}(X, t)v)u. \tag{60}$$

(See the Appendix for details about third-order tensors.) We require in addition that g be twice continuously differentiable. For each t we call the triple $(g(\cdot, t,), G(\cdot, t), \mathbb{G}(\cdot, t))$ a *second-order structured deformation* from $\kappa_r(\mathcal{B})$.

From the definition above, it is clear that Theorems 1 and 2 for first-order structured deformations apply to the pair (g, G) within the triple $(g, G, \mathbb{G})$, so that the significance of the fields G and $M = \nabla g - G$ carry over to the present context. Moreover, Theorems 1 and 2 have counterparts in this second-order context.

Theorem 3 *[11] Approximation Theorem: If at each time t the triple*

$$(g(\cdot, t,), G(\cdot, t), \mathbb{G}(\cdot, t))$$

is a second-order structured deformation from $\kappa_r(\mathcal{B})$, then at every time t there exists a sequence $n \longmapsto f_n(\cdot, t)$ of injective, piecewise twice-continuously differentiable mappings on $\kappa_r(\mathcal{B})$ such that

$$\lim_{n\to\infty} f_n(\cdot . t) = g(\cdot, t), \ \lim_{n\to\infty} \nabla f_n(\cdot . t) = G(\cdot, t) \quad \text{and} \quad \lim_{n\to\infty} \nabla^2 f_n(\cdot . t) = \mathbb{G}(\cdot, t) \tag{61}$$

with convergence in the sense of essentially uniform convergence (L^∞).

The second and third relations in (61) justify calling not only G, but also $\mathbb{G}$, measures of deformations without disarrangements, because neither $\mathbb{G}$ nor G reflects any of the discontinuities associated with the approximating deformations f_n. To distinguish the two fields, we call $\mathbb{G}$ the deformation without gradient-disarrangements. The key ingredient that permits the incorporation of gradient-disarrangements is:

Theorem 4 *[11] If $(g(\cdot, t), G(\cdot, t), \mathbb{G}(\cdot, t))$ is a second-order structured deformation at each time t, then for each sequence $n \longmapsto f_n(\cdot, t)$ as in (61) and for each $X \in \kappa_r(\mathcal{B})$ there holds*

$$\nabla G(X, t) - \mathbb{G}(X, t) = \lim_{r\to 0} \lim_{n\to\infty} \frac{\int_{J(\nabla f_n(\cdot, t)) \cap B(X, r)} [\nabla f_n(\cdot, t)](Y) \otimes \nu_{\nabla f_n(\cdot, t)}(Y) dA_Y}{|B(X, r)|} \tag{62}$$

where $J(\nabla f_n(\cdot,t))$ is the jump set of $\nabla f_n(\cdot,t)$, $[\nabla f_n(\cdot,t)](Y)$ is its jump at the point Y, $\nu_{\nabla f_n(\cdot,t)}(Y)$ is the normal to $J(\nabla f_n(\cdot,t))$ at Y, $B(X,r)$ is the sphere centered at X of radius r, and $|B(X,r)|$ is its volume. The tensor product $\otimes$ in (62) is defined by

$$(A \otimes u)v = (u \cdot v)A$$

for all second-order tensors A and vectors u, v. (This definition of $\otimes$ corrects an inconsequential misprint in the version [11]).

The formulas (5) for M and (62) for $\nabla G - \mathbb{G}$ permit us not only to call M the deformation due to disarrangements but also to call $\nabla G - \mathbb{G}$ the deformation due to gradient-disarrangements. We now are in a position to provide an analog for second-order structured deformations of the additive decomposition (6) for first-order structured deformations. To this end, for a second-order structured deformation $(g, G, \mathbb{G})$ we define the gradient-disarrangement tensor

$$\mathbb{M} := \nabla G - \mathbb{G}, \tag{63}$$

so that, by $(61)_2$ and (62), the relation

$$\nabla G = \mathbb{G} + \mathbb{M} \tag{64}$$

becomes an additive decomposition of ∇G into the part $\mathbb{G}$ without gradient-disarrangements and the part $\mathbb{M}$ due to gradient-disarrangements.

3.2 A multiscale identification relation for $\operatorname{curl} G$

We consider a second-order structured deformation $(g, G, \mathbb{G})$ from a region $\mathcal{A}$ and follow [1] by letting c be a smooth, closed curve in $\mathcal{A}$. The vector

$$\begin{aligned} \mathfrak{b}_c &= \oint_c M(x)dx = \oint_c (\nabla g(x) - G(x))dx \\ &= -\oint_c G(x)dx \end{aligned} \tag{65}$$

is obtained by integrating the disarrangement tensor M along c and so can be shown to represent the displacement after deformation of the initially coincident endpoints of c due to disarrangements. As noted in [1], the vector $\mathfrak{b}_c$ is an analog of the Burgers vector employed in crystalline plasticity to measure effects of dislocations. Moreover, if $\mathcal{S}$ is any smooth surface in $\mathcal{A}$ that spans c, then we have

$$\mathfrak{b}_c = \int_{\mathcal{S}} \operatorname{curl} M(x) n(x) dA_x = -\int_{\mathcal{S}} \operatorname{curl} G(x) n(x) dA_x \tag{66}$$

where the second-order tensor field $\operatorname{curl} G$ (sometimes called the "row-curl of G") denotes the curl of the tensor field G:

$$(\operatorname{curl} G)^T v = \operatorname{curl}(G^T v) \quad \text{for all } v \in \mathcal{V}. \tag{67}$$

Consequently, $\mathfrak{b}_c$ is a measure of the defectiveness associated with c for the structured deformation $(g, G, \mathbb{G})$ and is determined directly by $\operatorname{curl} M$ or, equivalently, by $\operatorname{curl} G$ through (66), and we call $\operatorname{curl} G$ the *defectiveness density* for $(g, G, \mathbb{G})$. The formula

$$(\operatorname{curl} G)(w \times v) = ((\nabla G)v)w - ((\nabla G)w)v \tag{68}$$

valid for all vectors $v, w \in \mathcal{V}$, the formula (64), and the symmetry of $\mathbb{G}$ (60) imply that

$$\begin{aligned} -(\operatorname{curl} M)(w \times v) &= (\operatorname{curl} G)(w \times v) \\ &= ((\mathbb{G} + \mathbb{M})v)w - (((\mathbb{G} + \mathbb{M}))w)v \\ &= (\mathbb{M}v)w - (\mathbb{M}w)v. \end{aligned} \tag{69}$$

We conclude that the gradient-disarrangement tensor $\mathbb{M}$ determines through (69) the defectiveness density tensor $\operatorname{curl} G$, and the formula (62) for $\mathbb{M}$ then shows that $\operatorname{curl} G$ is determined by the combinations

$$\Delta_{\nabla f_n, v, w}(Y) := (\nu_{\nabla f_n}(Y) \cdot v)[(\nabla f_n)w](Y) - (\nu_{\nabla f_n}(Y) \cdot w)[(\nabla f_n)v](Y) \tag{70}$$

of gradient-disarrangements through the formula

$$(\operatorname{curl} G)\,|_X\ (w \times v) = \lim_{r \to 0} \lim_{n \to \infty} \frac{\int_{J(\nabla f_n(\cdot, t)) \cap B(X,r)} \Delta_{\nabla f_n, v, w}(Y) dA_Y}{|B(X, r)|} \tag{71}$$

valid for all vectors $v, w \in \mathcal{V}$. This provides an explicit formula showing the manner in which submacroscopic gradient-disarrangements $[\nabla f_n]$ determine the defectiveness density $\operatorname{curl} G$.

3.3 Structured configurations and continuum fluxes, revisited

In the context of second-order structured deformations $(g, G, \mathbb{G})$, classical deformations retain the defining property of not causing disarrangements of any kind, i.e., $M = 0$ and $\mathbb{M} = 0$. Therefore, the equations $\nabla g - G = M = 0$ and $\nabla G - \mathbb{G} = \mathbb{M} = 0$ characterize classical deformations. Thus, a second-order structured deformation is a classical deformation if and only if it is of the form $(g, \nabla g, \nabla^2 g)$. The factorization (9) of first-order structured deformations has a direct counterpart [11] for second-order structured deformations of the form

$$(g, G, \mathbb{G}) = (g, \nabla g, \nabla^2 g) \diamond (i_{\kappa_r(\mathcal{B})}, K, \mathbb{G}_0) \tag{72}$$

with $K = (\nabla g)^{-1} G$ (as in the first-order case) and with $\mathbb{G}_0$ a third-order tensor field whose specific relation to g, G, and $\mathbb{G}$ need not be given here. Therefore, in the context of second-order structured deformations of a material body $\mathcal{B}$, the specification at each time t of a second-order structured deformation $(g, G, \mathbb{G})$ from the virgin configuration $\kappa_r(\mathcal{B})$ provides not only the deformed configuration $\kappa_{(g,G,\mathbb{G})}$ but also the submacroscopically disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K,\mathbb{G}_0)}$ that is macroscopically identically to the virgin configuration $\kappa_r(\mathcal{B})$. Just as in the first-order case, the deformed configuration $\kappa_{(g,G,\mathbb{G}_0)}$ is obtained from the submacroscopically disarranged configuration $\kappa_{(i_{\kappa_r(\mathcal{B})},K,\mathbb{G}_0)}$ by means of the classical deformation $(g, \nabla g, \nabla^2 g)$ that introduces no additional disarrangements.

The purely submacroscopic first-order structured deformation $(i_{\kappa_r(\mathcal{B})}, K)$ appears in the factorization (72), and this permits us to identify, as in Subsection 2.3 for the first order case, the induced fluxes w, $w_\backslash$, and w_d for a given spatial continuum flux w_s on $g(\kappa_r(\mathcal{B}))$. Consequently, we may use the definitions and interpretations established in Subsection 2.3, as well as the consistency relation (16). Because the transformation properties of surface integrals under smooth mappings do not depend upon second derivatives of the mappings, no modifications of these relations are required for the case of second-order structured deformations.

Let $t \longmapsto (g(\cdot,t), G(\cdot,t), \mathbb{G}(\cdot,t))$ be a second-order structured motion of the body $\mathcal{B}$. We shall introduce subsequently a continuum flux on the deformed configuration $\kappa_{(g,G,\mathbb{G})}$ of the form $\mathbb{S}_s(\cdot,t) : g((\kappa_r(\mathcal{B}))\,,t) \to Lin(\mathcal{V}, Lin\mathcal{V})$, i.e., a field with values that are third-order tensors. From the definition of the transpose in the Appendix, we have that $\mathbb{S}_s^T(\cdot,t) : g((\kappa_r(\mathcal{B}))\,,t) \to Lin(Lin\mathcal{V}, \mathcal{V})$ so that, for each $A \in Lin\mathcal{V}$, the field

$$w_s^A(\cdot,t) := \mathbb{S}_s^T(\cdot,t)A$$

is a vector field on $g((\kappa_r(\mathcal{B}))\,,t)$ to which the analysis of Subsection 2.3 applies. Consequently, we write (omiting for brevity the dependence on t and using the notation $\circ_1$ to denote composition in the spatial variable)

$$\begin{aligned} w^A \;&:= \det(\nabla g)(\nabla g)^{-1}(w_s^A \circ_1 g) \\ w_\backslash^A \;&:= (\det K)K^{-1}w^A \\ w_d^A \;&:= (\det K)w^A - w_\backslash^A. \end{aligned}$$

We so obtain as in Subsection 2.3 not only the additive decomposition

$$(\det K)w^A = w_\backslash^A + w_d^A \tag{73}$$

but also the consistency relation

$$Kw_\backslash^A = w_\backslash^A + w_d^A. \tag{74}$$

We use the definition of w_s^A to write

$$\begin{aligned} w^A &= \det(\nabla g)(\nabla g)^{-1}(\mathbb{S}_s^T \circ_1 g)A \\ &= \{(\mathbb{S}_s \circ_1 g)(\nabla g)^*\}^T A, \end{aligned}$$

with $(\nabla g)^* := \det(\nabla g)(\nabla g)^{-T}$. Denoting by $\mathbb{S} := (\mathbb{S}_s \circ_1 g)(\nabla g)^*$ the flux on the submacroscopically disarranged configuration, we have

$$\begin{aligned}w^A &= \mathbb{S}^T A\\ w_\backslash^A &= (\det K)K^{-1}\mathbb{S}^T A = (\mathbb{S}K^*)^T A\\ w_d^A &= ((\det K)\mathbb{S} - \mathbb{S}K^*)^T A,\end{aligned}$$

and the consistency relation (74) becomes

$$K(\mathbb{S}K^*)^T A = (\mathbb{S}K^*)^T A + ((\det K)\mathbb{S} - \mathbb{S}K^*)^T A. \tag{75}$$

Because the consistency relation in the last form (75) holds for every choice of A in $Lin\mathcal{V}$, we obtain the consistency relation for the third-order tensor flux $\mathbb{S}$

$$\mathbb{S}_\backslash K^T = \mathbb{S}_\backslash + \mathbb{S}_d \tag{76}$$

where

$$\mathbb{S}_\backslash := \mathbb{S}K^* \quad \text{and} \quad \mathbb{S}_d := (\det K)\mathbb{S} - \mathbb{S}_\backslash \tag{77}$$

are the fluxes of $\mathbb{S}$ with and without disarrangements, respectively. Similarly, the additive decompositon (73) yields the formula

$$(\det K)\mathbb{S} = \mathbb{S}_\backslash + \mathbb{S}_d. \tag{78}$$

We note that (76), the consistency relation for the third-order tensor flux $\mathbb{S}$, has the same form as (17), the consistency relation for the Piola-Kirchhoff stress field S. The definition of $K = (\nabla g)^{-1}G$ yields the equivalent form of (76)

$$\mathbb{S}_\backslash M^T + \mathbb{S}_d(\nabla g)^T = 0. \tag{79}$$

Consistency in the form (79) for the third-order tensor field $\mathbb{S}$ amounts to twenty-seven scalar equations, namely the relations

$$(\mathbb{S}_\backslash)_{ijp}M_{kp} + (\mathbb{S}_d)_{ijp}\frac{\partial g_k}{\partial x_p} = 0, \tag{80}$$

while consistency for the second-order tensor field S in the form (18) represents the nine scalar equations

$$(S_\backslash)_{ip}M_{jp} + (S_d)_{ip}\frac{\partial g_j}{\partial x_p} = 0. \tag{81}$$

In both relations, we use the Einstein summation convention, and we may write $M_{kp} = \frac{\partial g_k}{\partial x_p} - G_{kp}$.

3.4 The symmetry of $\mathbb{G}$ and the inequality $\det(G+M) - \det G \geq 0$ as internal constraints

Our discussion in Section 2 of the internal constraint $\det(G+M) - \det G \geq 0$ led us to the conclusion that the corresponding reaction stresses are determined by a non-negative scalar field λ^r that vanishes when $\det(G+M) - \det G > 0$ and that enters into the field relations for elasticity with disarrangements. Here we reexamine this internal constraint in the context of gradient-disarrangements, and we study the symmetry condition (60) on $\mathbb{G}$ as an additional internal constraint. These two constraints restrict the arrays $(G, M, \mathbb{G}, \mathbb{M})$ that at the outset lie in the $9+9+27+27$ dimensional space $Lin\mathcal{V} \times Lin\mathcal{V} \times Lin^2\mathcal{V} \times Lin^2\mathcal{V}$ to lie in the region

$$\begin{aligned} \mathcal{R} \quad = \quad & \{(G, M, \mathbb{G}, \mathbb{M}) : \det(G+M) - \det G \geq 0, \\ & \text{and } (\mathbb{G}u)v = (\mathbb{G}v)u \text{ for all } u, v \in \mathcal{V}\}. \end{aligned} \tag{82}$$

We consider the reaction arrays $(R_\backslash, R_d, \mathbb{R}_\backslash, \mathbb{R}_d) \in Lin\mathcal{V} \times Lin\mathcal{V} \times Lin^2\mathcal{V} \times Lin^2\mathcal{V}$ associated points $(G, M, \mathbb{G}, \mathbb{M})$ in $\mathcal{R}$. Adapting the "dissipation axiom" from Section 2.4 to the present context, we require for each $P_0 = (G_0, M_0, \mathbb{G}_0, \mathbb{M}_0)$ in $\mathcal{R}$ that the reaction array $(R_\backslash, R_d, \mathbb{R}_\backslash, \mathbb{R}_d)\mid_{P_0}$ satisfy the condition

$$(R_\backslash, R_d, \mathbb{R}_\backslash, \mathbb{R}_d)\mid_{P_0} \cdot (G'(0), M'(0), \mathbb{G}'(0), \mathbb{M}'(0)) \geq 0 \tag{83}$$

for every smooth curve $\tau \longmapsto (G(\tau), M(\tau), \mathbb{G}(\tau), \mathbb{M}(\tau))$ lying in $\mathcal{R}$. Here, the inner product "$\cdot$" is taken "entrywise", using the appropriate inner product for corresponding entries and then summing the four resulting numbers. An elementary argument shows that the reactions are restricted as follows:

$$(\mathbb{R}_\backslash(P_0)u)v = -(\mathbb{R}_\backslash(P_0)u)v \text{ for all } u, v \in \mathcal{V}\ , \ \ \mathbb{R}_d(P_0) = 0, \tag{84}$$

$$R_\backslash(P_0) = \lambda^r(F_0^{-T} - G_0^{-T}),\ \ R_d(P_0) = \lambda^r F_0^{-T} \tag{85}$$

with $\lambda^r \geq 0$ satisfying $\lambda^r(\det(G_0+M_0) - \det G_0) = 0$. Thus, the reactive stresses $R_\backslash$ and R_d satisfy the same conditions (25) and (26) obtained for elasticity with disarrangements in Section 2.4, while the reactive hyperstress $\mathbb{R}_\backslash$ is skew-symmetric (84) and $\mathbb{R}_d = 0$.

We continue to use $(24)_1$ to define the reactive part S^r of the stress S, and we define the reactive part $\mathbb{S}^r$ of the hyperstress $\mathbb{S}$ throught the formula

$$(\det K)\mathbb{S}^r = \mathbb{R}_\backslash + \mathbb{R}_d. \tag{86}$$

The differences $\mathbb{S} - \mathbb{S}^r$, $\mathbb{S}_\backslash - \mathbb{R}_\backslash$, $\mathbb{S}_d - \mathbb{R}_d$ are called the constitutively determined parts of $\mathbb{S}$, $\mathbb{S}_\backslash$, $\mathbb{S}_d$, respectively, and we continue to call $S - S^r$, $S_\backslash - R_\backslash$, $S_d - R_d$ the constitutively determined parts of S, $S_\backslash$, S_d, respectively.

3.5 Power and balance laws

For a body undergoing second-order structured motions

$$(X, t) \longmapsto (g(X,t), G(X,t), \mathbb{G}(X,t))$$

from the virgin configuration $\kappa_r(\mathcal{B})$, we modify the formulas (29) and (30) for the external and internal power as follows:

$$\begin{aligned} P_{ext}(\mathcal{P}, t, g, G) \quad = \quad & \int_{\partial\mathcal{P}} (S(X,t)n(X) \cdot \dot{g}(X,t) + \mathbb{S}(X,t)n(X) \cdot \dot{G}(X,t)) dA_X \\ & + \int_{\mathcal{P}} (b_{tot}(X,t) \cdot \dot{g}(X,t) + B(X,t) \cdot \dot{G}(X,t)) dV_X \end{aligned} \tag{87}$$

and

$$P_{int}(\mathcal{P}, t, g, G) = \int_{\mathcal{P}} (S(X,t) \cdot \nabla \dot{g}(X,t) + \mathbb{S}(X,t) \cdot \nabla \dot{G}(X,t) dV_X. \tag{88}$$

Here, the third-order tensor field $\mathbb{S}$ has the dimensions of force per unit length or, equivalently, moment per unit area. Its action $\mathbb{S}n$ on the unit normal vector field n for the boundary $\partial\mathcal{P}$ of $\mathcal{P}$ represents the area density of short-range interactions that do work against $\dot{G}$. The second-order tensor field B has the dimensions of force per unit area or, equivalently, moment per unit volume and represents the volume density of long-range interactions that do work against $\dot{G}$. The presence of the non-classical interactions $\mathbb{S}$ and B in these definitions have led others (see, e.g., [35]) in analogous situations to apply the term "generalized continuum" in order to distinguish the present setting from the one described in Subsection 2.5.

Given a second-order structured motion $t \longmapsto (g(\cdot,t), G(\cdot,t), \mathbb{G}(\cdot,t))$ of $\mathcal{B}$, a smooth velocity field $v : \kappa_r(\mathcal{B}) \to \mathcal{V}$, and a tensor field $V : \kappa_r(\mathcal{B}) \to Lin\mathcal{V}$ we define the virtual motion g_v of $\mathcal{B}$ by

$$g_v(X,t) = g(X,t) + tv(X), \tag{89}$$

and the virtual deformation field G_V

$$G_V(X,t) = G(X,t) + tV(X) \tag{90}$$

and require that, for the given fields S, b_r, $\mathbb{S}$, B, ρ_r, g, and G, there holds

$$P_{int}(\mathcal{P}, t, g_v, G_V) = P_{ext}(\mathcal{P}, t, g_v, G_V) \quad \text{for all } v,\, V, t, \text{ and } \mathcal{P}. \tag{91}$$

We require as well that the internal power be frame-indifferent, i.e.,

$$P_{int}(\mathcal{P}, t, r_Q \circ_1 g, QG) = P_{int}(\mathcal{P}, t, g, G) \tag{92}$$

for every rigid motion r:

$$r_Q(y,t) = Q(t)(y - y_o) + w(t), \tag{93}$$

where the tensor field QG in (92) is defined by the relation $(QG)(X,t) := Q(t)G(X,t)$.

The requirement (91) and the arbitrariness of v and V imply via a standard argument the balance laws

$$\operatorname{div} S + b_r = \rho_r \ddot{g} \quad \text{and} \quad \operatorname{div} \mathbb{S} + B = 0, \tag{94}$$

and the frame-indifference of the internal power is equivalent to the symmetry of the second-order tensor $\mathbb{S} \odot \nabla G + S(\nabla g)^T$:

$$(\mathbb{S} \odot \nabla G \;+ S(\nabla g)^T)^T = \mathbb{S} \odot \nabla G \;+ S(\nabla g)^T \tag{95}$$

Here, the second-order tensor $\mathbb{S} \odot \nabla G$ is the "contracted composition" defined in (143). A more detailed discussion of frame-indifference will be given following our discussion of constitutive relations.

The component form of $(94)_2$ is $\frac{\partial \mathbb{S}_{ijk}}{\partial x_k} + B_{ij} = 0$, while the component form of (95) is $\mathbb{S}_{ipq} \frac{\partial G_{jp}}{\partial x_q} + S_{ip} \frac{\partial g_j}{\partial x_p} = \mathbb{S}_{jpq} \frac{\partial G_{ip}}{\partial x_q} + S_{jp} \frac{\partial g_i}{\partial x_p}$. (In both relations, we have used the Einstein summation convention.)

3.6 Additive decomposition of the stress-power

The stress-power in the present setting, $S \cdot \nabla \dot{g} + \mathbb{S} \cdot \nabla \dot{G}$, or, more precisely, the internal power expended per unit volume is given by the integrand in the defining relation (88). We procede in a manner analogous to the discussion in Subsection 2.6 by exploiting the additive decompositions (6), (20), (64), (78) of ∇g, ∇G, $(\det K)S$ and $(\det K)\mathbb{S}$ obtained in Sections 2 and 3 in order to obtain the following additive decomposition of the density $(\det K)(S \cdot \nabla \dot{g} + \mathbb{S} \cdot \nabla \dot{G})$ of internal power:

$$
\begin{aligned}
(\det K)(S \cdot \nabla \dot{g} + \mathbb{S} \cdot \nabla \dot{G}) \quad = \quad & S_\backslash \cdot \dot{G} + S_d \cdot \dot{M} + \mathbb{S}_\backslash \cdot \mathbb{G}^{\cdot} + \mathbb{S}_d \cdot \mathbb{M}^{\cdot} + \\
& S_\backslash \cdot \dot{M} + S_d \cdot \dot{G} + \mathbb{S}_\backslash \cdot \mathbb{M}^{\cdot} + \mathbb{S}_d \cdot \mathbb{G}^{\cdot}. \qquad (96)
\end{aligned}
$$

The first four terms on the right-hand side are "pure" in the sense that each term has two factors having the same attribute: both factors are "without disarrangements" or both factors are "due to disarrangements." By contrast, the fifth through eighth terms are "mixed" in that in each term the two factors have different attributes and so represent the case where contact forces or contact moments are remotely located relative to where geometrical changes occur. This refined representation of the internal power density permits us in the next section to identify through consititutive relations which terms in the additive decomposition (96) will contribute to the internal dissipation and which to the storage of energy.

3.7 Constitutive relations

The generalized balance laws (94) amount to $3 + 9$ scalar relations restricting the fields g, G, and $\mathbb{G}$ as well as the skew-symmetric reaction stresses $\mathbb{R}_\backslash$ that together have $3 + 9 + 18 + 9$ scalar components. (The symmetry of $\mathbb{G}$ reduces the number of independent components from 27 to 18.) There are in the present context two consistency relations that can be used to cover the deficit of twenty-seven scalar equations: the consistency relation for $S_\backslash$ and S_d (18) that amounts to nine scalar equations and the consistency relation for $\mathbb{S}_\backslash$ and $\mathbb{S}_d$ (79) that amounts to twenty-seven scalar relations. In this subsection, we make constitutive assumptions directly on the stress $S - S^r$, on $\mathbb{S}_\backslash - \mathbb{R}_\backslash$ and on $\mathbb{S}_d - \mathbb{R}_d$. The constitutive assumption on $S - S^r$ then determines through (19) the constitutive relations on $S_\backslash - R_\backslash$ and on $S_d - R_d$ in such a way that the consistency relation (18) is satisfied identically, while the constitutive relations on $\mathbb{S}_\backslash - \mathbb{R}_\backslash$ and on $\mathbb{S}_d - \mathbb{R}_d$ through the consistency relation (79) provide 27 scalar restrictions on g, G, and $\mathbb{G}$.

By analogy with the case of elasticity with disarrangements, we assume that ψ, the free energy per unit volume in the virgin configuration $\kappa_r(\mathcal{B})$, is determined by the fields g, G, and $\mathbb{G}$ in the specific form:

$$
\psi(X, t) = \Psi(G(X, t), M(X, t), \mathbb{G}(X, t), \mathbb{M}(X, t)) \qquad (97)
$$

where $M = \nabla g - G$ and $\mathbb{M} = \nabla G - \mathbb{G}$ are the disarrangement tensor and the gradient-disarrangement tensor, respectively. Without making independent constitutive assumptions on $S_\backslash$ and on S_d, we preserve the stress relation (39) from elasticity with disarrangements by assuming directly

$$S - S^r = D_G\Psi + D_M\Psi, \tag{98}$$

and we use the definitions (19) to obtain the relations

$$S_\backslash = (\det K)(D_G\Psi{+}D_M\Psi{+}S^r)K^{-T},\ \ S_d = (\det K)(D_G\Psi{+}D_M\Psi{+}S^r)(I{-}K^{-T}). \tag{99}$$

These formulas and $(24)_1$ permit us to identify the reactive part $R_\backslash$ of $S_\backslash$ and the reactive part R_d of S_d by selecting the terms that are not consititutively determined, so that

$$\begin{aligned} R_\backslash &= (\det K)S^r K^{-T} = (R_\backslash + R_d)K^{-T} \\ R_d &= (\det K)S^r(I - K^{-T}) = (R_\backslash + R_d)(I - K^{-T}). \end{aligned}$$

Each of these relations is equivalent to the relation

$$R_\backslash K^T = R_\backslash + R_d \tag{100}$$

which amounts to the assertion that the reaction stresses obey the consistency relation (18), with $S_\backslash$ and S_d replaced by $R_\backslash$ and R_d, respectively. By means of the formulas (85) for $R_\backslash$ and R_d, the relation (100) becomes

$$\lambda^r\{(G^{-T}F^T)^{-1} + G^{-T}F^T - 3I\} = 0,$$

and arguments given in [3] imply that $\lambda^r = 0$. Thus, the consititutive assumption (98) and the definition $(24)_1$ of S^r imply that the reaction stresses associated with the constraint $\det(G+M) - \det G \geq 0$ all vanish: $R_\backslash = R_d = S^r = 0$. Moreover, (99) with $S^r = 0$ yields the constitutive relations for $S_\backslash$ and S_d :

$$\begin{aligned} S_\backslash &= \det K(D_G\Psi + D_M\Psi)K^{-T} \\ S_d &= \det K(D_G\Psi + D_M\Psi)(I - K^{-T}), \end{aligned} \tag{101}$$

and the arguments that led to (100) together with (101) tell us that $S_\backslash$ and S_d satisfy the consistency relation (17). In other words, the relation (98) and the definition $(24)_1$ of S^r imply that the consistency relation (17) is satisfied in every structured motion and, hence, places no restrictions on the fields g, G, and $\mathbb{G}$.

By analogy with the constitutive assumptions (38) in Section 2, we assume

$$\mathbb{S}_\backslash - \mathbb{R}_\backslash = (\det K)D_{\mathbb{G}}\Psi \quad \text{and} \quad \mathbb{S}_d - \mathbb{R}_d = (\det K)D_{\mathbb{M}}\Psi, \tag{102}$$

where $\mathbb{R}_\backslash$ and $\mathbb{R}_d$ satisfy (84). In the constitutive relations (98), (99), and (102) the functions on the left-hand sides as well as K are evaluated at pairs (X,t) while those on the remaining functions on the right-hand sides are evaluated at the arrays

$$(G(X,t), M(X,t), \mathbb{G}(X,t), \mathbb{M}(X,t)).$$

As noted above the induced consitutive relations (99) on $S_\backslash$ and S_d imply that the consistency relation $S_\backslash K^T = S_\backslash + S_d$ is satisfied identically, i.e., places no restrictions on the fields G, M, $\mathbb{G}$, and $\mathbb{M}$. In contrast, the directly imposed constitutive relations (102) on $\mathbb{S}_\backslash$ and $\mathbb{S}_d$ together with the higher-order consistency relation (79) do restrict G, M, $\mathbb{G}$, and $\mathbb{M}$. Moreover, they imply by means of (78) and (84) the following formula for $\mathbb{S}$ analogous to (98):

$$\mathbb{S} = D_{\mathbb{G}}\Psi + D_{\mathbb{M}}\Psi + \tilde{\mathbb{R}}_\backslash, \tag{103}$$

with $\tilde{\mathbb{R}}_\backslash = \mathbb{R}/(\det K)$ an arbitrary skew third-order tensor. Our constitutive assumptions also permit us to write the internal power density $S \cdot \nabla\dot{g} + \mathbb{S} \cdot \nabla\dot{G}$ in a form analogous to (96)

$$\begin{aligned} S \cdot \nabla\dot{g} + \mathbb{S} \cdot \nabla\dot{G} &= (D_G\Psi + D_M\Psi) \cdot (\dot{G} + \dot{M}) + \\ &\quad (D_{\mathbb{G}}\Psi + D_{\mathbb{M}}\Psi + \tilde{\mathbb{R}}_\backslash) \cdot (\mathbb{G}^{\cdot} + \mathbb{M}^{\cdot}) \\ &= \dot{\psi} + D_G\Psi \cdot \dot{M} + D_M\Psi \cdot \dot{G} + \\ &\quad D_{\mathbb{G}}\Psi \cdot \mathbb{M}^{\cdot} + D_{\mathbb{M}}\Psi \cdot \mathbb{G}^{\cdot} + \tilde{\mathbb{R}}_\backslash \cdot \mathbb{M}^{\cdot} \end{aligned} \tag{104}$$

that identifies the mixed power

$$\pi_{mix} := D_G\Psi \cdot \dot{M} + D_M\Psi \cdot \dot{G} + D_{\mathbb{G}}\Psi \cdot \mathbb{M}^{\cdot} + D_{\mathbb{M}}\Psi \cdot \mathbb{G}^{\cdot} + \tilde{\mathbb{R}}_\backslash \cdot \mathbb{M}^{\cdot} \tag{105}$$

as the (volume density of) internal dissipation $S \cdot \nabla\dot{g} + \mathbb{S} \cdot \nabla\dot{G} - \dot{\psi}$ during a second-order structured motion. In order that the evolution of the body satisfy the Second Law during isothermal processes, we impose as a restriction on processes the inequality

$$\pi_{mix} = D_G\Psi \cdot \dot{M} + D_M\Psi \cdot \dot{G} + D_{\mathbb{G}}\Psi \cdot \mathbb{M}^{\cdot} + D_{\mathbb{M}}\Psi \cdot \mathbb{G}^{\cdot} + \tilde{\mathbb{R}}_\backslash \cdot \mathbb{M}^{\cdot} \geq 0 \tag{106}$$

Because the reaction hyperstress is skew, the reaction term $\tilde{\mathbb{R}}_\backslash \cdot \mathbb{M}^{\cdot}$ in (105) and (106) depends only upon the skew part of the third-order tensor $\mathbb{M}$. The discussion in Section 3.2 shows that the skew part of $\mathbb{M}$ and the defectiveness density $\operatorname{curl} G$ are equivalent as kinematical fields, and we conclude that for the materials described here the rate of change of defectiveness density enters in the last term in the dissipation inequality . The constitutive assumptions (97), (98), (102), and (106) specify an elastic body undergoing gradient-disarrangements.

3.8 Frame-indifference

Constitutive equations are required to be independent of observer, i.e., frame-indifferent. Therefore, we need to record how the variables that enter into the constitutive relations introduced above transform under superposed rigid motions (93). The approximation theorem implies the following transformation rules for G, M, $\mathbb{G}$, $\mathbb{M}$:

$$g \to r_Q \circ g \Longrightarrow G \to QG, M \to QM, \mathbb{G} \to Q\mathbb{G}, \mathbb{M} \to Q\mathbb{M},$$

and the physical and geometrical properties of free energy, stress, and hyperstress imply the transformation rules:

$$g \to r_Q \circ g \Longrightarrow \psi \to \psi, S \to QS, \mathbb{S} \to Q\mathbb{S}.$$

Frame-indifference of the constitutive relation (97) for ψ implies by means of a standard argument the transformation rules for the partial derivatives of ψ: $g \to r_Q \circ g \Longrightarrow D_A\Psi \to QD_A\Psi$ where A stands for G, M, $\mathbb{G}$, and $\mathbb{M}$. Consequently, the reaction stresses and hyperstresses transform in the same manner as the stress and hyperstress, themselves. These transformation properties then tell us that the constitutive relations (98) and (102) are frame-indifferent, because both sides of the equality symbol have the same transformation properties.

It remains to require that the dissipation inequality (106) be frame-indifferent. This requirement implies that the mixed power π_{mix} on the left-hand side of (106) is invariant under superposed rigid motions. With the above transformation properties at hand, this amounts to the assertion

$$QD_G\Psi\cdot(QM)^{\cdot}+QD_M\Psi\cdot(QG)^{\cdot}+QD_{\mathbb{G}}\Psi\cdot(Q\mathbb{M})^{\cdot}+QD_{\mathbb{M}}\Psi\cdot(Q\mathbb{G})^{\cdot}+Q\tilde{\mathbb{R}}_{\backslash}\cdot(Q\mathbb{M})^{\cdot}$$

$$= D_G\Psi \cdot \dot{M} + D_M\Psi \cdot \dot{G} + D_{\mathbb{G}}\Psi \cdot \mathbb{M}^{\cdot} + D_{\mathbb{M}}\Psi \cdot \mathbb{G}^{\cdot} + \tilde{\mathbb{R}}_{\backslash} \cdot \mathbb{M}^{\cdot}, \tag{107}$$

which is easily shown to be equivalent to the condition that the second-order tensor $D_G\Psi M^T + D_M\Psi G^T + D_{\mathbb{G}}\Psi \odot \mathbb{M} + D_{\mathbb{M}}\Psi \odot \mathbb{G} + \tilde{\mathbb{R}}_{\backslash} \odot \mathbb{M}$ be symmetric, i.e.,

$$sk\{D_G\Psi M^T + D_M\Psi G^T + D_{\mathbb{G}}\Psi \odot \mathbb{M} + D_{\mathbb{M}}\Psi \odot \mathbb{G} + \tilde{\mathbb{R}}_{\backslash} \odot \mathbb{M}\} = 0. \tag{108}$$

It is worth noting that the frame-indifference of the free energy response and the frame-indifference of the mixed power (108) imply the symmetry condition (95) which, in turn, is equivalent to the frame-indifference of the internal power. The verification of this assertion follows a similar argument to the one provided in [3] showing that the frame indifference of the free energy response and of the mixed power imply the symmetry of the Cauchy stress tensor.

3.9 Field relations for elasticity with gradient-disarrangements

We now are in a position to assemble the field relations derived above for an elastic body undergoing gradient-disarrangements through second-order structured motions $t \longmapsto (g(\cdot,t), G(\cdot,t), \mathbb{G}(\cdot,t))$, with $\det G \le \det(G+M)$ and the symmetry of $\mathbb{G}$ treated as internal constraints. The preassigned free energy response function $(G, M, \mathbb{G}, \mathbb{M}) \longmapsto \Psi(G, M, \mathbb{G}, \mathbb{M})$ in (97) provides the constitutive input that complements the balance laws, consistency relations, dissipation inequality, and conditions of frame-indifference already discussed in detail. In recording the field relations below, we recall the additive decompositions (6), $\nabla g = G + M$, and (64), $\nabla G = \mathbb{G} + \mathbb{M}$, that relate the second-order tensor fields G and M to the macroscopic deformation gradient ∇g and the third-order tensor fields $\mathbb{G}$ and $\mathbb{M}$ to the gradient of deformation without disarrangements ∇G.

- balance laws:
$$\rho_r \ddot{g} = \operatorname{div}(D_G \Psi \; + D_M \Psi) + \rho_r b_r \tag{109}$$
with ρ_r and b_r the mass density and body force per unit volume in the virgin configuration, and
$$\operatorname{div}(D_{\mathbb{G}} \Psi \; + D_{\mathbb{M}} \Psi + \tilde{\mathbb{R}}_\backslash) + B = 0 \tag{110}$$
with B the body moments per unit volume in the virgin configuration,

- consistency relation:
$$(D_{\mathbb{G}} \Psi + \tilde{\mathbb{R}}_\backslash) \; M^T + D_{\mathbb{M}} \Psi \; (\nabla g)^T = 0 \tag{111}$$

- frame-indifference of the mixed power:
$$sk\{D_G \Psi \, M^T + D_M G^T + D_{\mathbb{G}} \Psi \; \odot \; \mathbb{M} + D_{\mathbb{M}} \Psi \odot \mathbb{G} + \tilde{\mathbb{R}}_\backslash \odot \mathbb{M}\} = 0, \tag{112}$$

- dissipation inequality:
$$D_G \Psi \cdot \dot{M} + D_M \cdot \dot{G} + D_{\mathbb{G}} \Psi \; \cdot \mathbb{M}^{\cdot} + D_{\mathbb{M}} \Psi \cdot \mathbb{G}^{\cdot} + \tilde{\mathbb{R}}_\backslash \cdot \mathbb{M}^{\cdot} \geq 0, \tag{113}$$

- (weakened) accommodation inequality:
$$0 < \det G \leq \det \nabla g. \tag{114}$$

In (112) and (113), the stresses $S_\backslash$ and S_d are given in terms of Ψ through the formulas (101). In all of the field relations listed above, the partial derivatives of the free energy response function Ψ are to be evaluated at the arrays of fields $(G, M, \mathbb{G}, \mathbb{M}) = (G, \nabla g - G, \mathbb{G}, \nabla G - \mathbb{G})$. The equations (109), (110), (111), and (112) are then partial-differential equations for the macroscopic deformation g, the deformation without disarrangements G, and the deformation without gradient-disarrangements $\mathbb{G}$. We call (109) - (114) the field relations for elasticity with gradient-disarrangements. The relations (109) - (114) amount to $3+9+27+3$ scalar equations and two inequalities restricting the fields g, G, $\mathbb{G}$, $\tilde{\mathbb{R}}_\backslash$ that amount to $3+9+18+9$ scalar unknowns. ($\mathbb{G}$ represents 18 rather than 27 scalar unknowns due to the symmetry required in (60); $\tilde{\mathbb{R}}_\backslash$ represents 9 rather than 27 scalar unknows due to the skew-symmetry required in $(84)_1$.) The stress relation (98) (with $S^r = 0$) and the analogous higher-order relation (103) relate the area densities of contact forces Sn and contact moments $\mathbb{S}n$ to the second-order structured motion $t \longmapsto (g(\cdot, t), G(\cdot, t), \mathbb{G}(\cdot, t))$. As discussed earlier in Subsection 3.2, the skew part of the third order tensor field $\mathbb{M} = \nabla G - \mathbb{G}$, or, equivalently, the second-order tensor field $\operatorname{curl} G$, provides the density of defects arising in the motion.

We close this subsection with the observation that, even in the presence of the internal constraints $\det G \leq \det \nabla g$ and the symmetry condition on $\mathbb{G}$, the frame-indifference of the free energy response and the frame-indifference of the mixed power (112) continue to imply the symmetry condition (95) which, in turn, is equivalent to the frame-indifference of the internal power.

3.10 Coherent, submacroscopically affine motions and strain-gradient elasticity

The case $\mathbb{G} = 0$ and $M = 0$ is of particular interest for applications of elasticity with gradient-disarrangements. The relation $\mathbb{G} = 0$ and $(61)_3$ tell us that the second gradients $\nabla^2 f_n$ of approximating deformations f_n vanish in the limit, so that the piecewise smooth approximations f_n become more nearly piecewise affine as n tends to infinity. Accordingly, we use the term "submacroscopically affine" to describe a second-order structured deformation of the form $(g(\cdot,t), G(\cdot,t), 0)$. The additional restriction $M = 0$ and (5) show that, on average, the jumps $[f_n]$ in approximating deformations converge to zero as n tends to ∞, and we use the term "coherent" to describe this aspect of the present case. We may think of the submacroscopic geometrical changes associated with a coherent, submacroscopically affine deformation as causing the body to be comprised of tiny domains, each approximately affinely deformed, with jumps in only the deformation gradient and with those jumps occuring across domain walls. The case of fine mixtures of phases [16] - [19] mentioned in the introduction fits well into this geometrical setting.

In this context the decomposition (64) takes the simpler form

$$\nabla^2 g = \nabla G = \mathbb{M}, \tag{115}$$

so that $\nabla^2 g$ captures at the macrolevel the disarrangements associated with $[\nabla f_n]$. These considerations show that, for a given coherent, submacroscopically affine structured deformation $(g(\cdot,t), \nabla g(\cdot,t), 0)$, the fields $G = \nabla g$ and $\nabla G = \nabla^2 g$ provide specific and complementary geometrical information: ∇g reflects through (7) the average smooth submacroscopic deformation of the domains, and $\nabla^2 g$ reflects through (62) the jumps in displacement gradient across the domain walls. In this manner, the case $\mathbb{G} = 0$ and $M = 0$ provides a precisely defined but restricted geometrical setting for elasticity with gradient-disarrangements. We note that a classical deformation $(g, \nabla g, \nabla^2 g)$ is submacroscopically affine if and only if g is itself an affine deformation.

Because $\mathbb{G} = 0$ and $M = 0$ are kinematical constraints that imply satisfaction of the kinematical constraints $\det G \leq \det \nabla g$ (with equality) and $(\mathbb{G}u)v = (\mathbb{G}v)u$ for all $u, v \in V$, we may incorporate all four constraints into the present theory by introducing mechanical reactions $R_\backslash$, R_d, $\mathbb{R}_\backslash$, $\mathbb{R}_d$, corresponding to the two constraints $\mathbb{G} = 0$ and to $M = 0$, that are not constitutively determined and that satisfy an obvious analogue of the "dissipation axiom". It is then clear that the second-order tensor of reactive stresses R_d and the third-order tensor of reactive hyperstresses $\mathbb{R}_\backslash$ are arbitrary, while $R_\backslash$ and $\mathbb{R}_d$ vanish. The stress relations (99) and (102) then become:

$$S_\backslash = (\det K)(D_G\Psi + D_M\Psi)K^{-T}, \tag{116}$$

$$S_d = (\det K)((D_G\Psi + D_M\Psi)(I - K^{-T}) + R_d \tag{117}$$

$$\mathbb{S}_\backslash = (\det K)(D_{\mathbb{G}}\Psi + \tilde{\mathbb{R}}_\backslash), \ \ \mathbb{S}_d = (\det K)D_{\mathbb{M}}\Psi, \tag{118}$$

and the relations (98) and (103) now read

$$S = D_G\Psi + D_M\Psi + \tilde{R}_d, \quad \mathbb{S} = D_{\mathbb{G}}\Psi + D_{\mathbb{M}}\Psi + \tilde{\mathbb{R}}_\backslash. \tag{119}$$

with $\tilde{R}_d = R_d/(\det K)$ and $\tilde{\mathbb{R}}_\backslash = \mathbb{R}_\backslash/(\det K)$. Together with $(119)_1$ the relations (19) among the stresses S, $S_\backslash$, and S_d tell us that $\tilde{R}_d K^{-T} = 0$ and $\tilde{R}_d(I - K^{-T}) = \tilde{R}_d$, so that the reaction stress $\tilde{R}_d$ is zero in $(119)_1$. In $(119)_2$ $\tilde{\mathbb{R}}_\backslash$ is an arbitrary third-order tensor, and all of the partial derivatives of Ψ are evaluated at the array $(\nabla g, 0, 0, \nabla^2 g)$.

The field relations (109) - (113) are now replaced by

- balance equations

$$\rho_r \ddot{g} = \operatorname{div}(D_G\Psi \ + D_M\Psi) + \rho_r b_r \tag{120}$$

$$\operatorname{div}(D_{\mathbb{G}}\Psi + \tilde{\mathbb{R}}_\backslash) + B = 0 \tag{121}$$

- consistency relation

$$D_{\mathbb{M}}\Psi = 0 \tag{122}$$

- frame-indifference

$$sk\{D_M\Psi(\nabla g)^T + (D_{\mathbb{G}}\Psi + \tilde{\mathbb{R}}_\backslash) \odot \nabla^2 g\} = 0 \tag{123}$$

- dissipation inequality

$$D_M\Psi \cdot \nabla\dot{g} + (D_{\mathbb{G}}\Psi + \tilde{\mathbb{R}}_\backslash) \cdot \nabla^2\dot{g} \geq 0. \tag{124}$$

In these relations the partial derivatives of Ψ continue to be evaluated at the array $(\nabla g, 0, 0, \nabla^2 g)$. The field relations amount to $3 + 9 + 27 + 3$ scalar equations and one inequality that, together, restrict the unknown fields g and $\tilde{\mathbb{R}}_\backslash$ that amount to $3 + 27$ unknown scalar fields.

If we assume further that the body moment field B vanishes, then we can choose the arbitrary reaction stress field $\tilde{\mathbb{R}}_\backslash$ so that $D_{\mathbb{G}}\Psi + \tilde{\mathbb{R}}_\backslash = 0$. Consequently, the hyperstress balance (121) is satisfied, and the field relations reduce to

$$\rho_r \ddot{g} = \operatorname{div}(D_G\Psi + D_M\Psi) \ + \rho_r b_r \tag{125}$$

$$D_{\mathbb{M}}\Psi = 0 \tag{126}$$

$$sk\{D_M\Psi(\nabla g)^T\} = 0 \tag{127}$$

$$D_M\Psi \cdot \nabla\dot{g} \geq 0 \tag{128}$$

In these relations, D_G, D_M, $D_{\mathbb{M}}$ denote, respectively, differentiation with respect to the first, second, and fourth variables of the free energy response function $(G, M, \mathbb{G}, \mathbb{M}) \longmapsto \Psi(G, M, \mathbb{G}, \mathbb{M})$. The derivative $D_{\mathbb{G}}\Psi$ no longer appears because we have chosen $\tilde{\mathbb{R}}_\backslash$ so that $\mathbb{S}_\backslash = D_{\mathbb{G}}\Psi + \tilde{\mathbb{R}}_\backslash = 0$. The consistency relation (126) amounts to the condition that, for a given macroscopic deformation gradient field ∇g, the gradient-disarrangement density field $\nabla^2 g$ renders the free energy stationary at each point of the body. The resulting stress field $(D_G\Psi + D_M\Psi)(\nabla g, 0, 0, \nabla^2 g)$ then satisfies balance of linear momentum according to the first relation.

A further simplification arises when the free energy response satisfies

$$D_M\Psi(G, 0, 0, \mathbb{M}) = 0 \text{ for all } G \text{ and } \mathbb{M},$$

in which case the frame-indifference condition (127) is satisfied and the dissipation inequality (128) is satisfied with equality. The field equations (125) and (126) comprise the following system of partial differential equations for the macroscopic deformation field g

$$\left\{ \begin{array}{c} \rho_r \ddot{g} = \operatorname{div} D_G\Psi(\nabla g, 0, 0, \nabla^2 g) \; + \rho_r b_r \\ D_{\mathbb{M}}\Psi(\nabla g, 0, 0, \nabla^2 g) = 0. \end{array} \right. \tag{129}$$

Here, as above, D_G and $D_{\mathbb{M}}$ denote differentiation with respect to the first and fourth entries in the list of variables $(G, M, \mathbb{G}, \mathbb{M})$ upon which Ψ depends. We observe that, in the presence of the internal constraints $M = 0$ and $\mathbb{G} = 0$, the frame-indifference of the free energy response and the frame-indifference of the mixed power (127) continue to imply the symmetry condition (95) that, because $\mathbb{S} = 0$, reduces in this case to the statement that the Cauchy stress is symmetric.

We suppose now that at submacroscopic levels the material response favors particular values of ∇f_n that correspond to phases of the body and also favors jumps between the favored values of ∇f_n. For coherent submacroscopically affine deformations, ∇g reflects through (7) the average smooth submacroscopic deformation, and $\nabla^2 g$ reflects through (62) averages of the jumps in deformation gradient ∇f_n. The response $(\nabla g, \nabla^2 g) \longmapsto \Psi(\nabla g, 0, 0, \nabla^2 g)$ then can be assigned in such a way that these preferences are captured in the field equations (129). In this manner, these field equations are expected to describe fine mixtures of coherent phases and to supply through the presence of $\nabla^2 g$ a length scale not available in models that do not capture the effects of gradient-disarrangements.

The relation $(129)_1$ is a field equation central to theories of "strain-gradient elasticity" used to describe a variety of phenomena associated with microstructural changes in elastic bodies and with localization of macroscopic deformation. The reader is referred to [25],[31] - [34] for a variety of ways of broadening elasticity by the inclusion of higher-order gradients of deformation. The article [25] provides such a broadening in which, like the present approach, the Cauchy stress continues to be symmetric.

Relation $(129)_2$ in the context of strain-gradient elasticity is new and can place a strong restriction on the structured motions $(g, \nabla g, 0)$ available to the body. For example, if

$$\Psi(G, 0, 0, \mathbb{M}) = \Psi_1(G) + \Psi_2(\mathbb{M}) \;\text{ for all } G \text{ and } \mathbb{M}$$

and if

$$D\Psi_2(\mathbb{M}) = 0 \quad \text{if and only if} \quad \mathbb{M} = 0,$$

then the consistency relation $(129)_2$ amounts to the requirement $\nabla^2 g = \nabla G - \mathbb{G} = \mathbb{M} = 0$. This leads one to seek solutions g of $(129)_1$ that are piecewise affine, a setting already studied in the context of phase mixtures [37] through a theory of piecewise rigid body mechanics.

3.11 Defect-dominant gradient energetics and strain-gradient plasticity

We consider now an alternative specialization of the full field relations for elasticity with gradient-disarrangements (109) - (114) in which the only contribution of ∇G to the stored energy is through the defectiveness density $\operatorname{curl} G$. By the relations (69) and $\nabla G = \mathbb{G} + \mathbb{M}$, it is equivalent to consider in place of $\operatorname{curl} G$ the skew, third-order tensor field $\tilde{\mathbb{M}}$ defined from the gradient-disarrangements field $\mathbb{M}$ by

$$(u, v) \longmapsto \tilde{\mathbb{M}}(u, v) := (\mathbb{M}u)v - (\mathbb{M}v)u \tag{130}$$

and to assume that the free energy response is of the form:

$$\Psi(G, M, \mathbb{G}, \mathbb{M}) = \tilde{\Psi}(G, M, \tilde{\mathbb{M}}). \tag{131}$$

This relation and (69) make explicit the assumption that ∇G contributes to the stored energy only through the defectiveness density. We use the term defect-dominant gradient energetics to describe the particular specialization embodied in (131).

We note that (131) yields the simplifications

$$D_{\mathbb{G}}\Psi = 0, \;\; D_{\mathbb{M}}\Psi(G, M, \mathbb{G}, \mathbb{M}) = D_{\tilde{\mathbb{M}}}\tilde{\Psi}(G, M, \tilde{\mathbb{M}})$$

and implies in the case $B = 0$ and with the choice $\tilde{\mathbb{R}}_{\backslash} = -D_{\tilde{\mathbb{M}}}\tilde{\Psi}$ the following simpler version of the field relations (109) - (114):

$$\begin{aligned} \rho_r \ddot{g} &= \operatorname{div}(D_G \tilde{\Psi} + D_M \tilde{\Psi}) + \rho_r b_r && (132) \\ D_{\tilde{\mathbb{M}}}\tilde{\Psi}(G, \nabla g - G, \tilde{\mathbb{M}}) &= 0 && (133) \end{aligned}$$

$$sk\{D_G \tilde{\Psi}\,(\nabla g - G)^T + D_M \tilde{\Psi} G^T\} = 0 \tag{134}$$

$$D_G \tilde{\Psi} \cdot (\nabla \dot{g} - \dot{G}) + D_M \tilde{\Psi} \cdot \dot{G} \geq 0 \tag{135}$$

$$0 < \det G \leq \det \nabla g. \tag{136}$$

In deriving these relations we have used the symmetry of $\mathbb{G}$ and the skew-symmetry of $\tilde{\mathbb{M}}$ and of $D_{\tilde{\mathbb{M}}}\tilde{\Psi}$. Moreover, the stresses $S_{\backslash}$ and S_d are given in terms of $\tilde{\Psi}$ through the formulas (101) with Ψ replaced by $\tilde{\Psi}$, and the hyperstresses all vanish: $\mathbb{S} = \mathbb{S}_{\backslash} = \mathbb{S}_d = 0$. The field relations consist of $3 + 9 + 3$ equations and two inequalities that restrict the unknown fields g and G that amount to $3+9$ scalar unknowns. As remarked above, the field $\tilde{\mathbb{M}}$ is determined by $\operatorname{curl} G$ that is determined, in turn, by the field G, so that the presence of $\tilde{\mathbb{M}}$ in the field equations does not increase the number of unknowns beyond the fields g and G. Moreover, the field relations above imply that the Cauchy stress is symmetric.

The field relations (132) - (136) can be restated in notation more familiar in theories of strain-gradient plasticity. Following [1] we note that the additive decomposition $\nabla g = G+M$ implies the multiplicative decomposition $\nabla g = F_e F_p$ in which $F_e = G$ and $F_p = I + G^{-1}M$. Therefore, we have $\operatorname{curl} G = \operatorname{curl} F_e$, and our analysis in Section 3.2 shows that the 9 independent components of $\tilde{\mathbb{M}}$ determine the 9 components of $\operatorname{curl} F_e$. In this manner, we can recast (132) - (136) entirely in terms of the two fields g and F_e. Moreover, as is the case in theories of plasticity, we may assume that $\det F_p = 1$ or, equivalently, that $\det F_e = \det \nabla g$, i.e., that the accommodation inequality (136) is satisfied with equality. This assumption places the kinematical setting of the theory within the class of invertible structured deformations [1]. That setting was used in [39] to derive standard representations of the field $\dot{F}_p F_p^{-1}$ in terms of fundamental metrics of multiple slip in single crystals. As noted in [39], such representations are assumed at the outset in standard treatments of multiple slip in single crystals. We note in closing that the consistency relation (133) now reads

$$D_{\operatorname{curl} F^e} \tilde{\Psi}(F^e, \nabla g - F^e, \operatorname{curl} F^e) = 0 \tag{137}$$

and amounts to the assertion that, given ∇g and F^e at each (X,t), the defectiveness density $\operatorname{curl} F^e$ at each (X,t) renders stationary the free energy at (X,t).

4 Appendix on third-order tensors

We let $\mathcal{V}$ denote the three-dimensional translation space of physical space and $Lin\mathcal{V}$ the space of linear mappings A from $\mathcal{V}$ into itself. $\mathcal{V}$ is an inner-product space, as is $Lin\mathcal{V}$, and we use without danger of confusion the symbol $\cdot$ to denote both inner products. The two inner products are related by

$$A \cdot B = \sum_{i=1}^{3} Ae_i \cdot Be_i = tr(A^T B) \tag{138}$$

for every A, $B \in Lin\mathcal{V}$ and for every orthnormal basis e_1,e_2,e_3 of $\mathcal{V}$. Here, A^T is the unique element C of $Lin\mathcal{V}$ that satisfies

$$Av \cdot w = v \cdot Cw \quad \text{for all } v, w \in \mathcal{V},$$

i.e., A^T is defined unambiguosly by the relation

$$Av \cdot w = v \cdot A^T w \quad \text{for all } v, w \in \mathcal{V}. \tag{139}$$

It is natural in the present context to view second-order tensors as elements A of $Lin\mathcal{V}$ and to view third-order tensors as elements Λ of $Lin(\mathcal{V}, Lin\mathcal{V})$, i.e., as linear mappings on $\mathcal{V}$ whose values are in $Lin\mathcal{V}$, so that $\Lambda v \in Lin\mathcal{V}$ for every $v \in \mathcal{V}$. For each third-order tensor Λ, we may define, along the lines of (139), the transpose Λ^T to be the unique element $\Omega \in Lin(Lin\mathcal{V}, \mathcal{V})$ that satisfies

$$\Lambda v \cdot A = v \cdot \Omega A \quad \text{for every } v \in \mathcal{V} \ \text{ and } \ A \in Lin\mathcal{V},$$

so that Λ^T is defined unambiguosly by the relation

$$\Lambda v \cdot A = v \cdot \Lambda^T A \quad \text{for every } v \in \mathcal{V} \ \text{ and } \ A \in Lin\mathcal{V}. \tag{140}$$

It follows for every $\Lambda, \Xi \in Lin(Lin\mathcal{V}, \mathcal{V})$ that $\Lambda^T \Xi \in Lin\mathcal{V}$ is a second-order tensor, and we define

$$\Lambda \cdot \Xi = tr(\Lambda^T \Xi). \tag{141}$$

It is straightforward to show that this formula defines an inner product on the vector space $Lin(\mathcal{V}, Lin\mathcal{V})$ of third-order tensors .

For each second-order tensor A and third-order tensor Λ, the symbol $A\Lambda \in Lin(\mathcal{V}, Lin\mathcal{V})$ denotes the composition of the linear mappings A and Λ, i.e.,

$$(A\Lambda)v = A(\Lambda v) \quad \text{for all } v \in \mathcal{V}. \tag{142}$$

so that the composition of a second-order and a third-order tensor is a third-order tensor. With this in mind, we define for each pair of third-order tensors $\Lambda, \Xi \in Lin(\mathcal{V}, Lin\mathcal{V})$ the second-order tensor $\Lambda \odot \Xi \in Lin\mathcal{V}$ to be the unique $B \in Lin\mathcal{V}$ satisfying

$$(A\Xi) \cdot \Lambda = A \cdot B \quad \text{for every } A \in Lin\mathcal{V},$$

so that

$$(A\Xi) \cdot \Lambda = A \cdot (\Lambda \odot \Xi) \quad \text{for every } A \in Lin\mathcal{V}. \tag{143}$$

The component forms (with respect to an orthonormal basis e_1, e_2, e_3) of quantities identified above are:

$$A_{ij} := e_i \cdot A e_j \tag{144}$$

$$\Lambda_{ijk} := e_i \cdot ((\Lambda e_k) e_j) \tag{145}$$

$$\Lambda^T_{ijk} := \Lambda_{jki} \tag{146}$$

$$\Lambda \cdot \Xi = \sum_{i,j,k=1}^{3} \Lambda_{ijk} \Xi_{ijk} \tag{147}$$

$$(\Lambda \odot \Xi)_{ij} = \sum_{k,l=1}^{3} \Lambda_{ikl} \Xi_{jkl}. \tag{148}$$

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References

[1] Del Piero, G., Owen, D.R.: Structured deformations of continua, Arch. Rational. Mech. Anal. 124 99-155 (1993)

[2] Del Piero, G., Owen, D.R.: Integral-gradient formulae for structured deformations, Arch. Rational. Mech. Anal. 131 121-138 (1995)

[3] Deseri, L., Owen, D.R.: Toward a field theory for elastic bodies undergoing disarrangements, J. Elas. 70 197-236 (2003)

[4] Del Piero, G., Owen, D.R: Structured Deformations, XXII Scuola Estiva di Fisica Matematica, Ravello - Settembre 1997, Quaderni dell' Istituto Nazionale di Fisica Matematica (2000)

[5] Owen, D.R., Field equations for elastic constituents undergoing disarrangements and mixing, in Quaderni di matematica, Volume 20, edited by Miroslav Šilhavý, Seconda Università di Napoli (2007)

[6] Deseri, L., Owen, D.R.: Submacroscopically stable equilibria of elastic bodies undergoing dissipation and disarangements, Math. Mech. Solids 15 611-638 (2010)

[7] Deseri, L., Owen, D.R.: Moving interfaces that separate loose and compact phases of elastic aggregates: a mechanism for drastic reduction or increase in macroscopic deformation, Cont. Mech. Thermo. 25 311-341 (2013) DOI:10.1007/s00161-012-0260-y

[8] Deseri, L., Owen, D.R.: Stable disarrangement phases of elastic aggregates: a setting for the emergence of no-tension materials with non-linear response in compression, Meccanica. 49 2907-2932 (2014) DOI:10.1007/s1 1012–0042-7014

[9] Deseri, L., Owen, D.R.: Stable disarrangement phases arising from expansion/contraction or from simple shearing of a model granular medium, Int. J. Eng. Sci. 96 111-140 (2015)

[10] Deseri, L., Owen, D.R.: Submacroscopic disarrangements induce a unique, additive and universal decomposition of continuum fluxes, J. Elasticity 122 223-230 (2016)

[11] Owen, D.R., Paroni, R.: Second-order structured deformations, Arch. Rational Mech. Anal. 155 215-235 (2000)

[12] Rittel, D., Roman, I.: Tensile deformation of coarse-grained cast austenitic manganese steels, Mat. Sci. Engng A110 77-87 (1989)

[13] Rittel, D., Aharonov, R., Feigin, G., Roman, I: Experimental investigation of surface instabilities in cylindrical tensile metallic specimens, Acta Metall. Mater. 39 719-724 (1991)

[14] Hill, R.: On discontinuous plastic states, with special reference to localized necking in thin sheets, J. Mech. Phys. Solids 1 19-30 (1952)

[15] Bigoni, D.: Nonlinear Solid Mechanics - Bifurcation Theory and Material Instability, Cambridge University Press (2012)

[16] Schryvers, D., Ma, Y., Toth, L., Tanner, L.: Electron microscopy study of the formation of Ni_5Al_3 in a $Ni_{62.5}Al_{37.5}B2$ alloy II:plate crystallography, Acta Metall. Mater. 43 4057-4065 (1995)

[17] Tan, S., Xu, H.: Observations on a CuAlNi single crystal, Cont. Mech. Thermodyn. 2 241-244 (1990)

[18] James, R.D., Hane, K.F.: Martensitic transformations and shape-memory materials, Acta Mater. 48 197-222 (2000)

[19] Truskinovsky, L., Zanzotto, G.: Finite-scale microstructure and metastability in one-dimensional elasticity, Meccanica 30 577-589 (1995)

[20] Fleck, N.A., Muller, G.M., Ashby, M.F., Hutchinson, J.W.: Strain gradient plasticity: theory and experiment, Acta Metall. Mater. 42 475-487 (1994)

[21] Hall, E.O.: The deformation and aging of mild steel, Proc. Phys. Soc. London B 64 747-753 (1951)

[22] Petch, N.J.: The cleavage strength of polycrystals, J. Iron Steel Inst. 174 25-28 (1953)

[23] Carlton, C.E., Ferreira, P.J.: What is behind the inverse Hall-Petch effect in nanocrystalline materials? Acta Mater. 55 3749-3756 (2007)

[24] Aifantis, E.C., Serrin, J.B.: Equilibrium solutions in the mechanical theory of fluid microstructures, J. Coll. and Interf. Sci. 96 530-547 (1983)

[25] Triantafyllidis, N., Aifantis, E.C.: A gradient approach to localization of deformation. I Hyperelastic materials, J. Elasticity 16 225-237 (1986)

[26] Aifantis, E.C.: Remarks on media with microstructures, Int. J. Eng. Sci. 22 961-968 (1984)

[27] Fleck, N.A., Hutchinson, J.W.: Strain gradient plasticity, Advances in Applied Mechanics 33 295-361 (1997)

[28] Acharya, A., Bassani, J.I.: Lattice incompatibility and a gradient theory of crystal plasticity, J. Mech. Phys. Solids 48 1565-1595 (2000)

[29] Gurtin, M.E., Fried, E., Anand, L.: The Mechanics and Thermodynamics of Continua, Cambridge University Press, 2010

[30] Frommert, M., Gottstein, G.: Mechanical behavior and microstructure evolution during steady-state dynamic recrystallization in the austenitic steel, Mat. Sci. Eng. A 506 101-110 (2008)

[31] Toupin, R.A.: Elastic materials with couple-stresses, Arch. Rat. Mech. Anal. 11 385-414 (1962)

[32] Mindlin, R.D.: Micro-structures in Linear Elasticity, Arch. Rat. Mech. Anal. 16 51-78 (1964)

[33] Green, A.E., Rivlin, R.S.: Multipolar Continuum Mechanics, Arch. Rat. Mech. Anal. 17 113-147 (1964)

[34] Lam, D.C.C., Yang, F., Chang, A.C.M., Wang, J., Tong, P.: Experiments and theory in strain gradient elasticity, J. Mech. Phys. Solids 51 1477-1508 (2003)

[35] Del Piero, G.: Nonclassical continua, pseudobalance, and the law of action and reaction, Math. Mech. Complex Sys. 2 71-107 (2014)

[36] Noll, W.: The geometry of contact, separation, and reformation of continuous bodies, Arch. Rational Mech. Anal. 122, 197-212 (1993)

[37] James, R.D., Rizzoni, R.: Piecewise rigid body mechanics, J. Nonlinear Sci. 13 65-114 (2003)

[38] Capriz, G., Podio-Guidugli, P.: Internal Constraints, Appendix 3A in Rational Thermodynamics, Second Edition by C. Truesdell, Springer, 1982

[39] Deseri, L., Owen, D. R.: Invertible structured deformations and the geometry of multiple slip in single crystals, Int. J. Plasticity 18 833-849 (2002)