Regularity Results for an Optimal Design Problem with Quasiconvex Bulk Energies

Menita Carozza, Irene Fonseca & Antonia Passarelli di Napoli

January 24, 2018

Abstract

Regularity results for equilibrium configurations of variational problems involving both bulk and surface energies are established. The bulk energy densities are uniformly strictly quasiconvex functions with quadratic growth, but are otherwise not subjected to any further structure conditions. For a minimal configuration (u, E), partial Hölder continuity of the gradient of the deformation u is proved, and partial regularity of the boundary of the minimal set E is obtained.

AMS Classifications. 49N15, 49N60, 49N99.

Key words. regularity, nonlinear variational problem, free interfaces.

1 Introduction and statements

In this paper we study a large class of multidimensional vectorial variational problems involving both bulk and surface energies, relevant to a plethora of problems issuing from material science and imaging science. The regularity of solutions to these problems is a rather subtle issue even in the scalar setting. In [4, 24] the authors established existence and regularity of minimal configurations of the model problem

$$\int_{\Omega} \sigma_E(x)|\nabla u|^2 \, dx + P(E, \Omega) \tag{1.1}$$

with $u = 0$ on $\partial\Omega$ and $\sigma_E(x) := a\chi_E + b\chi_{\Omega\setminus E}$ for $a > b$ positive constants, where $\Omega \subset \mathbb{R}^n$ is an open, bounded domain, $E \subset \Omega$, and $P(E, \Omega)$ stands for the perimeter of the set E in Ω. In [25] the authors treated more general bulk interfacial energies of the form

$$\mathcal{I}(u, E) := \int_{\Omega} \left(F(x, u, \nabla u) + \chi_E G(x, u, \nabla u)\right) \, dx + P(E, \Omega) \, ,$$

subject to the constraints

$$u = \Phi \ \text{ on } \ \partial\Omega \ \text{ and } \ |E| = d,$$

requiring that F and G satisfy restrictive structure assumptions and are convex and with quadratic growth with respect to the gradient variable. Recently in [8] we still dealt with constrained convex scalar problems, without requiring any additional structure assumption on the bulk energies, and considering a general p-growth condition with respect to the gradient.

This work is a natural extension of the above mentioned papers to the vectorial setting under the assumption of quasiconvexity on the bulk energies. To be precise, we consider an energy of the type

$$\mathcal{I}(v, A) := \int_\Omega \left(F(Dv) + \chi_{_A} G(Dv)\right)\, dx + P(A, \Omega)\,, \tag{1.2}$$

where $A \subset \Omega$ is a set of finite perimeter, $u \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N)$, χ_A is the characteristic function of the set A and $P(A,\Omega)$ denotes the perimeter of A in Ω. We assume that $F,\, G\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ are C^2 integrands satisfying, for $p > 1$ and for positive constants $\ell_1,\ \ell_2,\ L_1,\ L_2 > 0$ and $\mu \geq 0$, the following growth and uniformly strict p-quasiconvexity hypotheses,

$$0 \leq F(\xi) \leq L_1(\mu^2 + |\xi|^2)^{\frac{p}{2}}\,, \tag{F1$_{\rm p}$}$$

$$\int_\Omega F(\xi + D\varphi)\, dx \geq \int_\Omega \left(F(\xi) + \ell_1 |D\varphi|^2(\mu^2 + |D\varphi|^2)^{\frac{p-2}{2}}\right) dx\,, \tag{F2$_{\rm p}$}$$

and

$$0 \leq G(\xi) \leq L_2(\mu^2 + |\xi|^2)^{\frac{p}{2}}\,, \tag{G1$_{\rm p}$}$$

$$\int_\Omega G(\xi + D\varphi)\, dx \geq \int_\Omega \left(G(\xi) + \ell_2 |D\varphi|^2(\mu^2 + |D\varphi|^2)^{\frac{p-2}{2}}\right) dx \tag{G2$_{\rm p}$}$$

for every $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in C^1_0(\Omega;\mathbb{R}^N)$.
We will say that a pair (u, E) is a local minimizer of $\mathcal{I}$ in Ω, if for every open set $U \Subset \Omega$ and every pair (v, A) where A is a set of finite perimeter with $A\Delta E \Subset U$ and $v - u \in W^{1,p}_0(U;\mathbb{R}^N)$, we have

$$\int_U \left(F(\nabla u) + \chi_{_E} G(\nabla u)\right)\, dx + P(E, U) \leq \int_U \left(F(\nabla v) + \chi_{_A} G(\nabla v)\right)\, dx + P(A, U).$$

Existence and regularity of local minimizers of integral functionals of the type

$$\int_\Omega F(x, Du),$$

with uniformly strict p-quasiconvex integrand F and smooth dependence on the x variable, have been widely investigated (we refer to [1, 2, 9, 10, 11, 18, 26] and for an exhaustive treatment to [17, 20]). However, as far as we know, neither the existence nor the regularity of the local minimizers for functionals involving both bulk and surface energies of the form (1.2), are available in literature. Our results here are a first step to fill this gap.

We first establish the existence of minimizers of $\mathcal{I}$.

Theorem 1.1. *Let $p > 1$ and assume that* (F1)$_{\rm p}$, (F2)$_{\rm p}$, (G1)$_{\rm p}$ *and* (G2)$_{\rm p}$ *hold. Then, for $v \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N)$ and a set of finite perimeter in Ω, $A \subset \Omega$, and for every sequence (v_k, A_k) such that v_k weakly converges to v in $W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N)$ and $\chi_{_{A_k}}$ strongly converges to $\chi_{_A}$ in $L^1_{\rm loc}(\Omega)$, we have*

$$\mathcal{I}(v, A) \leq \liminf_{k\to+\infty} \mathcal{I}(v_k, A_k).$$

In particular, $\mathcal{I}$ admits minimal configurations $(u, \chi_E) \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N) \times BV_{\rm loc}(\Omega;[0,1])$.

Next we establish a partial regularity result for minimal configurations of the functional $\mathcal{I}(v, A)$. Here we focus on the case of quadratic growth (i.e. $p = 2$). The case of general p-growth, which will be treated in a forthcoming paper. We note that when $p = 2$, $(\mathrm{F1})_{\mathrm{p}}$, $(\mathrm{F2})_{\mathrm{p}}$, $(\mathrm{G1})_{\mathrm{p}}$ and $(\mathrm{G2})_{\mathrm{p}}$ reduce to

$$0 \le F(\xi) \le L_1(\mu^2 + |\xi|^2)\,, \tag{F1}$$

$$\int_\Omega F(\xi + D\varphi)\, dx \ge \int_\Omega \Big(F(\xi) + \ell_1 |D\varphi|^2\Big)\, dx\,, \tag{F2}$$

and

$$0 \le G(\xi) \le L_2(\mu^2 + |\xi|^2)\,, \tag{G1}$$

$$\int_\Omega G(\xi + D\varphi)\, dx \ge \int_\Omega \Big(G(\xi) + \ell_2 |D\varphi|^2\Big)\, dx. \tag{G2}$$

Theorem 1.2. *Assume that (F1)-(F2) and (G1)-(G2) hold, and let (u, E) be a local minimizer of $\mathcal{I}$. Then there exist an exponent $\beta \in (0, 1)$ and an open set $\Omega_0 \subset \Omega$ with full measure such that $u \in C^{1,\beta}(\Omega_0)$. Also, $\partial^* E \cap \Omega_0$ is a $C^{1,\frac{1}{2}}$-hypersurface in Ω_0, and $\mathcal{H}^s((\partial E \setminus \partial^* E) \cap \Omega_0) = 0$ for all $s > n - 8$.*
If, in addition,

$$\frac{L_2}{\ell_1 + \ell_2} < 1 \tag{H}$$

then there exists an open set $\Omega_1 \subset \Omega$ with full measure such that $u \in C^{1,\alpha}(\Omega_1)$ for every $\alpha \in (0, \frac{1}{2})$.

As it is usual in the vectorial setting, the proof is based on a comparison argument with solutions of a suitable linearized system, aiming at establishing decay estimates of some excess functions. The essential tool here is the use of suitable "hybrid" excess functions $U_*(x_o, \rho)$ and $U_{**}(x_0, \rho)$ (see (5.1) and (5.50) respectively) that describe the oscillations of the gradient of the minimal deformation u and of the perimeter of the minimal set E in a ball. The decay estimates are achieved by considering points in Ω at which the excess is small, and using a blow-up argument reducing the problem to the study of convergence of the minimal configurations (u_h, E_h) of a suitable rescaled functionals in the unit ball. This argument is hinged on two Caccioppoli type inequalities for minimizers of suitable perturbed rescaled functionals. Due to the particular form of our functional, these Caccioppoli type inequalities (see (5.18) and (5.60)) also involve quantities depending on the perimeter of the rescaled minimal set E_h. In order to ensure that these terms vanish in the passage to the limit, we need to establish suitable a priori estimates for the perimeter of E_h.

Remark 1.1. *Theorems 1.1 and 1.2 apply, in particular, to energies of the form*

$$(u, E) \mapsto \int_\Omega (\chi_E F_1(Du) + (1 - \chi_E) F_2(Du))\; dx + P(E, \Omega),$$

obtained from $\mathcal{I}$ by setting $F := F_2$ and $G := F_1 - F_2$, with F_1 and $F_1 - F_2$ strict 2-quasiconvex functions, and

$$F_1(\xi) \ge F_2(\xi) \qquad \text{for all } \xi \in \mathbb{R}^{N \times n}.$$

Note that such assumptions are the natural extension to the vectorial setting of the model case (1.1) *treated in [4, 24], recalling that there $a > b$.*

We end the Introduction by refering to [6] where regularity for vector-valued free interface variational problems is treated within the context of k-th order homogeneous partial differential operators $\mathcal{A}$ (for a detailed study of $\mathcal{A}$-quasiconvexification see [15]), and $\sigma_E|\nabla u|^2$ in (1.1) becomes $\tilde{\sigma}_E(x)\mathcal{A}u \cdot u$, with $\tilde{\sigma}_E := \sigma_1\chi_E + \sigma_2\chi_{\Omega\setminus E}$, σ_1 and σ_2 being two positive symmetric tensord not necessarily well-ordered. Themorem 1.5 in [6] provides $C^{1,\eta/2}$ regularity for some $\eta \in [0,1]$ while in Theorem 1.2 we achieve $C^{1,1/2}$.

2 Notations and Preliminary Results

We denote by c a generic constant that may vary form expression to expression in the same formula and between formulas. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norms we use on $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{N\times n}$ are the standard Euclidean norms, denoted by $|\cdot|$. In particular, for matrices $\xi, \eta \in \mathbb{R}^{N\times n}$ we write $\langle \xi, \eta\rangle := \text{trace}(\xi^T\eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi\rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm. When $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ we write $a \otimes b \in \mathbb{R}^{N\times n}$ for the tensor product defined as the matrix that has the element $a_r b_s$ in its r-th row and s-th column. Observe that $(a \otimes b)x = (b \cdot x)a$ for $x \in \mathbb{R}^n$, and $|a \otimes b| = |a||b|$.

Let $B_r(x_0)$ be the ball centered at x_0 with radius r, and set

$$(u)_{x_0,r} = \fint_{B_r(x_0)} u(x)\, dx.$$

We omit the dependence on the center when it is clear from the context.

When $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is sufficiently differentiable, we write

$$D_\xi F(\xi)[\eta] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\xi + t\eta) \quad \text{and} \quad D_{\xi\xi}F(\xi)[\eta, \eta] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} F(\xi + t\eta)$$

for ξ, $\eta \in \mathbb{R}^{N\times n}$. Hereby, $F'(\xi)$ is interpreted both as an $N \times n$ matrix and as the corresponding linear form on $\mathbb{R}^{N\times n}$, though $|F'(\xi)|$ will always denote the Euclidean norm of the matrix $F'(\xi)$. The second derivative, $F''(\xi)$, is a real bilinear form on $\mathbb{R}^{N\times n}$.

It is well–known that for quasiconvex C^1 integrands, the assumptions (F1) and (G1) yield the upper bounds

$$|D_\xi F(\xi)| \le c_1 L_1(\mu^2 + |\xi|^2)^{\frac{1}{2}} \qquad \text{and} \qquad |D_\xi G(\xi)| \le c_2 L_2(\mu^2 + |\xi|^2)^{\frac{1}{2}} \tag{2.1}$$

for all $\xi \in \mathbb{R}^{N\times n}$, with c_1 and c_2 constants (see [26] or Lemma 5.2 in [20]). Further, if F and G are C^2, then (F2) and (G2) imply the following strong Legendre-Hadamard conditions

$$D^2F(Q)\lambda_i\lambda_j\mu_\alpha\mu_\beta \ge c_3|\lambda|^2|\mu|^2, \qquad D^2G(Q)\lambda_i\lambda_j\mu_\alpha\mu_\beta \ge c_4|\lambda|^2|\mu|^2$$

for all $Q \in \mathbb{R}^{N\times n}$, $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^N$, where $c_3 = c_3(\ell_1)$ and $c_4 = c_4(\ell_2)$ are positive constants (see Proposition 5.2 in [20]).

We will need the following regularity result (see [20, 17])

Proposition 2.1. *Let $v \in W^{1,2}(\Omega; \mathbb{R}^N)$ be such that*

$$\int_\Omega Q^{ij}_{\alpha\beta} D_\alpha v_i D_\beta \varphi_j\, dx = 0$$

for every $\varphi \in C^\infty(\Omega;\mathbb{R}^N)$, where $Q_{ij}^{\alpha\beta}$ are real valued numbers such that $|Q_{\alpha\beta}^{ij}| \le L$ and the strong Legendre–Hadamard condition

$$Q_{\alpha\beta}^{ij}\lambda_i\lambda_j\mu_\alpha\mu_\beta \ge \ell|\lambda|^2|\mu|^2$$

is satisfied for all $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^N$, for some $\ell, L > 0$. Then $v \in C^\infty$, and for any ball $B_R(x_0) \subset \Omega$ the following estimate holds

$$\fint_{B_{\frac{R}{2}}(x_0)} |Dv - (Dv)_{x_0,\frac{R}{2}}|^2\,dx \le cR^2 \fint_{B_R(x_0)} |Dv - (Dv)_{x_0,R}|^2\,dx,$$

where $c = c(n, N, \ell, L)$.

The next iteration lemma has important applications in the regularity theory (for the proof we refer to [20], pp. 191–192).

Lemma 2.1. *Let $0 < \rho < R$ and let $\Phi\colon [\rho, R] \to \mathbb{R}$ be a bounded nonnegative function. Assume that for all $\rho \le s < t \le R$ we have*

$$\Phi(s) \le \vartheta\Phi(t) + A + \frac{B}{(s-r)^\alpha}$$

where $\vartheta \in (0,1)$, α, A, $B \ge 0$ are constants. Then there exists a constant $c = c(\vartheta, \alpha)$ such that

$$\Phi(\rho) \le c\left(A + \frac{B}{(R-\rho)^\alpha}\right)$$

Given a Borel set E in $\mathbb{R}^n$, $P(E,\Omega)$ denotes the perimeter of E in Ω, defined as

$$P(E,\Omega) := \sup\left\{\int_E \mathrm{div}\phi\,dx :\ \ \phi \in C_0^1(\Omega;\mathbb{R}^N),\ |\phi| \le 1\right\}.$$

It is known that, if E is a set of finite perimeter, then

$$P(E,\Omega) = \mathcal{H}^{n-1}(\partial^* E),$$

where

$$\partial^* E := \left\{x \in \Omega :\ \limsup_{\rho\to 0^+} \frac{P(E, B_\rho(x))}{\rho^{n-1}} > 0\right\}$$

is the reduced boundary of E (for more details we refer to [5]). Given a set $E \subset \Omega$ of finite perimeter in Ω, for every ball $B_r(x) \Subset \Omega$ we measure how far E is from being an area minimizer in the ball by setting

$$\psi(E, B_r(x)) := P(E, B_r(x)) - \inf\{P(A, B_r(x)) :\ A\Delta E \Subset B_r(x),\ \chi_A \in BV(\mathbb{R}^n)\}\ .$$

The following regularity result, due to Tamanini (see [28]), asserts that if the excess $\psi(E, B_r(x))$ decays fast enough when $r \to 0$, then E has essentially the same regularity properties of an area minimizing set.

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R}^n$ and let E be a set of finite perimeter satisfying, for some $\sigma \in (0,1)$,*

$$\psi(E, B_r(x)) \le cr^{n-1+2\sigma}$$

for every $x \in \Omega$ and every $r \in (0, r_0)$, with $c = c(x), r_0 = r_0(x)$ local positive constants. Then $\partial^* E$ is a $C^{1,\sigma}$-hypersurface in Ω and $\mathcal{H}^s\left((\partial E \setminus \partial^* E) \cap \Omega)\right) = 0$ for all $s > n-8$.

In order to perform the blow up procedure, it will be convenient to introduce suitable translations of the integrands F and G. To be precise, given a C^1 function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$, $Q \in \mathbb{R}^{N\times n}$ and $\lambda > 0$, set

$$f_{Q,\lambda}(\xi) := \frac{f(Q+\lambda\xi) - f(Q) - D_\xi f(Q)\lambda\xi}{\lambda^2}. \tag{2.2}$$

Lemma 2.2. *Let f be a $C^2(\mathbb{R}^{N\times n})$ function such that*

$$|f(\xi)| \le C|\xi|^2 \qquad \textit{and} \qquad |D_\xi f(\xi)| \le C|\xi|$$

and let $f_{Q,\lambda}(\xi)$ be the function defined in (2.2). Then

$$|f_{Q,\lambda}(\xi)| \le c|\xi|^2 \qquad \textit{and} \qquad |D_\xi f_{Q,\lambda}(\xi)| \le c|\xi| \tag{2.3}$$

for all $\xi \in \mathbb{R}^{N\times n}$ and for some positive constant depending on $|Q|$.

The proof can be found in [1], Lemma II.3, pag. 264.

3 Lower Semicontinuity

This section is devoted to the proof of Theorem 1.1. We recall that a weakly convergent sequence can be truncated in order essentially to obtain an equi-integrable sequence still weakly converging to the same limit. This result is the decomposition lemma proved by Fonseca, Müller and Pedregal (see Lemma 2.3 in [16], see also [1], [12],[22]) .

Lemma 3.1. *Let $(v_k) \subset W^{1,p}(\Omega;\mathbb{R}^N)$ be weakly converging to u. Then, there exists a subsequence (v_{k_j}) and a sequence $(u_j) \subset W^{1,\infty}(\Omega;\mathbb{R}^N)$ such that*
(i) $\mathcal{L}^n(\{v_{k_j} \ne u_j\}) = o(1)$ and $u_j \to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^N)$;
(ii) $(|Du_j|^p)$ is equi-integrable.

We now prove Theorem 1.1.

Proof of Theorem 1.1. Fix $v \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $A \subset \Omega$ a set of finite perimeter in Ω and consider (v_k) weakly converging to $v \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, and (χ_{A_k}) strongly converging in $L^1_{\mathrm{loc}}(\Omega)$ to χ_A, with $\liminf_{k\to\infty} \mathcal{I}(v_k, A_k) < +\infty$. Without loss of generality (and up to the extraction of a subsequence not relabeled), assume that the limits below exist,

$$\lim_{k\to\infty} \int_\Omega F(Dv_k)\,dx, \qquad \lim_{k\to\infty} \int_\Omega \chi_{A_k} G(Dv_k)\,dx \quad \text{and} \quad \lim_{k\to\infty} P(A_k,\Omega).$$

In view of the lower semicontinuity property of the perimeter, and by the quasiconvexity and growth assumption on F (see $(\mathrm{F1})_\mathrm{p}$, $(\mathrm{F2})_\mathrm{p}$), for all $\Omega' \Subset \Omega$

$$P(A,\Omega') \le \liminf_{k\to+\infty} P(A_k,\Omega') \le \lim_{k\to+\infty} P(A_k,\Omega) \tag{3.1}$$

and

$$\int_{\Omega'} F(Dv)\,dx \le \liminf_{k\to\infty} \int_{\Omega'} F(Dv_k)\,dx \le \lim_{k\to\infty} \int_\Omega F(Dv_k)\,dx. \tag{3.2}$$

Moreover, up to the extraction of a further sequence (not relabeled), there exists $(u_k) \in W^{1,\infty}(\Omega;\mathbb{R}^N)$ such that (i)-(ii) in Lemma 3.1 hold. Hence

$$\begin{aligned}
\lim_{k\to\infty}\int_\Omega \chi_{A_k} G(Dv_k)\,dx &\geq \limsup_{k\to\infty}\int_{\{u_k=v_k\}\cap\Omega'} \chi_{A_k} G(Du_k)\,dx \\
&\geq \limsup_{k\to\infty}\int_{\Omega'} \chi_{A_k} G(Du_k)\,dx - \limsup_{k\to\infty}\int_{\{u_k\neq v_k\}\cap\Omega'} \chi_{A_k} G(Du_k)\,dx \\
&\geq \limsup_{k\to\infty}\int_{\Omega'} \chi_{A_k} G(Du_k)\,dx - L_2\limsup_{k\to\infty}\int_{\{u_k\neq v_k\}\cap\Omega'} (\mu^2+|Du_k|^2)^{\frac{p}{2}}\,dx \\
&= \limsup_{k\to\infty}\int_{\Omega'} \chi_{A_k} G(Du_k)\,dx,
\end{aligned} \tag{3.3}$$

where we used $(\mathrm{G1})_\mathrm{p}$ in the second inequality, and the equi-integrability of $(|Du_k|^p)$ and condition (i) in Lemma 3.1 to obtain the last equality. Now,

$$\limsup_{k\to\infty}\int_{\Omega'}\chi_{A_k} G(Du_k)\,dx \geq \limsup_{k\to\infty}\int_{\Omega'}\chi_{A} G(Du_k)\,dx - \limsup_{k\to\infty}\int_{\Omega'}|\chi_{A_k}-\chi_A| G(Du_k)\,dx, \tag{3.4}$$

with

$$\int_{\Omega'}|\chi_{A_k}-\chi_A|G(Du_k)\,dx \leq L_2\int_{\{|\chi_{A_k}-\chi_A|\geq 1\}\cap\Omega'}(\mu^2+|Du_k|^2)^{\frac{p}{2}}\,dx \to 0 \text{ as } k\to\infty,$$

since by Chebyshev's inequality

$$\big|\{|\chi_{A_k}-\chi_A|\geq 1\}\cap\Omega'\big| \leq \int_{\Omega'}|\chi_{A_k}-\chi_A| \to 0 \text{ as } k\to\infty.$$

By the quasiconvexity and growth properties of G we have

$$\limsup_{k\to\infty}\int_{\Omega'}\chi_A G(Du_k)\,dx = \limsup_{k\to\infty}\int_{\Omega'\cap A} G(Du_k)\,dx \geq \int_{\Omega'\cap A} G(Du)\,dx, \tag{3.5}$$

and the conclusion now follows from (3.1)–(3.5), and by letting $\Omega' \nearrow \Omega$. □

4 A higher integrability result

This section is devoted to the proof of a higher integrability result for the gradient of the function u of the minimal configuration (u,E).

Theorem 4.1. *Assume that (F1)-(F2) and (G1)-(G2) hold, and let (u,E) be a local minimizer of $\mathcal{I}$. Then there exists $\delta=\delta(n,N,\ell_1,\ell_2,L_1,L_2)>0$ such that for every ball $B_{2r}(x_0)\Subset\Omega$ it holds*

$$\left(\fint_{B_r(x_0)}|Du|^{2(1+\delta)}\,dx\right)^{\frac{1}{1+\delta}} \leq C\fint_{B_{2r}(x_0)}|Du|^2\,dx + C\mu^2$$

where $C=C(n,N,\ell_1,\ell_2,L_1,L_2)>0$.

Proof. Consider $0 < r < s < t < 2r$ and let $\eta \in C_0^\infty(B_t)$ be a cut-off function between B_s and B_t, i.e., $0 \le \eta \le 1$, $\eta \equiv 1$ in B_s and $|\nabla \eta| \le \frac{c}{t-s}$.
Set

$$\psi_1 := \eta(u - (u)_{x_0,2r}) \qquad\qquad \psi_2 := (1-\eta)(u - (u)_{x_0,2r}).$$

By the uniformly strict quasiconvexity of F in (F2), we have

$$\ell_1 \int_{B_t} |D\psi_1(x)|^2\,dx \le \int_{B_t} F(D\psi_1)\,dx = \int_{B_t} F(Du - D\psi_2)\,dx. \tag{4.1}$$

We write

$$\begin{aligned}
\int_{B_t} F(Du - D\psi_2)\,dx &\le \int_{B_t} F(Du)\,dx + \int_{B_t} F(Du - D\psi_2)\,dx - \int_{B_t} F(Du)\,dx \\
&= \int_{B_t} F(Du)\,dx - \int_{B_t}\int_0^1 DF(Du - \theta D\psi_2)D\psi_2\,d\theta\,dx \\
&\le \int_{B_t} \Big[F(Du) + \chi_E G(Du)\Big]dx - \int_{B_t}\int_0^1 DF(Du - \theta D\psi_2)D\psi_2\,d\theta\,dx \\
&\le \int_{B_t} \Big[F(Du - D\psi_1) + \chi_E G(Du - D\psi_1)\Big]dx \\
&\qquad - \int_{B_t}\int_0^1 DF(Du - \theta D\psi_2)D\psi_2\,d\theta\,dx.
\end{aligned} \tag{4.2}$$

where we used the fact that $G(\xi) \ge 0$ and the minimality of (u, E) with respect to $(u - \psi_1, E)$. Inserting estimate (4.2) in (4.1), and using the upper bound on DF in (2.1), we obtain

$$\begin{aligned}
\ell_1 \int_{B_s} |Du|^2\,dx &= \ell_1 \int_{B_s} |D\psi_1|^2\,dx \le \int_{B_t} F(D\psi_2)\,dx + \int_{B_t} \chi_E G\big(D\psi_2\big)\,dx \\
&\quad + c\int_{B_t\setminus B_s} (\mu^2 + |Du|^2 + |D\psi_2|^2)^{\frac{1}{2}} |D\psi_2|\,dx \\
&\le c\int_{B_t\setminus B_s} |D\psi_2|^2\,dy + c\int_{B_t\setminus B_s} |Du|^2\,dx + c\mu^2|B_t| \\
&\le c\int_{B_t\setminus B_s} |Du|^2\,dx \,+\, c\int_{B_t\setminus B_s} \left|\frac{(u - (u)_{x_0,2r})}{t-s}\right|^2 dx + c\mu^2|B_t|,
\end{aligned}$$

where we used assumptions (F1) and (G1), Young's inequality and the definition of ψ_2. Adding $c\int_{B_s} |Du|^2\,dx$ to both sides of the previous estimate we get

$$\begin{aligned}
(\ell_1 + c)\int_{B_s} |Du|^2\,dx &\le c\int_{B_t} |Du|^2\,dx + c\int_{B_t\setminus B_s} \frac{|u - (u)_{x_0,2r}|^2}{(t-s)^2}\,dx + c\mu^2|B_t| \\
&\le c\int_{B_t} |Du|^2\,dx + c\int_{B_{2r}} \left(\mu^2 + \frac{|u - (u)_{x_0,2r}|^2}{(t-s)^2}\right) dx
\end{aligned}$$

and by the iteration Lemma 2.1 with $\Phi(z) := \int_{B_z} |Du|^2\,dx$ for $z \in [r, 2r]$, $\theta := \frac{c}{\ell_1 + c}$, $A := \int_{B_{2r}} \mu^2$ and $B := \int_{B_{2r}} |u - (u)_{x_0,2r}|^2\,dx$, we deduce that

$$\fint_{B_r} |Du|^2\,dx \le c\fint_{B_{2r}} \left(\mu^2 + \left|\frac{u - u_{2r}}{r}\right|^2\right) dx.$$

The Sobolev-Poincaré inequality ([20], p.102) implies that

$$\fint_{B_r} |Du|^2\,dx \le c\left(\fint_{B_{2r}} |Du|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}} + c\mu^2,$$

with the constant c depending only on n and not on r, and the conclusion follows by virtue of Giaquinta-Modica Theorem ([20], p. 203). □

5 The Decay Estimates

Consider the excess function defined as

$$U(x_0,r) := \fint_{B_r(x_0)} |Du(x) - (Du)_{x_0,r}|^2\,dx,$$

for $B_r(x_0) \subset \Omega$, and let the "hybrid" excess be given by

$$U_*(x_0,r) := \fint_{B_r(x_0)} |Du(x) - (Du)_{x_0,r}|^2\,dx + \frac{P(E,B_r(x_0))}{r^{n-1}} + r. \tag{5.1}$$

Proposition 5.1. *Let (u,E) be a local minimizer of $\mathcal{I}$ under the assumptions (F1), (F2), (G1), (G2) and (H). For every $M > 0$ and every $0 < \tau < \frac{1}{4}$ there exist $\varepsilon_0 = \varepsilon_0(\tau, M)$ and $c_* = c_*(M,\ell_1,L_1,\ell_2,L_2,n,N)$ such that whenever $B_r(x_0) \Subset \Omega$ verifies*

$$|(Du)_{x_0,r}| \le M \qquad \text{and} \qquad U_*(x_0,r) \le \varepsilon_0,$$

then

$$U_*(x_0,\tau r) \le c_*\,\tau\,U_*(x_0,r). \tag{5.2}$$

Proof. In order to prove (5.2), we argue by contradiction. Let $M > 0$ and $\tau \in (0,1/4)$ be such that for every $h \in \mathbb{N}$, $C_* > 0$, there exists a ball $B_{r_h}(x_h) \Subset \Omega$ such that

$$|(Du)_{x_h,r_h}| \le M, \qquad U_*(x_h,r_h) \to 0 \tag{5.3}$$

and

$$U_*(x_h,\tau r_h) \ge C_*\tau U_*(x_h,r_h). \tag{5.4}$$

The constant C_* will be determined later. Remark that we can confine ourselves to the case in which $E \cap B_{r_h}(x_h) \neq \emptyset$, since the case in which $B_{r_h}(x_h) \subset \Omega \setminus E$ is easier because then $U = U_*$.

Step 1. *Blow-up.*

Set $\lambda_h^2 := U_*(x_h,r_h)$, $A_h := (Du)_{x_h,r_h}$, $a_h := (u)_{x_h,r_h}$, and define

$$v_h(y) := \frac{u(x_h + r_h y) - a_h - r_h A_h y}{\lambda_h r_h} \tag{5.5}$$

for $y \in B_1 := B_1(0)$. One can easily check that $(Dv_h)_{0,1} = 0$ and $(v_h)_{0,1} = 0$.

Set

$$E_h := \frac{E - x_h}{r_h}, \qquad E_h^* := \frac{E - x_h}{r_h} \cap B_1.$$

Note that

$$\begin{aligned} U_*(x_h, r_h) &= \fint_{B_1} |Du(x_h + r_h y) - A_h|^2\, dy + \frac{P(E, B(x_h, r_h))}{r_h^{n-1}} + r_h \\ &= \fint_{B_1} |\lambda_h Dv_h|^2\, dy + P(E_h, B_1) + r_h. \end{aligned} \tag{5.6}$$

By the definition of λ_h and by (5.6), it follows that

$$r_h \to 0, \qquad P(E_h, B_1) \to 0, \qquad \frac{r_h}{\lambda_h^2} \le 1, \qquad \fint_{B_1(0)} |Dv_h|^2 \le 1, \qquad \frac{P(E_h, B_1)}{\lambda_h^2} \le 1. \tag{5.7}$$

Therefore, by (5.3) and (5.7), there exist a subsequence of $\{v_h\}$ (not relabeled), $A \in \mathbb{R}^{N\times n}$ and $v \in W^{1,2}(B_1;\mathbb{R}^N)$, such that

$$\begin{aligned} &v_h \rightharpoonup v \text{ weakly in } W^{1,2}(B_1;\mathbb{R}^N), \quad v_h \to v \text{ strongly in } L^2(B_1;\mathbb{R}^N), \\ &A_h \to A, \qquad \lambda_h Dv_h \rightharpoonup 0 \text{ in } L^2(B_1) \text{ and pointwise a.e.,} \end{aligned} \tag{5.8}$$

where we used the fact that $(v_h)_{0,1} = 0$. Moreover, by (5.7) and (5.3), we also deduce that

$$\lim_h \frac{\Big(P(E_h, B_1)\Big)^{\frac{n}{n-1}}}{\lambda_h^2} = \lim_h \Big(P(E_h, B_1)\Big)^{\frac{1}{n-1}} \limsup_h \frac{P(E_h, B_1)}{\lambda_h^2} = 0. \tag{5.9}$$

Therefore, by the relative isoperimetric inequality in a ball (see [5]),

$$\lim_h \min\left\{\frac{|E_h^*|}{\lambda_h^2}, \frac{|B_1 \setminus E_h|}{\lambda_h^2}\right\} \le c \lim_h \frac{(P(E_h, B_1))^{\frac{n}{n-1}}}{\lambda_h^2} = 0. \tag{5.10}$$

We expand F and G around A_h as follows:

$$\begin{aligned} F_h(\xi) &:= \frac{F(A_h + \lambda_h \xi) - F(A_h) - D_\xi F(A_h)\lambda_h \xi}{\lambda_h^2}, \\ G_h(\xi) &:= \frac{G(A_h + \lambda_h \xi) - G(A_h) - D_\xi G(A_h)\lambda_h \xi}{\lambda_h^2}, \end{aligned} \tag{5.11}$$

and we consider the corresponding rescaled functionals

$$\mathcal{I}_h(w) := \int_{B_1(0)} \Big(F_h(Dw)dy + \chi_{E_h^*} G_h(Dw)\Big)\, dy + P(E_h, B_1)\,. \tag{5.12}$$

We claim that v_h satisfies the minimality inequality

$$\mathcal{I}_h(v_h) \le \mathcal{I}_h(v_h + \psi) + \frac{1}{\lambda_h} \int_{B_1} \chi_{E_h^*} D_\xi G(A_h) D\psi(y)\, dy, \tag{5.13}$$

for $\psi \in W_0^{1,2}(B_1)$. Indeed, using the change of variable $x = x_h + r_h y$, the minimality of (u, E) with respect to $(u+\varphi, E)$, for $\varphi \in W_0^{1,2}(B(x_h, r_h))$, setting $\psi(y) := \frac{\varphi(x_h+r_h y)}{r_h}$ yields

$$\begin{aligned}
&\int_{B_1} \Big(F_h(Dv_h(y)) + \chi_{E_h^*} G_h(Dv_h(y))\Big)\, dy \\
&\quad \le \int_{B_1} \Big(F_h(Dv_h(y) + D\psi(y)) + \chi_{E_h^*} G_h(Dv_h(y) + D\psi(y))\Big)\, dy \\
&\quad + \frac{1}{\lambda_h} \int_{B_1} \chi_{E_h^*} D_\xi G(A_h) D\psi(y)\, dy
\end{aligned} \tag{5.14}$$

and (5.13) follows by the definition of $\mathcal{I}_h$ in (5.12).

Next we claim that

$$\begin{aligned}
&\int_{B_1} \Big(F_h(Dv_h(y)) + G_h(Dv_h(y))\Big)\, dy \\
&\quad \le \int_{B_1} \Big(F_h(Dv_h(y) + D\psi(y)) + G_h(Dv_h(y) + D\psi(y))\Big)\, dy \\
&\quad + \frac{L_2}{\lambda_h^2} \int_{(B_1\setminus E_h)\cap \mathrm{supp}\psi} (\mu^2 + |A_h + \lambda_h Dv_h|^2)\, dy,
\end{aligned} \tag{5.15}$$

for all $\psi \in W_0^{1,2}(B_1)$. In fact, the minimality of (u, E) with respect to $(u+\varphi, E)$ for $\varphi \in W_0^{1,2}(B(x_h, r_h))$, implies that

$$\begin{aligned}
\int_{B(x_h,r_h)} (F+G)(Du)\, dx &= \int_{B(x_h,r_h)} \Big[F(Du) + \chi_E G(Du)\Big] dx + \int_{B(x_h,r_h)\setminus E} G(Du) dx \\
&\le \int_{B(x_h,r_h)} \Big[F(Du + D\varphi) + \chi_E G(Du + D\varphi)\Big] dx + \int_{B(x_h,r_h)\setminus E} G(Du) dx \\
&= \int_{B(x_h,r_h)} (F+G)(Du + D\varphi) dx + \int_{B(x_h,r_h)\setminus E} \Big[G(Du) - G(Du + D\varphi)\Big] dx \\
&\le \int_{B(x_h,r_h)} (F+G)(Du + D\varphi) dx + \int_{(B(x_h,r_h)\setminus E)\cap \mathrm{supp}\varphi} G(Du) dx,
\end{aligned} \tag{5.16}$$

where we used that last integral vanishes outside the support of φ and that $G(\xi) \ge 0$. Using the change of variable $x = x_h + r_h y$ in (5.16), we get

$$\begin{aligned}
\int_{B_1} (F+G)(Du(x_h + r_h y)) dy \;\; &\le \;\; \int_{B_1} \Big[(F+G)(Du(x_h + r_h y) + D\varphi(x_h + r_h y)) \\
&\quad + \int_{(B_1\setminus E_h)\cap \mathrm{supp}\psi} \Big[G(Du(x_h + r_h y))\Big] dy
\end{aligned}$$

where, we recall, $\psi(y) := \frac{\varphi(x_h+r_h y)}{\lambda_h r_h}$, or, equivalently, using the definitions of v_h,

$$\begin{aligned}
\int_{B_1} (F+G)(A_h + \lambda_h Dv_h) dy \;\; &\le \;\; \int_{B_1} \Big[(F+G)(A_h + \lambda_h(Dv_h + D\psi)) \\
&\quad + \int_{(B_1\setminus E_h)\cap \mathrm{supp}\psi} \Big[G(A_h + \lambda_h Dv_h)\Big] dy
\end{aligned}$$

for all $\psi \in W_0^{1,2}(B_1)$. Therefore, setting

$$H_h := F_h + G_h$$

by the definition of F_h and G_h in (5.11) and using the assumption (G1), we have that

$$\begin{aligned}\int_{B_1} H_h(Dv_h)dy &\leq \int_{B_1} H_h(Dv_h + D\psi)dy + \frac{1}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap \mathrm{supp}\psi} G(A_h + \lambda_h Dv_h)\,dy \\ &\leq \int_{B_1}\Big[H_h(Dv_h + D\psi)dy + \frac{L_2}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap \mathrm{supp}\psi} (\mu^2 + |A_h + \lambda_h Dv_h|^2)\,dy, \quad (5.17)\end{aligned}$$

i.e. (5.15).

Step 2. *A Caccioppoli type inequality.*

We claim that there exists a constant $c = c(M, \mu, \ell_1, \ell_2, L_1, L_2, n, N)$ such that for every $0 < \rho < 1$ there exists $h_0 \in \mathbb{N}$ such that for all $h > h_0$ we have

$$\int_{B_{\frac{\rho}{2}}} |Dv_h - (Dv_h)_{\frac{\rho}{2}}|^2\,dy \leq c\int_{B_\rho}\left|\frac{v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}}y}{\rho}\right|^2 dy + c\frac{P(E_h, B_1)^{\frac{n}{n-1}}}{\lambda_h^2}. \tag{5.18}$$

We divide the proof into two substeps.

Substep 2.a *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*|$.

Consider $0 < \frac{\rho}{2} < s < t < \rho < 1$ and let $\eta \in C_0^\infty(B_t)$ be a cut off function between B_s and B_t, i.e., $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_s and $|\nabla\eta| \leq \frac{c}{t-s}$. Set $b_h := (v_h)_{B_\rho}$, $B_h := (Dv_h)_{B_{\frac{\rho}{2}}}$, and set

$$w_h(y) := v_h(y) - b_h - B_h y.$$

Define

$$\begin{aligned}\widetilde{F}_h(\xi) &:= \frac{F(A_h + \lambda_h B_h + \lambda_h\xi) - F(A_h + \lambda_h B_h) - D_\xi F(A_h + \lambda_h B_h)\lambda_h\xi}{\lambda_h^2}, \\ \widetilde{G}_h(\xi) &:= \frac{G(A_h + \lambda_h B_h + \lambda_h\xi) - G(A_h + \lambda_h B_h) - D_\xi G(A_h + \lambda_h B_h)\lambda_h\xi}{\lambda_h^2}. \end{aligned} \tag{5.19}$$

It is easy to check that Lemma 2.2 applies to each $\widetilde{F}_h, \widetilde{G}_h$, for some constants that could depend on M (see (5.3)) and also on ρ through $|\lambda_h B_h|$. However, given ρ we may choose h large enough to have $|\lambda_h B_h| < \frac{\lambda_h}{\rho^{\frac{n}{2}}} < 1$. In fact, by (5.7) we have

$$|B_h| = \left|\fint_{B_{\frac{\rho}{2}}} Dv_h\right| \leq \left(\int_{B_{\frac{\rho}{2}}} |Dv_h|^2\right)^{\frac{1}{2}} \cdot \frac{1}{|B_{\frac{\rho}{2}}|^{\frac{1}{2}}} \leq \frac{c}{\rho^{\frac{n}{2}}}, \tag{5.20}$$

and so the constant in (2.3) can be taken independently of ρ.
Set

$$\psi_{1,h}(y) := \eta w_h \qquad \text{and} \qquad \psi_{2,h}(y) := (1 - \eta)w_h.$$

By the uniformly strict quasiconvexity of $\widetilde{F}_h$ we have

$$\ell_1 \int_{B_t} |D\psi_{1,h}(y)|^2\, dy \le \int_{B_t} \widetilde{F}_h(D\psi_{1,h})\, dy = \int_{B_t} \widetilde{F}_h(Dw_h - D\psi_{2,h})\, dy. \tag{5.21}$$

Using the change of variable $x = x_h + r_h y$, the fact that $G(\xi) \ge 0$ and the minimality of (u, E) with respect to $(u + \varphi, E)$ for $\varphi \in W_0^{1,2}(B(x_h, r_h))$, we have

$$\begin{aligned}
&\int_{B_1} F(Du(x_h + r_h y))dy \le \int_{B_1} \Big[F(Du(x_h + r_h y)) + \chi_{E_h^*} G(Du(x_h + r_h y))\Big] dy \\
&\quad \le \int_{B_1} \Big[F(Du(x_h + r_h y) + D\varphi(x_h + r_h y)) + \chi_{E_h^*} G(Du(x_h + r_h y) + D\varphi(x_h + r_h y))\Big] dy,
\end{aligned}$$

i.e., by the definitions of v_h and w_h,

$$\begin{aligned}
&\int_{B_1} F(A_h + \lambda_h B_h + \lambda_h Dw_h)dy \\
&\quad \le \int_{B_1} \Big[F(A_h + \lambda_h B_h + \lambda_h(Dw_h + D\psi)) + \chi_{E_h^*} G(A_h + \lambda_h B_h + \lambda_h(Dw_h + D\psi))\Big] dy
\end{aligned}$$

for $\psi := \frac{\varphi(x_h + r_h y)}{\lambda_h r_h} \in W_0^{1,2}(B_1)$. Therefore, recalling the definitions of $\widetilde{F}_h$ and $\widetilde{G}_h$ in (5.19), we have that

$$\begin{aligned}
\int_{B_1} \widetilde{F}_h(Dw_h)dy &\le \int_{B_1} \Big[\widetilde{F}_h(Dw_h + D\psi) + \chi_{E_h^*} \widetilde{G}_h(Dw_h + D\psi)\Big] dy \\
&\quad + \frac{1}{\lambda_h^2} \int_{B_1} \chi_{E_h^*} \Big[G(A_h + \lambda_h B_h) + D_\xi G(A_h + \lambda_h B_h)\lambda_h(Dw_h + D\psi)\Big)\Big]\, dy.
\end{aligned} \tag{5.22}$$

Choosing $-\psi_{1,h}(y)$ as test function in (5.22), we get

$$\begin{aligned}
\int_{B_t} \widetilde{F}_h(Dw_h)\, dy &\le \int_{B_t} \Big[\widetilde{F}_h\big(Dw_h - D\psi_{1,h}\big)\, dy + \chi_{E_h^*} \widetilde{G}_h\big(Dw_h - D\psi_{1,h}\big)\Big]\, dy \\
&\quad + \frac{1}{\lambda_h^2} \int_{B_1} \chi_{E_h^*} \Big[G(A_h + \lambda_h B_h) + D_\xi G(A_h + \lambda_h B_h)\lambda_h(Dw_h - D\psi_{1,h})\Big]\, dy \\
&= \int_{B_t \setminus B_s} \widetilde{F}_h(D\psi_{2,h})\, dy + \int_{B_t \setminus B_s} \chi_{E_h^*} \widetilde{G}_h\big(D\psi_{2,h}\big)\, dy \\
&\quad + \frac{1}{\lambda_h^2} \int_{B_1} \chi_{E_h^*} \Big[G(A_h + \lambda_h B_h) + D_\xi G(A_h + \lambda_h B_h)\lambda_h D\psi_{2,h})\Big]\, dy \\
&\le \int_{B_t \setminus B_s} \widetilde{F}_h(D\psi_{2,h})\, dy + \int_{B_t \setminus B_s} \chi_{E_h^*} \widetilde{G}_h\big(D\psi_{2,h}\big)\, dy \\
&\quad + c\frac{|E_h^*|}{\lambda_h^2} + \frac{c}{\lambda_h} \int_{E_h^*} |D\psi_{2,h}|\, dy \\
&\le \int_{B_t \setminus B_s} \widetilde{F}_h(D\psi_{2,h})\, dy + \int_{B_t \setminus B_s} \chi_{E_h^*} \widetilde{G}_h\big(D\psi_{2,h}\big)\, dy \\
&\quad + c\frac{|E_h^*|}{\lambda_h^2} + \frac{c}{\lambda_h} \left(\int_{E_h^*} |D\psi_{2,h}|^2\, dy\right)^{\frac{1}{2}} |E_h^*|^{\frac{1}{2}},
\end{aligned} \tag{5.23}$$

for a constant $c = c(M, \mu, L_2)$, and where we used the second estimate in (2.1), Hölder's inequality, and the fact that $|A_h + \lambda_h B_h| \le M + 1$ (see (5.20)). We write

$$\begin{aligned}\int_{B_t} \widetilde{F}_h(Dw_h - D\psi_{2,h})\,dy =& \int_{B_t} \widetilde{F}_h(Dw_h)\,dy + \int_{B_t} \widetilde{F}_h(Dw_h - D\psi_{2,h})\,dy - \int_{B_t} \widetilde{F}_h(Dw_h)\,dy \\ =& \int_{B_t} \widetilde{F}_h(Dw_h)\,dy - \int_{B_t}\int_0^1 D\widetilde{F}_h(Dw_h - \theta D\psi_{2,h})D\psi_{2,h}\,d\theta\,dy. \quad (5.24)\end{aligned}$$

Inserting estimate (5.23) in (5.24), and using the upper bound on $D\widetilde{F}_h$ in Lemma 2.2, we obtain

$$\begin{aligned}\int_{B_t} \widetilde{F}_h(Dw_h - D\psi_{2,h})\,dy \;\le\; & \int_{B_t\setminus B_s} \widetilde{F}_h(D\psi_{2,h})\,dy + \int_{B_t\setminus B_s} \chi_{E_h^*}\widetilde{G}_h\big(D\psi_{2,h}\big)\,dy \\ & + c\int_{B_t\setminus B_s} (|Dw_h| + |D\psi_{2,h}|)|D\psi_{2,h}|\,dy \\ & + c\frac{|E_h^*|}{\lambda_h^2} + \frac{c}{\lambda_h}\left(\int_{E_h^*} |D\psi_{2,h}(y)|^2\,dy\right)^{\frac{1}{2}} |E_h^*|^{\frac{1}{2}}. \quad (5.25)\end{aligned}$$

Hence, combining (5.21) with (5.25), using the properties of η and Lemma 2.2, we obtain

$$\begin{aligned}\ell_1 \int_{B_s} |Dw_h|^2\,dy =\; & \ell_1 \int_{B_s} |D\psi_{1,h}|^2\,dy \le \ell_1 \int_{B_t} |D\psi_{1,h}|^2\,dy \\ \le\; & \int_{B_t\setminus B_s} \widetilde{F}_h(D\psi_{2,h})\,dy + \int_{B_t\setminus B_s} \chi_{E_h^*}\widetilde{G}_h\big(D\psi_{2,h}\big)\,dy \\ & + c\int_{B_t\setminus B_s} (|Dw_h| + |D\psi_{2,h}|)|D\psi_{2,h}|\,dy \\ & + c\frac{|E_h^*|}{\lambda_h^2} + \frac{c}{\lambda_h}\left(\int_{E_h^*} |D\psi_{2,h}(y)|^2\,dy\right)^{\frac{1}{2}} |E_h^*|^{\frac{1}{2}} \\ \le\; & c\int_{B_t\setminus B_s} |D\psi_{2,h}|^2\,dy + c\int_{B_t\setminus B_s} \chi_{E_h^*}|D\psi_{2,h}|^2\,dy + c\int_{B_t\setminus B_s} |Dw_h|^2\,dy + c\frac{|E_h^*|}{\lambda_h^2} \\ \le\; & c\int_{B_t\setminus B_s} |Dw_h|^2\,dy + c\int_{B_t\setminus B_s} \left|\frac{w_h}{t-s}\right|^2\,dy + c\frac{|E_h^*|}{\lambda_h^2},\end{aligned}$$

where we used Young's inequality. Adding to both sides of previous estimate $c\int_{B_s} |Dw_h|^2\,dy$ we get

$$\begin{aligned}(\ell_1 + c)\int_{B_s} |Dw_h|^2\,dy \;\le\; & c\int_{B_t} |Dw_h|^2\,dy + \frac{c}{(t-s)^2}\int_{B_t\setminus B_s} |w_h|^2\,dy + c\frac{|E_h^*|}{\lambda_h^2} \\ \le\; & c\int_{B_t} |Dw_h|^2\,dy + \frac{c}{(t-s)^2}\int_{B_\rho} |w_h|^2\,dy + c\frac{|E_h^*|}{\lambda_h^2} \quad (5.26)\end{aligned}$$

and by the iteration Lemma 2.1, we deduce that

$$\int_{B_{\frac{\rho}{2}}} |Dw_h|^2\,dy \le c\int_{B_\rho} \left|\frac{w_h}{\rho}\right|^2\,dy + c\frac{|E_h^*|}{\lambda_h^2}.$$

Therefore, by the definition of w_h, we conclude that

$$\int_{B_{\frac{\varrho}{2}}} |Dv_h - (Dv_h)_{\frac{\varrho}{2}}|^2\,dy \le c\int_{B_\rho}\left|\frac{v_h-(v_h)_\rho-(Dv_h)_{\frac{\varrho}{2}}y}{\rho}\right|^2 dy + c\frac{|E_h^*|}{\lambda_h^2}, \tag{5.27}$$

which, by the relative isoperimetric inequality and using the hypothesis of this substep that $\min\{|E_h^*|, |B_1\setminus E_h|\} = |E_h^*|$, yields the estimate (5.18).

Substep 2.b *The case* $\min\{|E_h^*|, |B_1\setminus E_h|\} = |B_1\setminus E_h|$.

As in the previous substep, we fix $0 < \frac{\varrho}{2} < s < t < \rho < 1$ and let $\eta \in C_0^\infty(B_t)$ be a cut off function between B_s and B_t, i.e., $0 \le \eta \le 1$, $\eta \equiv 1$ on B_s and $|\nabla\eta| \le \frac{c}{t-s}$. Also, we set $b_h := (v_h)_{B_\rho}$, $B_h := (Dv_h)_{B_{\frac{\varrho}{2}}}$ and define

$$w_h(y) := v_h(y) - b_h - B_h y$$

and

$$\widetilde{H}_h := \widetilde{F}_h(\xi) + \widetilde{G}_h(\xi).$$

Remark that Lemma 2.2 applies to $\widetilde{H}_h$ and so

$$|\widetilde{H}_h(\xi)| \le c(M)|\xi|^2,$$

and by the uniformly strict quasiconvexity conditions (F2) and (G2)

$$\int_{B_1}\widetilde{H}_h(\xi + D\psi)\,dx \ge \int_{B_1}\left(\widetilde{H}_h(\xi) + \widetilde{\ell}|D\psi|^2\right)dx, \tag{5.28}$$

for all $\psi \in W_0^{1,2}(B_1)$, where $\widetilde{\ell}$ is such that

$$\widetilde{\ell} \ge \ell_1 + \ell_2\,.$$

Set

$$\psi_{1,h}(y) := \eta w_h \qquad\text{and}\qquad \psi_{2,h}(y) := (1-\eta)w_h.$$

By (5.28) and since $\widetilde{H}_h(0) = 0$, we have

$$\widetilde{\ell}\int_{B_t}|D\psi_{1,h}(y)|^2\,dy \le \int_{B_t}\widetilde{H}_h(D\psi_{1,h})\,dy = \int_{B_t}\widetilde{H}_h(Dw_h - D\psi_{2,h})\,dy. \tag{5.29}$$

By virtue of the minimality inequality in (5.17) and since $Dv_h = Dw_h + B_h$, we get

$$\begin{aligned}\int_{B_1} H_h(Dw_h + B_h)dy \quad &\le \quad \int_{B_1} H_h(Dw_h + B_h + D\psi)\\ &\quad + \frac{L_2}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap \mathrm{supp}\psi}(\mu^2 + |A_h + \lambda_h B_h + \lambda_h Dw_h|^2)\,dy,\end{aligned}$$

or, equivalently, by the definition of $\widetilde{H}_h$,

$$\int_{B_1}\widetilde{H}_h(Dw_h)dy \quad\le\quad \int_{B_1}\widetilde{H}_h(Dw_h + D\psi)$$

$$+\frac{L_2}{\lambda_h^2}\int_{(B_1\setminus E_h)\cap\mathrm{supp}\psi}(\mu^2+|A_h+\lambda_h B_h+\lambda_h Dw_h|^2)\,dy. \quad (5.30)$$

Choosing $-\psi_{1,h}(y)$ as test function in (5.30) and using the fact that $\widetilde{H}_h(0)=0$, we get

$$\begin{aligned}\int_{B_t}\widetilde{H}_h(Dw_h)\,dy &\le \int_{B_t}\widetilde{H}_h\big(Dw_h(y)-D\psi_{1,h}\big)\,dy\\ &\quad+\frac{L_2}{\lambda_h^2}\int_{(B_t\setminus E_h)}(\mu^2+|A_h+\lambda_h B_h+\lambda_h Dw_h|^2\,dy)\\ &= \int_{B_t\setminus B_s}\widetilde{H}_h\big(D\psi_{2,h}\big)\,dy\\ &\quad+\frac{L_2}{\lambda_h^2}\int_{B_t\setminus E_h}(\mu^2+|A_h+\lambda_h B_h+\lambda_h Dw_h|^2)\,dy. \end{aligned} \quad (5.31)$$

Now we have that

$$\begin{aligned}&\int_{B_t}\widetilde{H}_h(Dw_h-D\psi_{2,h})\,dy\\ &=\int_{B_t}\widetilde{H}_h(Dw_h)\,dy+\int_{B_t}\widetilde{H}_h(Dw_h-D\psi_{2,h})\,dy-\int_{B_t}\widetilde{H}_h(Dw_h)\,dy\\ &=\int_{B_t}\widetilde{H}_h(Dw_h)\,dy-\int_{B_t}\int_0^1 D\widetilde{H}_h(Dw_h-\theta D\psi_{2,h})D\psi_{2,h}\,d\theta\,dy. \end{aligned} \quad (5.32)$$

Hence, inserting the estimate (5.31) in (5.32), by the upper bound on $D\widetilde{H}_h$ given by Lemma 2.2 , we obtain

$$\begin{aligned}\int_{B_t}\widetilde{H}_h(Dw_h-D\psi_{2,h})\,dy &\le \int_{B_t\setminus B_s}\widetilde{H}_h(D\psi_{2,h})\,dy\\ &\quad+\frac{L_2}{\lambda_h^2}\int_{B_t\setminus E_h}(\mu^2+|A_h+\lambda_h B_h+\lambda_h Dw_h|^2)\,dy\\ &\quad+c\int_{B_t\setminus B_s}(|Dw_h|+|D\psi_{2,h}|)|D\psi_{2,h}|\,dy. \end{aligned} \quad (5.33)$$

Combining (5.29) with (5.33) and using again Lemma 2.2 and (5.20), we have

$$\begin{aligned}\widetilde{\ell}\int_{B_s}|D\psi_{1,h}|^2\,dy &\le \widetilde{\ell}\int_{B_t}|D\psi_{1,h}|^2\,dy\\ &\le \int_{B_t\setminus B_s}\widetilde{H}_h(D\psi_{2,h})\,dy+\frac{L_2}{\lambda_h^2}\int_{B_t\setminus E_h}(\mu^2+|A_h+\lambda_h B_h+\lambda_h Dw_h|^2\,dy)\\ &\quad+c\int_{B_t\setminus B_s}(|Dw_h|+|D\psi_{2,h}|)|D\psi_{2,h}|\,dy\\ &\le c\int_{B_t\setminus B_s}|D\psi_{2,h}|^2\,dy+\left(1+\frac{1}{\varepsilon}\right)\frac{c(M,L_2)}{\lambda_h^2}|B_1\setminus E_h|\\ &\quad+(1+\varepsilon)L_2\int_{B_t\setminus E_h}|Dw_h|^2\,dy+c\int_{B_t\setminus B_s}|Dw_h|^2\,dy, \end{aligned}$$

for every $\varepsilon > 0$, and thus

$$\begin{aligned}\widetilde{\ell}\int_{B_s}|Dw_h|^2\,dy &\leq \widetilde{\ell}\int_{B_t}|D\psi_{1,h}|^2\,dy\\ &\leq c\int_{B_t\setminus B_s}|Dw_h|^2\,dy+(1+\varepsilon)L_2\int_{B_t}|Dw_h|^2\,dy\\ &\qquad +c\int_{B_\rho}\left|\frac{w_h}{t-s}\right|^2\,dy+\frac{c(M,L_2)}{\lambda_h^2}|B_1\setminus E_h|.\end{aligned}$$

Using the hole filling technique as in (5.26) , we obtain

$$\begin{aligned}(c+\widetilde{\ell})\int_{B_s}|Dw_h|^2\,dy &\leq \Big(c+(1+\varepsilon)L_2\Big)\int_{B_t}|Dw_h|^2\,dy\\ &\qquad +c\int_{B_\rho}\left|\frac{w_h}{t-s}\right|^2\,dy+\frac{c(M,L_2)}{\lambda_h^2}|B_1\setminus E_h|.\end{aligned}$$

The assumption (H) implies that there exists $\varepsilon > 0$ such that $\frac{(1+\varepsilon)L_2}{\ell_1+\ell_2} < 1$. Therefore we have

$$\frac{c+(1+\varepsilon)L_2}{c+\widetilde{\ell}}\leq\frac{c+(1+\varepsilon)L_2}{c+\ell_1+\ell_2}<1$$

So, by virtue of the iteration Lemma 2.1, from the previous estimate we deduce that

$$\int_{B_{\frac{\rho}{2}}}|Dw_h|^2\,dy\leq c\int_{B_\rho}\left|\frac{w_h}{\rho}\right|^2\,dy+c\frac{|B_1\setminus E_h|}{\lambda_h^2},$$

where $c = c(M, \mu, \ell_1, L_1, \ell_2, L_2, n, N)$. Therefore, by the definition of w_h, we conclude that

$$\int_{B_{\frac{\rho}{2}}}|Dv_h-(Dv_h)_{\frac{\rho}{2}}|^2\,dy\leq c\int_{B_\rho}\left|\frac{v_h-(v_h)_\rho-(Dv_h)_{\frac{\rho}{2}}y}{\rho}\right|^2\,dy+c\frac{|B_1\setminus E_h|}{\lambda_h^2},$$

which, by the relative isoperimetric inequality and since we have $|B_1 \setminus E_h| = \min\{|E_h^*|, |B_1 \setminus E_h|\}$, gives the estimate (5.18).

Step 3. *We prove that*

$$\fint_{B_{\frac{\tau}{2}}}|Dv-(Dv)_{\frac{\tau}{2}}|^2\leq c\tau^2\fint_{B_\tau}|Dv-(Dv)_\tau|^2\,dx, \tag{5.34}$$

for $B_\tau = B_\tau(0)$ with $\tau < 1$. As before, we will divide the proof in two substeps. Let A and v be as introduced in (5.8).

Substep 3.a *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*|$.

We claim that v solves the linear system

$$\int_{B_1}D_{\xi\xi}F(A)DvD\psi\,dy=0,$$

for all $\psi \in C_0^1(B_1)$. Since v_h satisfies (5.13), we have that

$$0 \le \mathcal{I}_h(v_h + s\psi) - \mathcal{I}_h(v_h) + \frac{1}{\lambda_h}\int_{B_1} \chi_{E_h^*} D_\xi G(A_h) sD\psi(y)\, dy, \tag{5.35}$$

for every $\psi \in C_0^1(B_1(0))$ and $s \in (0,1)$. By the definition of $\mathcal{I}_h$ we get

$$\begin{aligned}
0 \ &\le \ \mathcal{I}_h(v_h + s\psi) - \mathcal{I}_h(v_h) + \frac{1}{\lambda_h}\int_{B_1} \chi_{E_h^*} D_\xi G(A_h) sD\psi(y)\, dy \\
&= \ \frac{1}{\lambda_h}\int\limits_{B_1}\left(\int\limits_0^1 \Big[D_\xi F(A_h + \lambda_h(Dv_h + tsD\psi))\Big] sD\psi\, dt \, - D_\xi F(A_h) sD\psi\right) dy \\
&\quad + \frac{1}{\lambda_h}\int\limits_{B_1}\left(\int\limits_0^1 \chi_{E_h^*}[D_\xi G(A_h + \lambda_h(Dv_h + tsD\psi))] sD\psi\, dt \, - \chi_{E_h^*} D_\xi G(A_h) sD\psi\right) dy \\
&\quad + \frac{1}{\lambda_h}\int_{B_1} \chi_{E_h^*} D_\xi G(A_h) sD\psi(y)\, dy \\
&= \ \frac{1}{\lambda_h}\int\limits_{B_1}\left(\int\limits_0^1 \Big[D_\xi F(A_h + \lambda_h(Dv_h + tsD\psi))\Big] sD\psi\, dt \, - D_\xi F(A_h) sD\psi\right) dy \\
&\quad + \frac{1}{\lambda_h}\int\limits_{B_1}\int\limits_0^1 \chi_{E_h^*} D_\xi G(A_h + \lambda_h(Dv_h + tsD\psi))] sD\psi\, dt\, dy
\end{aligned}$$

Dividing by s and taking the limit as $s \to 0$, we deduce that

$$\begin{aligned}
0 \ \ &\le \ \ \frac{1}{\lambda_h}\int\limits_{B_1} \left(D_\xi F(A_h + \lambda_h Dv_h) \, - D_\xi F(A_h)\right) D\psi dy \\
&\quad + \frac{1}{\lambda_h}\int\limits_{B_1} \chi_{E_h^*} D_\xi G(A_h + \lambda_h Dv_h) D\psi\, dy.
\end{aligned} \tag{5.36}$$

We partition the unit ball as

$$B_1 = \mathbf{B}_h^+ \cup \mathbf{B}_h^- = \{y \in B_1 : \ \lambda_h |Dv_h| > 1\} \cup \{y \in B_1 : \ \lambda_h |Dv_h| \le 1\}.$$

By (5.7), we get

$$|\mathbf{B}_h^+| \le \int_{\mathbf{B}_h^+} \lambda_h^2 |Dv_h|^2\, dy \le \lambda_h^2 \int_{\mathbf{B}_h^+} |Dv_h|^2\, dy \le c\lambda_h^2. \tag{5.37}$$

By virtue of the first estimate in (2.1) and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{\lambda_h}\left|\int_{\mathbf{B}_h^+} [D_\xi F(A_h + \lambda_h Dv_h) - D_\xi F(A_h)] D\psi\, dy\right| \\
&\quad \le \frac{c}{\lambda_h}|\mathbf{B}_h^+| + c\int_{\mathbf{B}_h^+} |Dv_h|\, dy
\end{aligned}$$

$$\leq c\lambda_h + c\left(\int_{\mathbf{B}_h^+} |Dv_h|^2\,dy\right)^{\frac{1}{2}} |\mathbf{B}_h^+|^{\frac{1}{2}} \leq c\lambda_h, \tag{5.38}$$

for a constant $c = c(L_1, M)$, thanks to (5.4) (to bound $|A_h| \leq M$), (5.7) and (5.37), and therefore

$$\lim_{h\to\infty} \frac{1}{\lambda_h}\left|\int_{\mathbf{B}_h^+} [D_\xi F(A_h + \lambda_h Dv_h) - D_\xi F(A_h)] D\psi\,dy\right| = 0. \tag{5.39}$$

On $\mathbf{B}_h^-$ we have

$$\begin{aligned}
&\frac{1}{\lambda_h}\int_{\mathbf{B}_h^-} [D_\xi F(A_h + \lambda_h Dv_h) - D_\xi F(A_h)] D\psi\,dy \\
&= \int_{\mathbf{B}_h^-}\int_0^1 D_{\xi\xi} F(A_h + t\lambda_h Dv_h)\,dt Dv_h D\psi\,dy \\
&= \int_{\mathbf{B}_h^-}\int_0^1 [D_{\xi\xi} F(A_h + t\lambda_h Dv_h) - D_{\xi\xi} f(A)]\;dt Dv_h D\psi\,dy \\
&\quad + \int_{\mathbf{B}_h^-}\int_0^1 D_{\xi\xi} f(A)\,dt Dv_h D\psi\,dy.
\end{aligned} \tag{5.40}$$

By (5.3) and the definition of $\mathbf{B}_h^-$ we have that $|A_h + \lambda_h Dv_h| \leq M + 1$ on $\mathbf{B}_h^-$. Hence the uniform continuity of $D_{\xi\xi}F$ on bounded sets implies

$$\begin{aligned}
&\lim_h \left|\int_{\mathbf{B}_h^-}\int_0^1 [D_{\xi\xi} F(A_h + t\lambda_h Dv_h) - D_{\xi\xi} f(A)]\;dt Dv_h D\psi\,dy\right| \\
&\leq \lim_h \int_{\mathbf{B}_h^-}\left|\int_0^1 [D_{\xi\xi} F(A_h + t\lambda_h Dv_h) - D_{\xi\xi} f(A)]\;dt\right| |Dv_h||D\psi|\,dy \\
&\leq \lim_h \left(\int_{\mathbf{B}_h^-}\left|\int_0^1 [D_{\xi\xi} F(A_h + t\lambda_h Dv_h) - D_{\xi\xi} f(A)]\;dt\right|^2 dy\right)^{\frac{1}{2}} ||Dv_h||_{L^2(B_1)}||D\psi||_{L^\infty(B_1)} \\
&\leq c\lim_h \left(\int_{\mathbf{B}_h^-}\left|\int_0^1 [D_{\xi\xi} F(A_h + t\lambda_h Dv_h) - D_{\xi\xi} f(A)]\;dt\right|^2 dy\right)^{\frac{1}{2}} = 0,
\end{aligned} \tag{5.41}$$

where we used (5.7) and the fact that by (5.8)

$$\lambda_h Dv_h \to 0 \qquad \text{a.e. in } B_1.$$

Note that (5.37) yields that $\chi_{\mathbf{B}_h^-} \to \chi_{B_1}$ in L^r for every $r < \infty$. Therefore by (5.7)

$$\begin{aligned}
&\lim_h \left|\int_{\mathbf{B}_h^-}\int_0^1 D_{\xi\xi} f(A)\,dt Dv_h D\psi\,dy - \int_{B_1}\int_0^1 D_{\xi\xi} f(A)\,dt Dv_h D\psi\,dy\right| \\
&\leq \lim_h \int_{B_1} |\chi_{\mathbf{B}_h^-} - \chi_{B_1}|\left|\int_0^1 D_{\xi\xi} f(A)\,dt\right| |Dv_h||D\psi|\,dy \\
&\leq c\lim_h ||\chi_{\mathbf{B}_h^-} - \chi_{B_1}||_{L^2(B_1)}||Dv_h||_{L^2(B_1)} = 0.
\end{aligned} \tag{5.42}$$

Hence using (5.41) and (5.42) in (5.40), we have that

$$\lim_h \frac{1}{\lambda_h}\int_{\mathbf{B}_h^-}[D_\xi F(A_h+\lambda_h Dv_h)-D_\xi F(A_h)]D\psi\,dy=\int_{B_1}D_{\xi\xi}F(A)DvD\psi\,dy. \tag{5.43}$$

By the second estimate in (2.1), we deduce that

$$\begin{aligned}\frac{1}{\lambda_h}\left|\int_{B_1}\chi_{E_h^*}[D_\xi G(A_h+\lambda_h Dv_h)D\psi\,dy\right| &\le \frac{1}{\lambda_h}\int_{B_1}\chi_{E_h^*}\left(\mu^2+|A_h+\lambda_h Dv_h|^2\right)^{\frac12}|D\psi|\,dy\\ &\le \frac{c}{\lambda_h}|E_h^*|+c\int_{E_h^*}|Dv_h|\,dy\\ &\le \frac{c}{\lambda_h}|E_h^*|+c\left(\int_{B_1}|Dv_h|^2\,dy\right)^{\frac12}|E_h^*|^{\frac12}\\ &\le \frac{c}{\lambda_h}|E_h^*|+c|E_h^*|^{\frac12},\end{aligned}$$

for a constant $c=c(L_2,M)$, thanks to (5.3) and (5.7). Since $\min\{|E_h^*|,|B_1\setminus E_h|\}=|E_h^*|$, by (5.10) we have

$$\lim_h\frac{|E_h^*|}{\lambda_h}=0$$

and so

$$\lim_{h\to\infty}\frac{1}{\lambda_h}\left|\int_{B_1}\chi_{E_h^*}D_\xi G(A_h+\lambda_h Dv_h)D\psi\,dy\right|=0. \tag{5.44}$$

By (5.39), (5.43) and (5.44), passing to the limit as $h\to\infty$ in (5.36) yields

$$0\le\int_{B_1}D_{\xi\xi}F(A)DvD\psi\,dy,$$

and with $-\psi$ in place of ψ we get

$$\int_{B_1}D_{\xi\xi}F(A)DvD\psi\,dy=0,$$

i.e., v solves a linear system with constant coefficients. By Proposition 2.1 we deduce that $v\in C^\infty$, and for every $0<\tau<1$, we have

$$⨍_{B_{\frac{\tau}{2}}}|Dv-(Dv)_{\frac{\tau}{2}}|^2\le c\tau^2⨍_{B_\tau}|Dv-(Dv)_\tau|^2\,dx\le c\tau^2,$$

since

$$||Dv||_{L^2(B_1)}\le\limsup_h||Dv_h||_{L^2(B_1)}\le 1.$$

Substep 3.b *The case* $\min\{|E_h^*|,|B_1\setminus E_h|\}=|B_1\setminus E_h|$.

We claim that v solves the linear system

$$\int_{B_1}D_{\xi\xi}(F+G)(A)DvD\psi\,dy=0,$$

for all $\psi \in C_0^1(B_1)$. Arguing as (5.16) and rescaling, we have that

$$\begin{aligned}\int_{B_1} H_h(Dv_h)dy &\leq \int_{B_1} H_h(Dv_h + sD\psi) + c\frac{1}{\lambda_h}\int_{B_1\setminus E_h} s|D\psi|dy \\ &\quad + c\int_{B_1\setminus E_h}\int_0^1 |Dv_h + tsD\psi||sD\psi|\,dt\,dy,\end{aligned}$$

for every $\psi \in C_0^1(B_1(0))$ and for every $s \in (0,1)$, and so

$$\begin{aligned}0 &\leq \int_{B_1} H_h(Dv_h + sD\psi) - \int_{B_1} H_h(Dv_h)dy \\ &\quad + c\frac{1}{\lambda_h}\int_{B_1\setminus E_h} s|D\psi|dy + c\int_{B_1\setminus E_h}\int_0^1 |Dv_h + tsD\psi||sD\psi|\,dt\,dy.\end{aligned}$$

Therefore

$$\begin{aligned}0 &\leq \int_{B_1}\int_0^1 D_\xi H_h(Dv_h + s\theta D\psi)d\theta sD\psi dy \\ &\quad + c\frac{1}{\lambda_h}\int_{B_1\setminus E_h} s|D\psi|dy + \int_{B_1\setminus E_h}\int_0^1 |Dv_h + tsD\psi||sD\psi|\,dt\,dy.\end{aligned}$$

Dividing the previous inequality by s and taking the limit as $s \to 0$, we obtain that

$$\begin{aligned}0 &\leq \int_{B_1} D_\xi H_h(Dv_h)D\psi dy + c\frac{1}{\lambda_h}\int_{B_1\setminus E_h} |D\psi|dy \\ &\quad + \int_{B_1\setminus E_h} |Dv_h||D\psi|\,dy.\end{aligned}$$

By the definition of H_h, we conclude that

$$\begin{aligned}0 &\leq \frac{1}{\lambda_h}\int_{B_1}\Big[D_\xi(F+G)(A_h + \lambda_h Dv_h)D\psi - D_\xi(F+G)(A_h)D\psi\Big]dy \\ &\quad + c\frac{1}{\lambda_h}\int_{B_1\setminus E_h} |D\psi|dy + c\int_{B_1\setminus E_h} |Dv_h||D\psi|\,dy.\end{aligned}$$

Just as before, we partition B_1 as

$$B_1(0) = \mathbf{B}_h^+ \cup \mathbf{B}_h^- = \{y \in B_1 :\ \lambda_h|Dv_h| > 1\} \cup \{y \in B_1 :\ \lambda_h|Dv_h| \leq 1\},$$

and arguing as in (5.39) , we obtain that

$$\lim_{h\to\infty}\frac{1}{\lambda_h}\left|\int_{\mathbf{B}_h^+}[D_\xi(F+G)(A_h + \lambda_h Dv_h) - D_\xi(F+G)(A_h)]D\psi\,dy\right| = 0, \tag{5.45}$$

and as in (5.43),

$$\lim_h \frac{1}{\lambda_h}\int_{\mathbf{B}_h^-}[D_\xi(F+G)(A_h + \lambda_h Dv_h) - D_\xi(F+G)(A_h)]D\psi\,dy$$

$$= \int_{B_1} D_{\xi\xi}(F+G)(A)DvD\psi\, dy. \tag{5.46}$$

Moreover, we have that

$$\begin{aligned} \frac{1}{\lambda_h}\int_{B_1\setminus E_h}|D\psi|dy + \int_{B_1\setminus E_h}|Dv_h||D\psi|\, dy &\le \frac{c}{\lambda_h}|B_1\setminus E_h| + c|B_1\setminus E_h|^{\frac{1}{2}}\left(\int_{B_1\setminus E_h}|Dv_h|^2 dy\right)^{\frac{1}{2}} \\ &\le \frac{c}{\lambda_h}|B_1\setminus E_h| + c|B_1\setminus E_h|^{\frac{1}{2}}, \end{aligned}$$

where we used (5.7). Since $\min\{|E_h^*|, |B_1\setminus E_h|\} = |B_1\setminus E_h|$, by (5.10), we have

$$\lim_h \frac{|B_1\setminus E_h|}{\lambda_h} = 0,$$

and we obtain

$$\lim_h \left[\frac{1}{\lambda_h}\int_{B_1\setminus E_h}|D\psi|dy + \int_{B_1\setminus E_h}|Dv_h||D\psi|\, dy\right] = 0. \tag{5.47}$$

By (5.45), (5.46) and (5.47), passing to the limit as $h\to\infty$ in (5.45) we conclude that

$$0 \le \int_{B_1} D_{\xi\xi}(F+G)(A)DvD\psi\, dy$$

and with $-\psi$ in place of ψ we finally get

$$\int_{B_1} D_{\xi\xi}(F+G)(A)DvD\psi\, dy = 0,$$

asserting the claim. By Proposition 2.1, we deduce also in this case that $v\in C^\infty$ and for every $0<\tau<1$ satisfies estimate (5.34).

Step 4. *An estimate for the perimeters.*
By the minimality of (u,E) with respect to $(u,\widetilde{E})$, where $\widetilde{E}$ is a set of finite perimeter such that $\widetilde{E}\Delta E \Subset B_{r_h}(x_h)$, where $B_{r_h}(x_h)$ are the balls of the contradiction argument, we get

$$\int_{B_{r_h}(x_h)} \chi_E G(Du) + P(E, B_{r_h}(x_h)) \le \int_{B_{r_h}(x_h)} \chi_{\widetilde{E}} G(Du) + P(\widetilde{E}, B_{r_h}(x_h)).$$

Using the change of variable $x = x_h + r_h y$ we have

$$r_h^n \int_{B_1} \chi_{E_h} G(Du(x_h+r_h y))dy + r_h^{n-1}P(E_h, B_1) \le r_h^n \int_{B_1} \chi_{\widetilde{E}_h} G(Du(x_h+r_h y))dy + r_h^{n-1}P(\widetilde{E}_h, B_1),$$

and so

$$r_h\int_{B_1}\chi_{E_h}G(A_h+\lambda_h Dv_h)dy + P(E_h,B_1) \le r_h\int_{B_1}\chi_{\widetilde{E}_h}G(A_h+\lambda_h Dv_h)dy + P(\widetilde{E}_h,B_1). \tag{5.48}$$

Assume first that $\min\{|B_1 \setminus E_h|, |E_h^*|\} = |B_1 \setminus E_h|$. Then by the relative isoperimetric inequality, we have

$$|B_1 \setminus E_h| \le c(n)P(E_h, B_1)^{\frac{n}{n-1}}.$$

By Fubini's Theorem and choosing as a representative of E_h the set of points of density one, we get

$$|B_1 \setminus E_h| \ge \int_\theta^{2\theta} \mathcal{H}^{n-1}(\partial B_\rho \setminus E_h) d\rho,$$

for every $\theta \in (0, 1/4)$, therefore we may choose $\rho \in (\theta, 2\theta)$ such that

$$\mathcal{H}^{n-1}(\partial B_\rho \setminus E_h) \le \frac{c}{\theta} P(E_h, B_1)^{\frac{n}{n-1}}.$$

Set $\widetilde{E}_h := E_h \cup B_\rho$ and observe that

$$P(\widetilde{E}_h, B_1) \le P(E_h, B_1 \setminus \bar{B}_\rho) + \mathcal{H}^{n-1}(\partial B_\rho \setminus E_h).$$

With such a choice of $\widetilde{E}_h$, (5.48) yields

$$P(E_h, B_1) \le r_h \int_{B_1} \chi_{B_\rho} G(A_h + \lambda_h Dv_h) dy + P(E_h, B_1 \setminus \bar{B}_\rho) + \mathcal{H}^{n-1}(\partial B_\rho \setminus E_h)$$

and so by (5.3) and (5.8)

$$\begin{aligned} P(E_h, B_\theta) &\le \frac{c}{\theta} P(E_h, B_1)^{\frac{n}{n-1}} + c(L_2) r_h \int_{B_{2\theta}} (\mu^2 + |A_h + \lambda_h Dv_h|^2)\, dx \\ &\le \frac{c}{\theta} P(E_h, B_1)^{\frac{n}{n-1}} + c(L_2, \mu, M) r_h \theta^n + c(L_2, \mu, M) r_h \lambda_h^2. \end{aligned} \tag{5.49}$$

We arrive at the same conclusion (5.49) if $\min\{|B_1 \setminus E_h|, |E_h^*|\} = |E_h^*|$, choosing as a competiting set $\widetilde{E}_h = E_h \setminus B_\rho$.

Step 5. *Conclusion.*

Using the change of variable $x = x_h + r_h y$ and the Caccioppoli inequality in (5.18), for every $0 < \tau < \frac{1}{4}$ we have

$$\begin{aligned} \limsup_{h\to\infty} \frac{U_*(x_h, \tau r_h)}{\lambda_h^2} &\le \limsup_{h\to\infty} \frac{1}{\lambda_h^2} \fint_{B_{\tau r_h}(x_h)} |Du(x) - (Du)_{x_h, \tau r_h}|^2\, dx \\ &\quad + \limsup_{h\to\infty} \frac{P(E, B(x_h, \tau r_h))}{\lambda_h^2 \tau^{n-1} r_h^{n-1}} + \limsup_{h\to\infty} \frac{\tau r_h}{\lambda_h^2} \\ &\le c \limsup_{h\to\infty} \fint_{B_\tau} |Dv_h - (Dv_h)_\tau|^2\, dy + \limsup_{h\to\infty} \frac{P(E_h, B_\tau)}{\lambda_h^2 \tau^{n-1}} + \tau \\ &\le c \limsup_{h\to\infty} \fint_{B_{2\tau}} \frac{|v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^2}{\tau^2}\, dy \\ &\quad + \frac{c}{\tau^n} \limsup_{h\to\infty} \frac{P(E_h, B_1)^{\frac{n}{n-1}}}{\lambda_h^2} + \frac{c}{\tau^{n-1}} \limsup_{h\to\infty} \left(\frac{r_h \tau^n}{\lambda_h^2} + r_h \right) + \tau, \end{aligned}$$

where we used (5.7) and estimate (5.49). By virtue of the strong convergence of $v_h \to v$ in $L^2(B_1)$, since $(Dv_h)_\tau \to (Dv)_\tau$ in $\mathbb{R}^{nN}$, by (5.8), (5.9), (5.10) and by the Poincaré–Wirtinger inequality, we get

$$\begin{aligned}\limsup_{h\to\infty} \frac{U_*(x_h,\tau r_h)}{\lambda_h^2} &\leq c \fint_{B_{2\tau}} \frac{|v-(v)_{2\tau}-(Dv)_\tau y|^2}{\tau^2}\,dy + c\tau \\ &\leq c \lim_{h\to\infty} \fint_{B_{2\tau}} |Dv-(Dv)_\tau|^2\,dy + c\tau \\ &\leq c\tau^2 + c\tau \leq C\tau,\end{aligned}$$

where we used the estimate (5.34), and where $C = C(M,\mu,\ell_1,L_1,\ell_2,L_2,n,N)$. The contradiction follows, by choosing C_* such that $C_* > C$, since by (5.4)

$$\liminf_h \frac{U_*(x_h,\tau r_h)}{\lambda_h^2} \geq C_*\tau.$$

□

Next, we obtain a suitable decay estimate that allow us to prove Theorem 1.2 without the assumption (H). To this aim, we introduce a new "hybrid" excess as

$$U_{**}(x_0,r) := \fint_{B_r(x_0)} |Du(x)-(Du)_{x_0,r}|^2\,dx + \left(\frac{P(E,B_r(x_0))}{r^{n-1}}\right)^{\frac{\delta}{1+\delta}} + r^\beta, \tag{5.50}$$

where δ has been determined in Theorem 4.1 and $0<\beta<\frac{\delta}{1+\delta}$.
In the proof of Proposition 5.2 we will only elaborate on the steps that substantially differ from the corresponding ones in the proof of Proposition 5.1.

Proposition 5.2. *Let (u,E) be a local minimizer of $\mathcal{I}$ under the assumptions (F1), (F2), (G1) and (G2). For every $M>0$ and every $0<\tau<\frac{1}{4}$ there exist $\varepsilon_0=\varepsilon_0(\tau,M)$ and $c_{**}=c_{**}(M,\ell_1,L_1,\ell_2,L_2,n,N)$ for which whenever $B_r(x_0)\Subset\Omega$ verifies*

$$|(Du)_{x_0,r}| \leq M \qquad \text{and} \qquad U_{**}(x_0,r)\leq \epsilon_0,$$

then

$$U_{**}(x_0,\tau r) \leq c_{**}\,\tau^\beta\, U_{**}(x_0,r). \tag{5.51}$$

Proof. In order to prove (5.51), we argue by contradiction. Let $M>0$ and $\tau\in(0,1/4)$ be such that that for every $h\in\mathbb{N}$, $C_{**}>0$, there exists a ball $B_{r_h}(x_h)\Subset\Omega$ such that

$$|(Du)_{x_h,r_h}| \leq M, \qquad U_{**}(x_h,r_h)\to 0 \tag{5.52}$$

but

$$U_{**}(x_h,\tau r_h) \geq C_{**}\tau^\beta U_{**}(x_h,r_h). \tag{5.53}$$

The constant C_{**} will be determined later. Remark that we can confine ourselves to the case in which $E\cap B_{r_h}(x_h)\neq\emptyset$, since the case in which $B_{r_h}(x_h)\subset\Omega\setminus E$ is easier because $U=U_{**}-r^\beta$ where, we recall ,

$$U(x_0,r) := \fint_{B_r(x_0)} |Du(x)-(Du)_{x_0,r}|^2\,dx$$

for $B_r(x_0)\subset\Omega$.

Step 1. *Blow-up.*

Set $\lambda_h^2:=U_{**}(x_h,r_h)$, $A_h:=(Du)_{x_h,r_h}$, $a_h:=(u)_{x_h,r_h}$, and define as before

$$v_h(y):=\frac{u(x_h+r_hy)-a_h-r_hA_hy}{\lambda_h r_h}$$

for $y\in B_1(0)$. One can easily check that $(Dv_h)_{0,1}=0$ and $(v_h)_{0,1}=0$.
Again, as before, we set

$$E_h:=\frac{E-x_h}{r_h},\qquad E_h^*:=\frac{E-x_h}{r_h}\cap B_1.$$

Note that

$$\begin{aligned} U_{**}(x_h,r_h) &= \fint_{B_1(0)}|Du(x_h+r_hy)-A_h|^2\,dy+\left(\frac{P(E,B(x_h,r_h))}{r_h^{n-1}}\right)^{\frac{\delta}{1+\delta}}+r_h^\beta \\ &= \fint_{B_1(0)}|\lambda_h Dv_h|^2\,dy+(P(E_h,B_1))^{\frac{\delta}{1+\delta}}+r_h^\beta. \end{aligned} \tag{5.54}$$

By the definition of λ_h, it follows that

$$r_h\to 0,\qquad P(E_h,B_1)\to 0,\qquad \frac{r_h^\beta}{\lambda_h^2}\le 1,\qquad \fint_{B_1(0)}|Dv_h|^2\le 1,\qquad \frac{(P(E_h,B_1))^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\le 1. \tag{5.55}$$

Therefore, by virtue of (5.52), (5.54) and (5.55), there exist a subsequence $\{v_h\}$ (not relabeled), $A\in\mathbb{R}^{N\times n}$ and $v\in W^{1,2}(B_1(0);\mathbb{R}^N)$, such that

$$\begin{aligned} &v_h\rightharpoonup v \text{ weakly in } W^{1,2}(B_1(0);\mathbb{R}^N),\quad v_h\to v \text{ strongly in } L^2(B_1(0);\mathbb{R}^N),\\ &A_h\to A,\qquad \lambda_hDv_h\to 0 \text{ in } L^2(B_1(0)) \text{ and pointwise a.e.}, \end{aligned} \tag{5.56}$$

where we used the fact that $(v_h)_{0,1}=0$. We also note that

$$\frac{r_h^{\frac{\delta}{1+\delta}}}{\lambda_h^2}=r_h^{\frac{\delta}{1+\delta}-\beta}\frac{r_h^\beta}{\lambda_h^2}\to 0, \tag{5.57}$$

since $0<\beta<\frac{\delta}{1+\delta}$. Moreover, by (5.55) and (5.52), we deduce that

$$\lim_h\frac{\Big(P(E_h,B_1)\Big)^{\frac{n}{n-1}\frac{\delta}{1+\delta}}}{\lambda_h^2}=\lim_h\Big(P(E_h,B_1)\Big)^{\frac{\delta}{(n-1)(1+\delta)}}\limsup_h\frac{(P(E_h,B_1))^{\frac{\delta}{1+\delta}}}{\lambda_h^2}=0. \tag{5.58}$$

Therefore, by the relative isoperimetric inequality in a ball (see [5]),

$$\lim_h\min\left\{\frac{|E_h^*|^{\frac{\delta}{1+\delta}}}{\lambda_h^2},\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\right\}\le c\lim_h\frac{\Big(P(E_h,B_1)^{\frac{\delta}{1+\delta}}\Big)^{\frac{n}{n-1}}}{\lambda_h^2}=0. \tag{5.59}$$

Step 2. *A Caccioppoli type inequality.*

We claim that there exists a constant $c = c(M, \mu, \ell_1, \ell_2, L_1, L_2)$ such that, for every $0 < \rho < 1$, there exists $h_0 \in \mathbb{N}$ such that for all $h > h_0$ we have

$$\int_{B_{\frac{\rho}{2}}} |Dv_h - (Dv_h)_{\frac{\rho}{2}}|^2\, dy \le c \int_{B_\rho} \left| \frac{v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}} y}{\rho} \right|^2 dy + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}} \frac{P(E_h, B_1)^{\frac{n\delta}{(n-1)(1+\delta)}}}{\lambda_h^2}. \quad (5.60)$$

We divide the proof into two substeps.

Substep 2.a *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*|$.

The proof of this substep goes exactly as that of Substep 2.a of Proposition 5.1 up to estimate (5.27). Next we observe that

$$\begin{aligned} \int_{B_{\frac{\rho}{2}}} |Dv_h - (Dv_h)_{\frac{\rho}{2}}|^2\, dy &\le c \int_{B_\rho} \left| \frac{v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}} y}{\rho} \right|^2 dy + c \frac{|E_h^*|}{\lambda_h^2} \\ &\le c \int_{B_\rho} \left| \frac{v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}} y}{\rho} \right|^2 dy + c |E_h^*|^{\frac{1}{1+\delta}} \frac{|E_h^*|^{\frac{\delta}{1+\delta}}}{\lambda_h^2} \\ &\le c \int_{B_\rho} \left| \frac{v_h - (v_h)_\rho - (Dv_h)_{\frac{\rho}{2}} y}{\rho} \right|^2 dy + c \frac{|E_h^*|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}, \end{aligned}$$

and this, by the relative isoperimetric inequality and using the hypothesis of this substep that $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |E_h^*|$, yields the estimate (5.60).

Substep 2.b *The case* $\min\{|E_h^*|, |B_1 \setminus E_h|\} = |B_1 \setminus E_h|$.

Fix $0 < \frac{\rho}{2} < s < t < \rho < 1$ and let $\eta \in C_0^\infty(B_t)$ be a cut off function between B_s and B_t, i.e., $0 \le \eta \le 1$, $\eta \equiv 1$ on B_s and $|\nabla \eta| \le \frac{c}{t-s}$. Also, we set $b_h := (v_h)_{B_\rho}$, $B_h := (Dv_h)_{B_{\frac{\rho}{2}}}$ and define

$$w_h(y) := v_h(y) - b_h - B_h y, \quad \psi_{1,h}(y) := \eta w_h \quad \text{and} \quad \psi_{2,h}(y) := (1-\eta) w_h.$$

We recall that

$$\begin{aligned} \int_{B_t} \widetilde{H}_h(Dw_h)\, dy &\le \int_{B_t} \widetilde{H}_h\big(Dw_h(y) - D\psi_{1,h}\big)\, dy \\ &\quad + \frac{L_2}{\lambda_h^2} \int_{(B_t \setminus E_h)} (\mu^2 + |A_h + \lambda_h Dv_h|^2)\, dy \\ &= \int_{B_t \setminus B_s} \widetilde{H}_h\big(D\psi_{2,h}\big)\, dy + \frac{L_2}{\lambda_h^2} \int_{B_t \setminus E_h} (\mu^2 + |A_h + \lambda_h Dv_h|^2)\, dy. \quad (5.61) \end{aligned}$$

We remark that the higher integrability result of Theorem 4.1, through the change of variable $x = x_h + r_h y$, translates into the following

$$\left(\fint_{B_t} |Du(x_h + r_h y)|^{2(1+\delta)}\, dy \right)^{\frac{1}{1+\delta}} \le c \fint_{B_{2t}} |Du(x_h + r_h y)|^2\, dy + c\mu^2,$$

or, equivalently,

$$\left(\fint_{B_t} |A_h + \lambda_h Dv_h|^{2(1+\delta)}\,dy\right)^{\frac{1}{1+\delta}} \le c\fint_{B_{2t}} |A_h + \lambda_h Dv_h|^2\,dy + c\mu^2. \tag{5.62}$$

Using Hölder's inequality and inequality (5.62) in the estimate (5.61), we get

$$\begin{aligned}
\int_{B_t} \widetilde{H}_h(Dw_h)\,dy \;&\le\; \int_{B_t\setminus B_s} \widetilde{H}_h\big(D\psi_{2,h}\big)\,dy \\
&\quad + c(\mu)\frac{L_2}{\lambda_h^2}\left(\int_{B_t\setminus E_h}\Big(1+|A_h+\lambda_h Dv_h|^{2(1+\delta)}\Big)\,dy\right)^{\frac{1}{1+\delta}} |B_t\setminus E_h|^{\frac{\delta}{1+\delta}} \\
&\le\; \int_{B_t\setminus B_s} \widetilde{H}_h\big(D\psi_{2,h}\big)\,dy \\
&\quad + c(\mu)\frac{L_2}{\lambda_h^2}t^{\frac{n}{1+\delta}}\left(\fint_{B_t}\Big(1+|A_h+\lambda_h Dv_h|^{2(1+\delta)}\Big)\,dy\right)^{\frac{1}{1+\delta}} |B_1\setminus E_h|^{\frac{\delta}{1+\delta}} \\
&\le\; \int_{B_t\setminus B_s} \widetilde{H}_h\big(D\psi_{2,h}\big)\,dy \\
&\quad + c(\mu)\frac{L_2}{\lambda_h^2}t^{\frac{n}{1+\delta}}\left(1+\fint_{B_{2t}}|A_h+\lambda_h Dv_h|^2\,dy\right)|B_1\setminus E_h|^{\frac{\delta}{1+\delta}} \\
&\le\; \int_{B_t\setminus B_s} \widetilde{H}_h\big(D\psi_{2,h}\big)\,dy + c(\mu,M)\frac{L_2}{\lambda_h^2}t^{\frac{n}{1+\delta}-n}|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}.
\end{aligned}$$

Therefore we have

$$\int_{B_t} \widetilde{H}_h(Dw_h)\,dy \le \int_{B_t\setminus B_s} \widetilde{H}_h\big(D\psi_{2,h}\big)\,dy + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}, \tag{5.63}$$

where we used the fact that $t > \frac{\rho}{2}$. Now we observe that

$$\begin{aligned}
\int_{B_t} \widetilde{H}_h(Dw_h - D\psi_{2,h})\,dy &= \int_{B_t} \widetilde{H}_h(Dw_h)\,dy \\
&\quad + \int_{B_t} \widetilde{H}_h(Dw_h - D\psi_{2,h})\,dy - \int_{B_t} \widetilde{H}_h(Dw_h)\,dy \\
&= \int_{B_t} \widetilde{H}_h(Dw_h)\,dy - \int_{B_t}\int_0^1 D\widetilde{H}_h(Dw_h - \theta D\psi_{2,h})D\psi_{2,h}\,d\theta\,dy.
\end{aligned} \tag{5.64}$$

Hence, inserting the estimate (5.63) in (5.64), by the upper bound on $D\widetilde{H}_h$ given by Lemma 2.2 , we obtain

$$\begin{aligned}
\int_{B_t} \widetilde{H}_h(Dw_h - D\psi_{2,h})\,dy \;&\le\; \int_{B_t\setminus B_s} \widetilde{H}_h(D\psi_{2,h})\,dy + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2} \\
&\quad + c\int_{B_t\setminus B_s} (|Dw_h| + |D\psi_{2,h}|)|D\psi_{2,h}|\,dy.
\end{aligned} \tag{5.65}$$

Combining (5.29) with (5.65), using Lemma 2.2 and Young's inequality, we have

$$\begin{aligned}
\widetilde{\ell}\int_{B_s}|D\psi_{1,h}|^2\,dy &\le \widetilde{\ell}\int_{B_t}|D\psi_{1,h}|^2\,dy \le \int_{B_t\setminus B_s}\widetilde{H}_h(D\psi_{2,h})\,dy + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\\
&\quad + c\int_{B_t\setminus B_s}(|Dw_h|+|D\psi_{2,h}|)|D\psi_{2,h}|\,dy\\
&\le c\int_{B_t\setminus B_s}|D\psi_{2,h}|^2\,dy + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2} + c\int_{B_t\setminus B_s}|Dw_h|^2\,dy. \quad (5.66)
\end{aligned}$$

By the properties of η, we obtain by (5.66)

$$\begin{aligned}
\widetilde{\ell}\int_{B_s}|Dw_h|^2\,dy \le \widetilde{\ell}\int_{B_t}|D\psi_{1,h}|^2\,dy &\le c\int_{B_t\setminus B_s}|Dw_h|^2\,dy + c\int_{B_t\setminus B_s}\left|\frac{w_h}{t-s}\right|^2\,dy\\
&\quad + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}.
\end{aligned}$$

Using the hole filling technique as in (5.26) , we get

$$\begin{aligned}
(c+\widetilde{\ell})\int_{B_s}|Dw_h|^2\,dy &\le c\int_{B_t}|Dw_h|^2\,dy + c\int_{B_t\setminus B_s}\left|\frac{w_h}{t-s}\right|^2\,dy + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}\\
&\le c\int_{B_t}|Dw_h|^2\,dy + c\int_{B_\rho}\left|\frac{w_h}{t-s}\right|^2\,dy + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2}.
\end{aligned}$$

By virtue of the iteration Lemma 2.1, from previous estimate we deduce that

$$\int_{B_{\frac{\rho}{2}}}|Dw_h|^2\,dy \le c\int_{B_\rho}\left|\frac{w_h}{\rho}\right|^2\,dy + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2},$$

where $c = c(M,\mu,\ell_1,L_1,\ell_2,L_2,n,N)$. Therefore, by the definition of w_h, we conclude that

$$\int_{B_{\frac{\rho}{2}}}|Dv_h-(Dv_h)_{\frac{\rho}{2}}|^2\,dy \le c\int_{B_\rho}\left|\frac{v_h-(v_h)_\rho-(Dv_h)_{\frac{\rho}{2}}y}{\rho}\right|^2\,dy + \frac{c}{\rho^{\frac{n\delta}{1+\delta}}}\frac{|B_1\setminus E_h|^{\frac{\delta}{1+\delta}}}{\lambda_h^2},$$

which, by the relative isoperimetric inequality and since we have $|B_1\setminus E_h| = \min\{|E_h^*|,|B_1\setminus E_h|\}$, gives the estimate (5.60).

The proofs of the Step 3 and 4 of Proposition 5.1 hold true also in this case.

Step 5. *Conclusion.*

Using the change of variable $x = x_h + r_h y$ and the Caccioppoli inequality in (5.60), for every $0<\tau<\frac{1}{4}$ we have

$$\begin{aligned}
\limsup_{h\to\infty}\frac{U_{**}(x_h,\tau r_h)}{\lambda_h^2} &\le \limsup_{h\to\infty}\frac{1}{\lambda_h^2}\fint_{B_{\tau r_h}(x_h)}|Du(x)-(Du)_{x_h,\tau r_h}|^2\,dx\\
&\quad + \limsup_{h\to\infty}\frac{1}{\lambda_h^2}\left(\frac{P(E,B(x_h,\tau r_h))}{\tau^{n-1}r_h^{n-1}}\right)^{\frac{\delta}{1+\delta}} + \limsup_{h\to\infty}\frac{\tau^\beta r_h^\beta}{\lambda_h^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq c \limsup_{h\to\infty} \fint_{B_\tau} |Dv_h - (Dv_h)_\tau|^2\, dy + \limsup_{h\to\infty} \frac{1}{\lambda_h^2} \left(\frac{P(E_h, B_\tau)}{\tau^{n-1}} \right)^{\frac{\delta}{1+\delta}} + \tau^\beta \\
&\leq c \limsup_{h\to\infty} \fint_{B_{2\tau}} \frac{|v_h - (v_h)_{2\tau} - (Dv_h)_\tau y|^2}{\tau^2}\, dy \\
&\quad + c(\tau,\delta) \limsup_{h\to\infty} \frac{P(E_h, B_1)^{\frac{n\delta}{(n-1)(1+\delta)}}}{\lambda_h^2} + c r_h \tau^{\frac{\delta}{1+\delta}} + \tau^\beta,
\end{aligned}
$$

where we used (5.55) and estimate (5.49). By virtue of the the strong convergence of $v_h \to v$ in $L^2(B_1)$, since $(Dv_h)_\tau \to (Dv)_\tau$ in $\mathbb{R}^{nN}$, by (5.55), (5.56), (5.57) (5.58), (5.59) and by the use of Poincaré - Wirtinger inequality, we get

$$
\begin{aligned}
\limsup_{h\to\infty} \frac{U_*(x_h, \tau r_h)}{\lambda_h^2} &\leq c \fint_{B_{2\tau}} \frac{|v - (v)_{2\tau} - (Dv)_\tau y|^2}{\tau^2}\, dy + \tau^\beta \\
&\leq c \lim_{h\to\infty} \fint_{B_{2\tau}} |Dv - (Dv)_\tau|^2\, dy + \tau^\beta \\
&\leq c\tau^2 + \tau^\beta \leq C\tau^\beta,
\end{aligned}
$$

where we used the estimate (5.34) and where $C = C(M, \mu, \ell_1, L_1, \ell_2, L_2, n, N)$. The contradiction follows by choosing C_{**} such that $C_{**} > C$, since by (5.4)

$$
\liminf_h \frac{U_*(x_h, \tau r_h)}{\lambda_h^2} \geq C_{**}\tau^\beta.
$$

□

6 Proof of the Main Theorem

Here we give the proof of Theorem 1.2 through a suitable iteration procedure.

6.1 An Iteration Lemma

In the next Lemma the constant c_* is that introduced in (5.2).

Lemma 6.1. *Let (u, E) be a minimizer of the functional $\mathcal{I}$. For every $M > 0$, for every $\alpha \in (0,1)$ and for every $\vartheta \in (0, \vartheta_0)$, with $\vartheta_0 := \min\left\{ c_*^{-\frac{1}{1-\alpha}}, \frac{1}{4} \right\}$, there exist $\varepsilon_1 > 0$ and $R > 0$ such that, if $r < R$ and $x_0 \in \Omega$ satisfy*

$$
B_r(x_0) \Subset \Omega, \qquad |Du|_{x_0,r} < M \qquad \textit{and} \qquad U_*(x_0, r) < \varepsilon_1,
$$

then

$$
(D_k) \qquad\qquad U_*(x_0, \vartheta^k r) \leq \vartheta^{k\alpha} U_*(x_0, r)
$$

for all $k \in \mathbb{N}$.

Proof. Fix $M > 0$ and $\vartheta \in (0, \vartheta_0)$. Let

$$\varepsilon_1 := \min\left\{\varepsilon_0, (1-\vartheta^{\frac{1}{2}})^2\vartheta^{n+1}\right\}$$

where ε_0 is determined in Proposition 5.1 corresponding to $M+1$ and to ϑ. We will argue by induction. First note that (D_1) holds true by virtue of Proposition 5.1 and since $\varepsilon_1 < \varepsilon_0$ and $c_*\vartheta \le \vartheta^\alpha$. Assume that (D_k) hold true for $k \le h$ and we prove that (D_{h+1}) is satisfied. Observe that

$$\begin{aligned}
|Du|_{x_0,\vartheta^h r} &\le |Du|_{x_0,r} + \sum_{j=1}^{h} \left||Du|_{x_0,\vartheta^j r} - |Du|_{x_0,\vartheta^{j-1}r}\right| \\
&\le |Du|_{x_0,r} + \sum_{j=1}^{h} |(Du)_{x_0,\vartheta^j r} - (Du)_{x_0,\vartheta^{j-1}r}| \\
&\le |Du|_{x_0,r} + \sum_{j=1}^{h} \fint_{B_{\vartheta^j r}} |Du - (Du)_{x_0,\vartheta^{j-1}r}| \\
&\le |Du|_{x_0,r} + \sum_{j=1}^{h} \left(\fint_{B_{\vartheta^j r}} |Du - (Du)_{x_0,\vartheta^{j-1}r}|^2\right)^{\frac{1}{2}} \\
&\le |Du|_{x_0,r} + \sum_{j=1}^{h} \left(\frac{|B_{\vartheta^{j-1}r}|}{|B_{\vartheta^j r}|}\right)^{\frac{1}{2}} \left(\fint_{B_{\vartheta^{j-1}r}} |Du - (Du)_{x_0,\vartheta^{j-1}r}|^2\right)^{\frac{1}{2}} \\
&\le |Du|_{x_0,r} + \vartheta^{-\frac{n}{2}} \sum_{j=1}^{h} U(x_0,\vartheta^{j-1}r)^{\frac{1}{2}} \\
&\le M + \vartheta^{-\frac{n}{2}} \sum_{j=1}^{h} U_*(x_0,\vartheta^{j-1}r)^{\frac{1}{2}} \\
&\le M + (c_*)^{\frac{1}{2}} \vartheta^{-\frac{n}{2}} U_*(x_0,r)^{\frac{1}{2}} \sum_{j=1}^{h} \vartheta^{\frac{j-1}{2}} \\
&\le M + \vartheta^{-\frac{n+1}{2}} \varepsilon_1^{\frac{1}{2}} \frac{(c_*)^{\frac{1}{2}}\vartheta^{\frac{1}{2}}}{1-\vartheta^{\frac{1}{2}}} \\
&\le M + \vartheta^{-\frac{n+1}{2}} \varepsilon_1^{\frac{1}{2}} \frac{1}{1-\vartheta^{\frac{1}{2}}} \\
&\le M + 1,
\end{aligned} \tag{6.1}$$

because, by our choice of ε_1, we have

$$\vartheta^{-\frac{n+1}{2}} \varepsilon_1^{\frac{1}{2}} \frac{1}{1-\vartheta^{\frac{1}{2}}} \le 1.$$

Moreover, since (D_h) holds true, we have that

$$U_*(x_0,\vartheta^h r) \le \vartheta^{h\alpha} U_*(x_0,r) < \varepsilon_1, \tag{6.2}$$

by our choice of ϑ and ε_1, and so by (6.1) we can apply Proposition 5.1 with $\vartheta^h r$ in place of r to deduce

$$U_*(x_0,\vartheta^{h+1}r) \le \vartheta^\alpha U_*(x_0,\vartheta^h r) \le \vartheta^{(h+1)\alpha} U_*(x_0,r),$$

by (6.2). Therefore (D_k) holds true for every $k \in \mathbb{N}$. □

Arguing exactly in the same way and by using Proposition 5.2 instead of Proposition 5.1, we have the following

Lemma 6.2. *Let (u, E) be a minimizer of the functional $\mathcal{I}$ and let β be the exponent of Lemma 5.2. For every $M > 0$ and for every $\vartheta \in (0, \vartheta_0)$, with $\vartheta_0 < \min\{c_{**}, \frac{1}{4}\}$, there exist $\varepsilon_1 > 0$ and $R > 0$ such that, if $r < R$ and $x_0 \in \Omega$ satisfy*

$$B_r(x_0) \Subset \Omega, \qquad |Du|_{x_0,r} < M \qquad \textit{and} \qquad U_{**}(x_0, r) < \varepsilon_1,$$

then

$$(D'_k) \qquad\qquad U_{**}(x_0, \vartheta^k r) \leq \vartheta^{k\beta} U_{**}(x_0, r)$$

for all $k \in \mathbb{N}$.

6.2 Proof of Theorem 1.2

Proof. Assume first that (F1), (F2),(G1), (G2) and (H) hold, consider the set

$$\Omega_0 := \{x \in \Omega : \limsup_{\rho \to 0} |(Du)_{x,\rho}| < +\infty \text{ and } \limsup_{\rho \to 0} U_*(x, \rho) = 0\},$$

and let $x_0 \in \Omega_0$. For every $M > 0$ and for ε_1 determined in Lemma 6.1 there exists a radius R_{M,ε_1} such that

$$|Du|_{x_0,r} < M \qquad \textit{and} \qquad U_{**}(x_0, r) < \varepsilon_1,$$

for every $0 < r < R_{M,\varepsilon_1}$. If $0 < \rho < \frac{\vartheta_0}{2} r < R$, let $h \in \mathbb{N}$ be such that $\vartheta^{h+1} r < \rho < \vartheta^h r$, and let $\vartheta = \frac{\vartheta_0}{2}$ where ϑ_0 is as in Lemma 6.1. By Lemma 6.1, we obtain

$$\begin{aligned}
U_*(x_0, \rho) &= \fint_{B_\rho} |Du - (Du)_{x_0,\rho}|^2 + \frac{P(E, B_\rho(x_0))}{\rho^{n-1}} + \rho \\
&= \fint_{B_\rho} |Du - (Du)_{\vartheta^h r} + (Du)_{\vartheta^h r} - (Du)_{x_0,\rho}|^2 + \frac{P(E, B_\rho(x_0))}{\rho^{n-1}} + \rho \\
&\leq 2\fint_{B_\rho} |Du - (Du)_{\vartheta^h r}|^2 + 2\fint_{B_\rho} |(Du)_{\vartheta^h r} - (Du)_{x_0,\rho}|^2 + \frac{P(E, B_\rho(x_0))}{\rho^{n-1}} + \rho \\
&\leq c\fint_{B_\rho} |Du - (Du)_{\vartheta^h r}|^2 + \frac{P(E, B_\rho(x_0))}{\rho^{n-1}} + \rho \\
&\leq c\left(\frac{\vartheta^h r}{\rho}\right)^n \fint_{B_{\vartheta^h r}} |Du - (Du)_{\vartheta^h r}|^2 + \frac{P(E, B_\rho(x_0))}{\rho^{n-1}} + \rho \\
&= c\left(\frac{\vartheta^{h+1} r}{\vartheta\rho}\right)^n \fint_{B_{\vartheta^h r}} |Du - (Du)_{\vartheta^h r}|^2 + \frac{P(E, B_\rho(x_0))}{\rho^{n-1}} + \rho \\
&\leq c\left(\frac{\rho}{\vartheta\rho}\right)^n \fint_{B_{\vartheta^h r}} |Du - (Du)_{\vartheta^h r}|^2 + \frac{P(E, B_\rho(x_0))}{(\vartheta^{h+1} r)^{n-1}} + \rho
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{c}{\vartheta^n} \fint_{B_{\vartheta^h r}} |Du - (Du)_{\vartheta^h r}|^2 + \frac{1}{\vartheta^{n-1}} \frac{P(E, B_{\vartheta^h r}(x_0))}{(\vartheta^h r)^{n-1}} + \vartheta^h r \\
&= \frac{c}{\vartheta_0^n} \fint_{B_{\vartheta^h r}} |Du - (Du)_{\vartheta^h r}|^2 + \frac{2^{n-1}}{\vartheta_0^{n-1}} \frac{P(E, B_{\vartheta^h r}(x_0))}{(\vartheta^h r)^{n-1}} + \vartheta^h r \\
&\leq CU_*(x_0, \vartheta^h r) \leq C_* \vartheta^{h\alpha} U_*(x_0, r) \leq C_* \left(\frac{\rho}{r}\right)^\alpha U_*(x_0, r), \qquad (6.3)
\end{aligned}
$$

where we used the fact that $\vartheta_0 := \min\left\{c_*^{-\frac{1}{1-\alpha}}, \frac{1}{4}\right\}$, $\vartheta = \frac{\vartheta_0}{2}$ and $\vartheta^{h+1} r < \rho < \vartheta^h r$. The previous estimate implies that

$$
U(x_0, \rho) = \fint_{B_\rho} |Du - (Du)_\rho|^2 dx \leq C_* \left(\frac{\rho}{r}\right)^\alpha U_*(x_0, r).
$$

Since $U_*(y, r)$ is continuous in y, we have that $U_*(y, r) < \varepsilon_1$ for all y in a suitable neighborhood I of x_0. For every $y \in I$ we then have that

$$
U(y, \rho) \leq C_* \left(\frac{\rho}{r}\right)^\alpha U_*(y, r).
$$

The last inequality implies, by the Campanato characterization of Hölder continuous functions ([20, Theorem 2.9]), that u is $C^{1,\alpha}$ in I for every $0 < \alpha < \frac{1}{2}$, and we conclude that the function u has Hölder continuous derivatives in an open set Ω_0 that contains all points y such that

$$
\limsup_{r \to 0} U_*(y, r) = 0.
$$

Next we prove a suitable decay estimate for the perimeter of the minimal set. For every $0 < \rho < \frac{\vartheta_0}{2} r$, let $h \in \mathbb{N}$ be such that $\vartheta^{h+1} r < \rho < \vartheta^h r$, where $\vartheta = \frac{\vartheta_0}{2}$ as before. We observe that

$$
|(Du)_\rho| \leq |Du|_\rho \leq \left(\frac{\vartheta^h r}{\rho}\right)^n \fint_{B_{\vartheta^h r}} |Du| \leq \frac{M+1}{\vartheta^n} = \frac{c(M)}{\vartheta_0^n} = c(M, c_*), \qquad (6.4)
$$

where we used (6.1). Consider A any set of finite perimeter such that $E \Delta A \subset\subset B_\rho(x_0)$. From the minimality of (u, E) we have that

$$
\begin{aligned}
&\int_\Omega \Big(F(Du) + \chi_E G(Du)\Big)\, dx + P(E, \Omega) \\
&\qquad \leq \int_\Omega \Big(F(Du) + \chi_A G(Du)\Big)\, dx + P(A, \Omega).
\end{aligned}
$$

Using the fact that $E \Delta A \subset\subset B_\rho(x)$, we deduce that

$$
\begin{aligned}
P(E, B_\rho(x_0)) - P(A, B_\rho(x_0)) &\leq \int_{B_\rho(x_0)} \Big(\chi_A(x) - \chi_E(x)\Big) G(Du)\, dx \\
&\leq L_2 \int_{B_\rho(x_0)} (\mu^2 + |Du|^2)\, dx \\
&\leq c \int_{B_\rho(x_0)} |Du - (Du)_\rho|^2\, dx + c(\mu, M, c_*) \rho^n
\end{aligned}
$$

$$
\begin{aligned}
&= c\rho^n U(x_0,\rho) + c(\mu, M, c_*)\rho^n \\
&\le C\rho^n \frac{\rho^\alpha}{r^\alpha} U_*(x_0, r) + c(\mu, M, c_*)\rho^n \\
&\le c(\mu, M, c_*, r)\rho^n\,,
\end{aligned}
$$

where we invoked the assumption (G2) and we used estimates (6.4) and (6.3) . At this point the result follows from Theorem 2.1.

When the assumption (H) is not enforced, the proof goes exactly in the same way provided we use Lemma 6.2 in place of Lemma 6.1, with

$$
\Omega_1 := \{x \in \Omega : \limsup_{\rho\to 0} |(Du)_{x_0,\rho}| < +\infty \text{ and } \limsup_{\rho\to 0} U_{**}(x_0,\rho) = 0\}.
$$

□

Acknowledgment. The authors warmly thank the Center for Nonlinear Analysis, where part of this research was carried out. The research of I. Fonseca was partially funded by the National Science Foundation under Grant No. DMS-1411646. The research of M. Carozza and A. Passarelli di Napoli was partially supported by MIUR through the Project PRIN (2012) “Calcolo delle Variazioni”. and by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] E. Acerbi & N. Fusco, *An approximation lemma for $W^{1;p}$ functions*, Material instabilities in continuum mechanics (Edinburgh, 19851986), Oxford Sci. Publ., Oxford Univ. Press, New York (1988), 1–5.

[2] E. Acerbi & N. Fusco, *A regularity theorem for minimizers of quasiconvex integrals*. Arch. Ration. Mech. Anal. **99** (1987), n.3, 261-281.

[3] H.W. Alt & L.A. Caffarelli, *Existence and regularity results for a minimum problem with free boundary*. J. Reine Angew. Math. **325** (1981), 107–144.

[4] L. Ambrosio & G. Buttazzo, *An optimal design problem with perimeter penalization*. Calc. Var. Partial Differential Equations **1** (1993), 55–69.

[5] L. Ambrosio, N. Fusco & D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press (2000).

[6] A. Arroyo-Rabasa, *Regularity for free interface variational problems in a general class of gradients*. Calc. Var. Partial Differential Equations 55 (2016), no. 6, Art. 154, 44 pp

[7] E. Bombieri, *Regularity theory for almost minimal currents*. Arch. Ration. Mech. Anal. **78** (1982), 99–130.

[8] M. Carozza, I. Fonseca & A. Passarelli Di Napoli, *Regularity results for an optimal design problem with a volume constraint*. ESAIM Control Optim. Calc. Var. **20** (2014), n.2, 460–487.

[9] M. Carozza & A. Passarelli Di Napoli, *A regularity theorem for minimisers of quasiconvex integrals: The case* $1 < p < 2$. Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), n.6, 1181–1200.

[10] M. Carozza & A. Passarelli Di Napoli, *Partial regularity of local minimizers of quasiconvex integrals with sub-quadratic growth*. Proc. Roy. Soc. Edinburgh Sect. A **133,6** (2003), 1249-1262.

[11] M. Carozza & A. Passarelli Di Napoli, *Partial regularity for anisotropic functionals of higher order*. ESAIM Control Optim. Calc. Var. **13** (2007), n.4, 692-706.

[12] C. De Lellis, M. Focardi & E.N. Spadaro, *Lower semicontinuous functionals for Almgren's multiple valued functions*, Ann. Acad. Sci. Fenn. Math. **36** (2011), n.2, 393–410.

[13] L.C. Evans, *Quasiconvexity and partial regularity in the calculus of variations*. Arch. Ration. Mech. Anal. **95 , 3** (1986) 227-252.

[14] L.C. Evans & F.R. Gariepy, *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL (1992).

[15] I. Fonseca & S. Müller, *$\mathcal{A}$-quasiconvexity, lower semicontinuity, and Young measures*, SIAM J. Math. Anal. **30** (1999), n.6, 1355–1390.

[16] I. Fonseca, S. Müller & P. Pedregal, *Analysis of concentration and oscillation effects generated by gradients*, SIAM J. Math. Anal. **29** (1998), n.3, 736–756.

[17] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.* Annals of Mathematical Studies **105**, Princeton University Press, Princeton NJ (1983).

[18] M. Giaquinta & G. Modica, *Partial regularity of minimizers of quasiconvex integrals.* Ann. Inst. H. Poincaré, Anal. Non Linéaire **3** (1986), 185–208.

[19] D. Gilbarg & N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Second edition, Grundlehren der Mathematischen Wissenschaften, **224**, Springer-Verlag, Berlin, (1983).

[20] E. Giusti. *Direct Methods in the Calculus of Variations*. World Scientific (2003).

[21] M. Gurtin, *On phase transitions with bulk, interfacial, and boundary energy*. Arch. Ration. Mech. Anal. **96** (1986), 243–264.

[22] J. Kristensen, *Lower semicontinuity in spaces of weakly differentiable functions*. Math. Ann. **313** (1999), n.4, 653–710.

[23] C.J. Larsen, *Regularity of components in optimal design problems with perimeter penalization*. Calc. Var. Partial Differential Equations **16** (2003), 17–29.

[24] F.H. Lin, *Variational problems with free interfaces*. Calc. Var. Partial Differential Equations **1** (1993), 149–168

[25] F.H. Lin & R.V. Kohn, *Partial regularity for optimal design problems involving both bulk and surface energies*. Chinese Ann. of Math. Ser. B **20** (1999), n.2, 137–158.

[26] P. Marcellini, *Approximation of quasiconvex functionals and lower semicontinuity of multiple integrals*. Manuscripta Math. **51** (1985), 1–28.

[27] V. Šverák & X. Yan. *Non-Lipschitz minimizers of smooth uniformly convex variational integrals*. Proc. Natl. Acad. Sci. USA **99** (2002), n.24, 15269–15276.

[28] I. Tamanini, *Boundaries of Caccioppoli sets with Hölder-continuous normal vector*. J. Reine Angew. Math. **334** (1982), 27–39.

Dipartimento di Ingegneria - Università del Sannio, Corso Garibaldi, 82100 Benevento, Italy
E-mail address: carozza@unisannio.it

Department of Mathematical Sciences , Carnegie Mellon University, Pittsburgh PA 15213-3890
E-mail address: fonseca@andrew.cmu.edu

Dipartimento di Mat. e Appl. "R. Caccioppoli"- Università di Napoli "Federico II" , Via Cintia, 80126 Napoli, Italy
E-mail address: antpassa@unina.it