THE WEIGHTED AMBROSIO - TORTORELLI APPROXIMATION SCHEME

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Abstract. The Ambrosio-Tortorelli approximation scheme with weighted underlying metric is investigated. It is shown that it Γ-converges to a Mumford-Shah image segmentation functional depending on the weight $\omega\, dx$, where $\omega \in SBV(\Omega)$, and on its value ω^-.

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1. Introduction and Main Results

A central problem in image processing is image denoising. Given an image u_0, we decompose it as

$$u_0 = u_g + n$$

where u_g represents a noisy-free ground truth picture, while n encodes noise or textures. Examples of models for such noise distributions are Gaussian noise in Magnetic Resonance Tomography, and Poisson noise in radar measurements [16]. Variational PDE methods have proven to be efficient to remove the noise n from u_0. Several successful variational PDEs have been proposed in the literature (see, for example [36, 42, 43, 44]) and, among these, the Mumford-Shah image segmentation functional

$$G(u,K) := \alpha \int_{\Omega\setminus K} |\nabla u|^2 dx + \alpha \mathcal{H}^{N-1}(K) + \int_{\Omega} (u-u_0)^2 dx \quad \text{where } u \in W^{1,2}(\Omega\setminus K),\ K\subset\Omega \text{ closed in } \Omega, \tag{1.1}$$

introduced in [41], is one of the most successful approaches. By minimizing the functional (1.1) one tries to find a "piecewise smooth" approximation of u_0. The existence of such minimizers can be proved by using compactness and lower semicontinuity theorems in $SBV(\Omega)$ (see [1, 2, 3, 4]). Furthermore, regularity results in [22, 24] give that minimizers u satisfy

$$u \in C^1(\Omega\setminus \overline{S}_u) \text{ and } \mathcal{H}^{N-1}(\overline{S}_u \cap \Omega \setminus S_u) = 0.$$

Here, as in what follows, S_u stands for the jump set of u.

The parameter $\alpha > 0$ in (1.1), determined by the user, plays an important role. For example, choosing $\alpha > 0$ too large will result in over-smoothing and the edges that should have been preserved will be lost, and choosing $\alpha > 0$ too small may keep the noise un-removed. The choice of the "best" parameter α then becomes an interesting task. In [25] the authors proposed a training scheme by using bilevel learning optimization defined in machine learning, which is a semi-supervised learning scheme that optimally adapts itself to the given "perfect data" (see [20, 21, 26, 27, 45, 46]). This learning scheme searches $\alpha > 0$ such that the recovered image u_α, obtained from(1.1), best fits the given clean image u_g, measured in terms of the L^2-distance. A simplified bilevel learning scheme $(\mathcal{B})$ from [25] is the following:

Level 1.

$$\bar{\alpha} := \operatorname*{arg\,min}_{\alpha>0} \int_\Omega |u_\alpha - u_g|^2\, dx, \tag{1.2}$$

Level 2.

$$u_\alpha := \operatorname*{arg\,min}_{u\in SBV(\Omega)} \left\{ \int_\Omega \alpha\, |\nabla u|^2\, dx + \alpha \mathcal{H}^{N-1}(S_u) + \int_\Omega |u-u_0|^2\, dx \right\},$$

In [25] the authors proved that the above bilevel learning scheme has at least one solution $\bar{\alpha} \in (0,+\infty]$, and a small modification rules out the possibility of $\bar{\alpha} = +\infty$.

The model proposed in [37] is aimed at improving the above learning scheme. It is a bilevel learning scheme which utilizes the scheme $(\mathcal{B})$ in each subdomain of Ω, and searches for the best combination of different subdomains from which a recovered image $\bar{u}$, which best fits u_g, might be obtained.

To present the model, we first fix some notation. For $K \in \mathbb{N}$, $Q_K \subset \mathbb{R}^N$ denotes a cube with its faces normal to the orthonormal basis of $\mathbb{R}^N$, and with side-length greater than or equal to $1/K$. Define $\mathcal{P}_K$ to be a collection of finitely many Q_K such that

$$\mathcal{P}_K := \left\{ Q_K \subset \mathbb{R}^N :\ Q_K \text{ are mutually disjoint, } \Omega \subset \bigcup Q_K \right\},$$

and $\mathcal{V}_K$ denotes the collection of all possible $\mathcal{P}_K$. For $K = 0$ we set $Q_0 := \Omega$, hence $\mathcal{P}_0 = \{\Omega\}$.

A simplified bilevel learning scheme $(\mathcal{P})$ in [37] is as follows:

Level 1.

$$\bar{u} := \underset{K\geq 0,\, \mathcal{P}_K \in \mathcal{V}_K}{\arg\min} \left\{ \int_\Omega |u_g - u_{\mathcal{P}_K}|^2 \, dx \right\} \tag{1.3}$$

$$\text{where } u_{\mathcal{P}_K} := \underset{u\in SBV(\Omega)}{\arg\min} \left\{ \int_\Omega \alpha_{\mathcal{P}_K}(x) \, |\nabla u|^2 \, dx + \int_{S_u} \alpha_{\mathcal{P}_k}(x) d\mathcal{H}^{N-1} + \int_\Omega |u - u_0|^2 \, dx \right\} \tag{1.4}$$

Level 2.

$$\begin{aligned} &\alpha_{\mathcal{P}_K}(x) := \alpha_{Q_K} \text{ for } x \in Q_K \in \mathcal{P}_K, \text{ where } \alpha_{Q_K} := \underset{\alpha>0}{\arg\min} \int_{Q_K} |u_\alpha - u_g|^2 \, dx, \\ &u_\alpha := \underset{u\in SBV(Q_K\cap\Omega)}{\arg\min} \left\{ \int_{Q_K\cap\Omega} \alpha \, |\nabla u|^2 \, dx + \alpha \mathcal{H}^{N-1}(S_u) + \int_{Q_K\cap\Omega} |u - u_0|^2 \, dx \right\} \end{aligned} \tag{1.5}$$

Scheme ($\mathcal{P}$) allows us to perform the denoising procedure "point-wisely", and it is an improvement of (1.2). Note that at step $K = 0$, (1.3) reduces to (1.2). It is well known that the Mumford-Shah model, as well as the ROF model in [44], leads to undesirable phenomena like the staircasing effect (see [10, 19]). However, such staircasing effect is significantly mitigated in (1.3), according to numerical simulations in [37] (a theoretical validation of such improvement is needed). We remark that the most important step is (1.4) for the following reasons:

1. (1.4) is the bridge connecting level 1 and level 2;
2. since $\alpha_{\mathcal{P}_K}$ is defined by locally optimizing the parameter α_{Q_K}, we expect $u_{\mathcal{P}_K}$ be "close" to u_g locally in Q_K;
3. the last integrand in (1.4) keeps $u_{\mathcal{P}_K}$ close to u_0 globally, hence we may expect $u_{\mathcal{P}_K}$ to have a good balance between local optimization and global optimization.

We may view (1.4) as a weighted version of (1.1) by changing the underlying metric from dx to $\alpha_{\mathcal{P}_K} dx$. By the construction of $\alpha_{\mathcal{P}_k}$ in (1.5), we know it is a piecewise constant function and, since $K > 0$ is finite, the discontinuity set of $\alpha_{\mathcal{P}_K}$ has finite $\mathcal{H}^{N-1}$ measure. However, $\alpha_{\mathcal{P}_K}$ is only defined $\mathcal{L}^N$-a.e., and hence the term

$$\int_{S_u} \alpha_{\mathcal{P}_k}(x) d\mathcal{H}^{N-1}$$

might be ill-defined.

In this paper, we deal with the well-definess of (1.4) by modifying $\alpha_{\mathcal{P}_K}$ accordingly, and by building a sequence of functionals which Γ-converges to (1.4). To be precise, we adopt the approximation scheme of Ambrosio and Tortorelli in [8] and change the underlying metric properly. In (1.1) Ambrosio and Tortorelli considered a sequence of functionals reminiscent of the Cahn-Hilliard approximation, and introduced a family of elliptic functionals

$$G_\varepsilon(u, v) := \int_\Omega \alpha \, |\nabla u|^2 \, v^2 dx + \int_\Omega \alpha \left[\varepsilon \, |\nabla v|^2 + \frac{1}{4\varepsilon}(v-1)^2 \right] dx + \int_\Omega (u - u_0)^2 \, dx,$$

where $u \in W^{1,2}(\Omega)$, $(v - 1) \in W_0^{1,2}(\Omega)$, and $u_0 \in L^2(\Omega)$. The additional field v plays the role of controlling variable on the gradient of u. In [8] a rigorous argument has been made to show that $G_\varepsilon \to G$ in the sense of Γ-convergence ([9]), where G is defined in (1.1).

In view of (1.5), we fix a weight function $\omega \in SBV(\Omega)$ such that ω is positive and $\mathcal{H}^{N-1}(S_\omega) < +\infty$.

Our new (weighted version) Mumford-Shah image segmentation functional is defined as

$$E_\omega(u) := \int_\Omega |\nabla u|^2 \,\omega\, dx + \int_{S_u} \omega^- \, d\mathcal{H}^{N-1}, \tag{1.6}$$

and the (weighted version) of Ambrosio - Tortorelli functionals are defined as

$$E_{\omega,\varepsilon}(u,v) := \int_\Omega |\nabla u|^2 v^2 \omega\, dx + \int_\Omega \left[\varepsilon |\nabla v|^2 + \frac{1}{4\varepsilon}(v-1)^2\right] \omega\, dx.$$

It is natural to take $u \in GSBV_\omega(\Omega)$ in (1.6) (see Definition 2.6. For basic definitions and theorems of weighted spaces we refer to [5, 6, 11, 14, 15, 18, 32, 33]). Moreover, since $K \geq 0$ is finite and $\alpha_{Q_K} > 0$ in (1.5) , it is not restricted to assume that

$$\operatorname{essinf}\{\omega(x), x \in \Omega\} \geq l, \text{ where } l > 0 \text{ is a constant.}$$

This condition implies that all weighted spaces considered in this paper are embedded in the corresponding non-weighted spaces, and hence we may apply some results that hold in the context of non-weighted spaces. For example, $BV_\omega \subset BV$ and $W^{1,2}_\omega \subset W^{1,2}$ (see Definition 2.6), and most theorems in [8] can be applied to $u \in SBV_\omega(\Omega)$ (for example, Theorem 2.3 in [8]).

Before we state our main result, we recall that similar problems have been studied for different types of weight functions ω (see, for example [12, 13, 35]). In particular, [12, 13] treat a uniformly continuous and strong A_∞ (defined in [23]) weight function on Modica-Mortola and Mumford-Shah-type functionals, respectively, and in [35] the authors considered a $C^{1,\beta}$-continuous weight function, with some other minor assumptions, in the one-dimensional Cahn-Hilliard model.

Our main result is the following:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be open bounded, let $\omega \in SBV(\Omega) \cap L^\infty(\Omega)$, and let $\mathcal{E}_{\omega,\varepsilon}$: $L^1_\omega(\Omega) \times L^1(\Omega) \to [0,+\infty]$ be defined by*

$$\mathcal{E}_{\omega,\varepsilon}(u,v) := \begin{cases} E_{\omega,\varepsilon}(u,v) & \text{if } (u,v) \in W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega),\ 0 \leq v \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the functionals $\mathcal{E}_{\omega,\varepsilon}$ Γ-converge, with respect to the $L^1_\omega \times L^1$ topology, to the functional

$$\mathcal{E}_\omega(u,v) := \begin{cases} E_\omega(u) & \text{if } u \in GSBV_\omega(\Omega) \text{ and } v = 1 \text{ a.e.}, \\ +\infty & \text{otherwise.} \end{cases}$$

The proof of Γ-convergence consists of two steps. The first step is to prove the "lim inf inequality"

$$\liminf_{\varepsilon \to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \geq E_\omega(u)$$

for every sequence $u_\varepsilon \to u$, $v_\varepsilon \to v$. This is obtained in Section 3.2 in the case $N = 1$ by using most of the arguments proposed in [8], and the properties of SBV functions in one dimension (see Lemma 2.5). The case $N > 1$ is studied in Section 4.3, and it uses a special slicing argument (see Lemma 4.6).

The second step is the construction of a recovery sequence ($u_\varepsilon \to u, v_\varepsilon \to 1$) such that the term

$$\int_\Omega \left[\varepsilon |\nabla v|^2 + \frac{1}{4\varepsilon}(v-1)^2\right] \omega\, dx \tag{1.7}$$

only captures the information of ω^-. We note that for small $\varepsilon > 0$, (1.7) only penalizes a ε-neighborhood around the jump point of u. By using fine properties of SBV functions (see Theorem 2.4), we are able to incorporate u and v in our model such that (1.7) will only penalize along the direction $-\nu_{S_\omega}$. This will be carried out in Lemma 3.7.

We remark that the techniques we developed in this paper can be adapted to other functional models. For example,

1. the weighted Cahn-Hilliard model defined as
$$CH_{\omega,\varepsilon}(u) := \int_I \left[\varepsilon\, |\nabla u(x)|^2 + \frac{1}{\varepsilon} W(u)\right] \omega\, dx,$$
for $u \in W^{1,2}_\omega(\Omega)$ and with a double well potential function $W\colon \mathbb{R} \to [0,+\infty)$ such that $\{W = 0\} = \{0,1\}$ with the Γ-limit
$$CH_\omega(u) := c_W P_\omega(u)$$
defined for $u = \chi_E \in BV_\omega(\Omega)$, where
$$c_W := 2\int_0^1 \sqrt{W(s)}\, ds \text{ and } P_\omega(u) := \int_{S_u} \omega^- d\mathcal{H}^{N-1};$$
2. higher order singular perturbation models defined by the Γ-limit
$$H_\omega(u) := \int_\Omega |\nabla u|^2 \omega\, dx + \int_{S_u} \omega^-(x)\, d\mathcal{H}^{N-1},$$
and approximation energies
$$H_{\omega,\varepsilon}(u,v) := \int_\Omega |\nabla u|^2 v^2 \omega\, dx + \frac{1}{C(k)} \int_\Omega \left[\varepsilon^{2k-1} \left|\nabla^{(k)} v\right|^2 + \frac{1}{\varepsilon}(v-1)^2\right] \omega\, dx,$$
where
$$C(k) := \min\left\{ \int_{\mathbb{R}^+} \left|v^{(k)}\right|^2 + (v-1)^2 dx,\ v(0) = v'(0) = \cdots = v^{(k-1)}(0) = 0,\ \lim_{t\to\infty} v(t) = 1 \right\}.$$

The analysis of items 1 and 2 above is forthcoming (see [38]).

This article is organized as follows: In Section 2 we introduce some definitions and we recall preliminary results. In Section 3 we prove the one dimensional version of Theorem 1.1. Section 4 is devoted to the proof of our main result.

2. Definitions and Preliminary Results

Throughout this paper, $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary, and $I := (-1,1)$.

Definition 2.1. *We say that $u \in BV(\Omega)$ is a special function of bounded variation, and we write $u \in SBV(\Omega)$, if the Cantor part of its derivative, $D^c u$, is zero, so that (see [4], (3.89))*
$$Du = D^a u + D^j u = \nabla u \mathcal{L}^N \lfloor \Omega + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \lfloor S_u. \tag{2.1}$$
Moreover, we say that $u \in GSBV(\Omega)$ if $K \wedge u \vee -K \in SBV(\Omega)$ for all $K \in \mathbb{N}$.

Here we always identify $u \in SBV(\Omega)$ with its approximation representative $\bar{u}$, where

$$\bar{u}(x) := \frac{1}{2}\left[u^+(x) + u^-(x)\right],$$

with

$$u^+(x) := \inf\left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{\mathcal{L}^N(B(x,r) \cap \{u > t\})}{r^N} = 0\right\},$$

and

$$u^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{\mathcal{L}^N(B(x,r) \cap \{u < t\})}{r^N} = 0\right\}.$$

We note that $\bar{u}$ is Borel measurable (see [29], Lemma 1, page 210), and it can be shown that $\bar{u} = u$ $\mathcal{L}^N$-a.e. $x \in \Omega$, and that

$$(\bar{u})^+(x) = u^+(x) \text{ and } (\bar{u})^-(x) = u^-(x)$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ (see [29], Corollary 1, page 216). Furthermore, we have that

$$- < u^-(x) \leq u^+(x) < +\infty \tag{2.2}$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ (see [29], Theorem 2, page 211). The inequality (2.2) uniquely determines the sign of ν_u in (2.1).

Definition 2.2. *(The weight function) We say that $\omega\colon \Omega \to (0, +\infty]$ belongs to $\mathcal{W}(\Omega)$ if $\omega \in L^1(\Omega)$ and has a positive lower bound, i.e., there exists $l > 0$ such that*

$$\operatorname{ess\,inf}\{\omega(x), x \in \Omega\} \geq l. \tag{2.3}$$

Without loss of generality, we take $l = 1$. Moreover, in this paper we will only consider the cases in which ω is either a continuous function or a SBV function. If $\omega \in SBV$ then, in addition, we require that

$$\mathcal{H}^{N-1}(S_\omega) < \infty \text{ and } \mathcal{H}^{N-1}(\overline{S_\omega} \setminus S_\omega) = 0.$$

We next fix some notation which will be used throughout this paper.

Notation 2.3. Let $\Gamma \subset \Omega$ be a $\mathcal{H}^{N-1}$-rectifiable set and let $x \in \Gamma$ be given.

1. We denote by $\nu_\Gamma(x)$ a normal vector at x with respect to Γ, and $Q_{\nu_\Gamma}(x,r)$ is the cube centered at x with side length r and two faces normal to $\nu_\Gamma(x)$;
2. T_{x,ν_Γ} stands for the hyperplane normal to $\nu_\Gamma(x)$ and passing through x, and $\mathbb{P}_{x,\nu_\Gamma}$ stands for the projection operator from Γ onto T_{x,ν_Γ};
3. we define the hyperplane
$$T_{x,\nu_\Gamma}(t) := T_{x,\nu_\Gamma} + t\nu_\Gamma(x)$$
for $t \in \mathbb{R}$;
4. we introduce the half-spaces
$$H_{\nu_\Gamma}(x)^+ := \left\{y \in \mathbb{R}^N : \ \nu_\Gamma(x)\cdot(y-x) \geq 0\right\}$$
and
$$H_{\nu_\Gamma}(x)^- := \left\{y \in \mathbb{R}^N : \ \nu_\Gamma(x)\cdot(y-x) \leq 0\right\}.$$
Moreover, we define the half-cubes
$$Q^{\pm}_{\nu_\Gamma}(x,r) := Q_{\nu_\Gamma}(x,r) \cap H_{\nu_\Gamma}(x)^{\pm};$$
5. for given $\tau > 0$, we denote by $R_{\tau,\nu_\Gamma}(x,r)$ the part of $Q_{\nu_\Gamma}(x,r)$ which lies strictly between the two hyperplanes $T_{x,\nu_\Gamma}(-\tau r)$ and $T_{x,\nu_\Gamma}(\tau r)$;

6. we set
$$A_\delta := \{x \in \Omega : \operatorname{dist}(x, A) < \delta\} \tag{2.4}$$
for every $A \subset \Omega$ and $\delta > 0$.

Theorem 2.4 ([29], Theorem 3, page 213)**.** *Assume that $u \in BV(\Omega)$. Then*

1. for $\mathcal{H}^{N-1}$-a.e. $x_0 \in \Omega \setminus S_u$,
$$\lim_{r\to 0} ⨍_{B(x_0,r)} |u(x) - \bar{u}(x_0)|^{\frac{N}{N-1}}\, dx = 0;$$

2. for $\mathcal{H}^{N-1}$-a.e. $x_0 \in S_u$,
$$\lim_{r\to 0} ⨍_{B(x_0,r)\cap H_{\nu_{S_u}}(x_0)^{\pm}} \left|u(x) - u^{\pm}(x_0)\right|^{\frac{N}{N-1}}\, dx = 0;$$

3. for $\mathcal{H}^{N-1}$ a.e. $x_0 \in S_u$
$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \int_{S_u\cap Q_{\nu_{S_u}}(x_0,\varepsilon)} \left|u^+(x) - u^-(x)\right| d\mathcal{H}^{N-1}(x) = \left|u^+(x_0) - u^-(x_0)\right|.$$

Lemma 2.5. *Let $\omega \in SBV(I)$ be such that $\mathcal{H}^0(S_\omega) < \infty$. For every $x \in I$ the following statements hold:*

1. if $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty \subset I$ are such that $x_n < x < y_n$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$, then
$$\liminf_{n\to\infty} \operatorname*{ess\,inf}_{y\in(x_n,y_n)} \omega(y) \geq \omega^-(x); \tag{2.5}$$

2.
$$\lim_{\substack{z_n\to x\\ \{z_n\}_{n=1}^\infty \subset H^{\pm}_{\nu_{S_\omega}(x)}}} \bar{\omega}(z_n) = \omega^{\pm}(x); \tag{2.6}$$

3.
$$\limsup_{d_{\mathcal{H}}(K_n,x)\to 0} \operatorname*{ess\,sup}_{\substack{z\in K_n\\ K_n\subset\subset H^{\pm}_{\nu_{S_\omega}(x)}}} \omega(z) = \omega^{\pm}(x), \tag{2.7}$$

where $K_n \subset\subset H^{\pm}_{\nu_{S_\omega}(x)}$ and $d_{\mathcal{H}}$ denotes the Hausdorff distance (see Definition A.1).

Proof. If $x \notin S_\omega$, then there exists $\delta > 0$ such that
$$S_\omega \cap (x - \delta, x + \delta) = \varnothing,$$
and so ω is absolutely continuous in $(x - \delta, x + \delta)$, and (2.5)-(2.7) are trivially satisfied with $\omega(x) = \omega^-(x)$ and with equality in place of the inequality in (2.5).

Let $x \in S_\omega$ and, without loss of generality, assume that $x = 0$, and let x_n, $y_n \to 0$ with $x_n < 0 < y_n$ for all $n \in \mathbb{N}$. Since $\mathcal{H}^0(S_\omega) < \infty$, choose $\bar{r} > 0$ such that
$$S_\omega \cap (0 - \bar{r}, 0 + \bar{r}) = 0.$$
As $\bar{\omega}$ is absolutely continuous in $(-\bar{r}, 0)$ and $(0, \bar{r})$, we may extend $\bar{\omega}$ uniquely to $x = 0$ from the left and the right (see Exercise 3.7 (1) in [34]) to define
$$\bar{\omega}(0^+) := \lim_{x\searrow 0^+} \bar{\omega}(x) \text{ and } \bar{\omega}(0^-) := \lim_{x\nearrow 0^-} \bar{\omega}(x). \tag{2.8}$$

Assume that (the case $\bar{\omega}(0^-) \geq \bar{\omega}(0^+)$ can be treated similarly)

$$\bar{\omega}(0^-) \leq \bar{\omega}(0^+). \tag{2.9}$$

We first claim that

$$\liminf_{n\to\infty} \inf_{x\in(x_n,y_n)} \bar{\omega}(x) \geq \bar{\omega}(0^-). \tag{2.10}$$

Let $\varepsilon > 0$ be given. By (2.8) find $\bar{r} > \delta > 0$ small enough such that

$$\left|\bar{\omega}(x) - \bar{\omega}(0^-)\right| \leq \frac{1}{2}\varepsilon \text{ for all } x \in (-\delta, 0), \text{ and } \left|\bar{\omega}(x) - \bar{\omega}(0^+)\right| \leq \frac{1}{2}\varepsilon \text{ for all } x \in (0, \delta).$$

This, together with (2.9), yields

$$\bar{\omega}(x) \geq \bar{\omega}(0^-) - \frac{1}{2}\varepsilon,$$

for all $x \in (-\delta, \delta)$. Since $x_n \to 0$ and $y_n \to 0$, we may choose n large enough such that $(x_n, y_n) \subset (-\delta, \delta)$ and hence

$$\inf_{x\in(x_n,y_n)} \bar{\omega}(x) \geq \bar{\omega}(0^-) - \varepsilon.$$

Thus, (2.10) follows by the arbitrariness of $\varepsilon > 0$.

We next claim that

$$\bar{\omega}(0^\pm) = \omega^\pm(0). \tag{2.11}$$

By Theorem 2.4 part 2 and the fact that $\bar{\omega} = \omega$ $\mathcal{L}^1$-a.e., we have

$$\omega^-(0) = \lim_{r\to 0} \frac{1}{r} \int_{-r}^{0} \omega(t)\, dt = \lim_{r\to 0} \frac{1}{r} \int_{-r}^{0} \bar{\omega}(t)\, dt = \bar{\omega}(0^-),$$

where at the last equality we used the properties of absolutely continuous function and the definition of $\bar{\omega}(0^-)$. The equation $\bar{\omega}(0^+) = \omega^+(0)$ can be proved similarly.

Therefore

$$\liminf_{n\to\infty} \operatorname*{ess\,inf}_{x\in(x_n,y_n)} \omega(x) = \liminf_{n\to\infty} \inf_{x\in(x_n,y_n)} \bar{\omega}(x) \geq \bar{\omega}(0^-) = \omega^-(0),$$

which concludes (2.5), and (2.6) and (2.7) hold by (2.8) and (2.11). □

Definition 2.6. *(Weighted function spaces) Let $\omega \in \mathcal{W}(\Omega)$ and $1 \leq p < \infty$:*

1. $L^p_\omega(\Omega)$ is the space of functions $u \in L^p(\Omega)$ such that

$$\int_\Omega |u|^p\, \omega\, dx < \infty,$$

endowed with the norm

$$\|u - v\|_{L^p_\omega} := \left(\int_\Omega |u - v|^p\, \omega\, dx\right)^{\frac{1}{p}}$$

if $u,\ v \in L^p_\omega(\Omega)$;

2. *$W^{1,p}_\omega(\Omega)$ is the space of functions $u \in W^{1,p}(\Omega)$ such that*

$$u \in L^p_\omega(\Omega) \quad \textit{and } \nabla u \in L^p_\omega(\Omega;\mathbb{R}^N),$$

endowed with the norm

$$\|u - v\|_{W^{1,p}_\omega} := \|u - v\|_{L^p_\omega} + \|\nabla u - \nabla v\|_{L^p_\omega}$$

if u, $v \in W^{1,p}_\omega(\Omega)$;

3. *$BV_\omega(\Omega)$ is the space of functions $u \in BV(\Omega)$ such that*

$$u \in L^1_\omega(\Omega) \textit{ and } \int_\Omega \omega \, d\,|Du| < \infty,$$

endowed with the norm

$$\|u - v\|_{BV_\omega} := \|u - v\|_{L^1_\omega} + \int_\Omega \omega \, d\,|Du - Dv|$$

if $u, v \in BV_\omega(\Omega)$;

4. *$u \in SBV_\omega(\Omega)$ if $u \in BV_\omega(\Omega) \cap SBV(\Omega)$, and $u \in GSBV_\omega(\Omega)$ if $K \wedge u \vee -K \in SBV_\omega(\Omega)$ for all $K \in \mathbb{N}$.*

Lemma 2.7. *Let $\omega \in \mathcal{W}(\Omega)$ be given, and suppose that $u \in SBV_\omega(\Omega)$. Then*

$$\mathcal{H}^{N-1}(S_u \cap \{\omega = +\infty\}) = 0.$$

Proof. By Definition 2.6 we have

$$\begin{aligned} +\infty > \int_\Omega \omega \, d\,|Du| &= \int_\Omega |\nabla u| \, \omega \, dx + \int_{S_u} \left|u^+ - u^-\right| \omega \, d\mathcal{H}^{N-1} \\ &\geq \int_{S_u \cap \{\omega = +\infty\}} \left|u^+ - u^-\right| \omega \, d\mathcal{H}^{N-1}. \end{aligned} \tag{2.12}$$

Since $|u^+ - u^-|\,(x) > 0$ for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$, it follows from (2.12) that $\mathcal{H}^{N-1}(S_u \cap \{\omega = +\infty\}) = 0$. □

Lemma 2.8. *The space L^2_ω is a Hilbert space endowed with the inner product*

$$(u, v)_{L^2_\omega} := (u, v\,\omega)_{L^2} = \int u\, v\, \omega \, dx. \tag{2.13}$$

Proof. It is clear that (2.13) is an inner product. Also, $(u, u)_{L^2_\omega} = (u\sqrt{\omega}, u\sqrt{\omega})_{L^2} \geq 0$, and if $(u, u)_{L^2_\omega} = 0$ then by (2.3)

$$\int_\Omega u^2 \omega \, dx \geq \int_\Omega u^2 dx = 0,$$

and thus $u = 0$ *a.e.*

To see that L^2_ω is complete, and therefore a Hilbert space, let $\{u_n\}_{n=1}^\infty$ be a Cauchy sequence in L^2_ω and notice that $\{u_n\sqrt{\omega}\}_{n=1}^\infty$ is a Cauchy sequence in L^2. Hence, there is a function $v \in L^2$ such that $u_n\sqrt{\omega} \to v$ in L^2. Defining $u := v/\sqrt{\omega}$, we have that $u \in L^2_\omega$ and $u_n \to u$ in L^2_ω. □

Lemma 2.9. *Let* $\{u_n\}_{n=1}^{\infty} \subset W_{\omega}^{1,2}(\Omega)$ *be such that* $u_n \to u$ *in* L_{ω}^1 *and*

$$\sup \int_{\Omega} |\nabla u_n|^2 \, \omega \, dx < \infty.$$

Then, for every measurable set $A \subset \Omega$

$$\liminf_{n\to\infty} \int_A |\nabla u_n|^2 \, \omega \, dx \geq \int_A |\nabla u|^2 \, \omega \, dx,$$

and $u \in W_{\omega}^{1,2}(\Omega)$.

Proof. By (2.3) we have that $\{\nabla u_n\}_{n=1}^{\infty}$ is uniformly bounded in $L^2(\Omega, \mathbb{R}^N)$ and $u_n \to u$ in $L^1(\Omega)$. Hence $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega; \mathbb{R}^N)$, and using standard lower semi-continuity of convex energies (see [31], Theorem 6.3.7), we conclude that

$$+\infty > \liminf_{n\to\infty} \int_A |\nabla u_n|^2 \, \omega \, dx \geq \int_A |\nabla u|^2 \, \omega \, dx,$$

for every measurable subset $A \subset \Omega$. In particular, with $A = \Omega$ and using the fact that $1 \leq \omega$ a.e., we deduce that $u \in W_{\omega}^{1,2}(\Omega)$. □

Lemma 2.10. *Let* $u \in L_{\omega}^1(\Omega)$ *be such that*

$$\int_{\Omega} |\nabla u|^2 \, \omega \, dx + \int_{S_u} \omega \, d\mathcal{H}^{N-1} < +\infty. \tag{2.14}$$

Then $\mathcal{H}^{N-1}(S_u) < +\infty$ *and* $u \in GSBV_{\omega}(\Omega)$.

Proof. By (2.14) and (2.3)

$$\int_{\Omega} |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(S_u) < +\infty,$$

and hence by [8] we have that $u \in GSBV(\Omega)$. To show that $u \in GSBV_{\omega}(\Omega)$, we only need to verify that

$$\int_{S_{u_K}} |u_K^+ - u_K^-| \, \omega \, d\mathcal{H}^{N-1} < +\infty$$

for every $K \in \mathbb{N}$ and with $u_K := K \wedge u \vee -K$. Indeed, by (2.14)

$$\int_{S_{u_K}} |u_K^+ - u_K^-| \, \omega \, d\mathcal{H}^{N-1} \leq 2K \int_{S_{u_K}} \omega \, d\mathcal{H}^{N-1} \leq 2K \int_{S_u} \omega \, d\mathcal{H}^{N-1} < +\infty.$$

□

3. The One Dimensional Case

3.1. The Case $\omega \in \mathcal{W}(I) \cap C(I)$.

Let $\omega \in \mathcal{W}(I) \cap C(I)$ be given. Consider the functionals

$$E_{\omega,\varepsilon}(u,v) := \int_I v^2 |u'|^2 \, \omega \, dx + \int_I \left[\frac{\varepsilon}{2} |v'|^2 + \frac{1}{2\varepsilon}(v-1)^2 \right] \omega \, dx$$

for $(u,v) \in W_{\omega}^{1,2}(I) \times W^{1,2}(I)$, and let

$$E_{\omega}(u) := \int_I |u'|^2 \, \omega \, dx + \sum_{x \in S_u} \omega(x)$$

be defined for $u \in GSBV_\omega(I)$ (Note that $E_{1,\varepsilon}(u,v)$ and $E_1(u)$ are, respectively, the non-weighted Ambrosio-Tortorelli approximation scheme and Mumford-Shah functional studied in [8]).

Theorem 3.1 (Γ-Convergence). *Let $\mathcal{E}_{\omega,\varepsilon}\colon L^1_\omega(I) \times L^1(I) \to [0,+\infty]$ be defined by*

$$\mathcal{E}_{\omega,\varepsilon}(u,v) := \begin{cases} E_{\omega,\varepsilon}(u,v) & \text{if } (u,v) \in W^{1,2}_\omega(I) \times W^{1,2}(I),\ 0 \le v \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the functionals $\mathcal{E}_{\omega,\varepsilon}$ Γ-converge, with respect to the $L^1_\omega \times L^1$ topology, to the functional

$$\mathcal{E}_{\omega}(u,v) := \begin{cases} E_{\omega}(u) & \text{if } u \in GSBV_\omega(I) \text{ and } v = 1 \text{ a.e.}, \\ +\infty & \text{otherwise.} \end{cases}$$

We begin with an auxiliary proposition.

Proposition 3.2. *Let $\{v_\varepsilon\}_{\varepsilon>0} \subset W^{1,2}(I)$ be such that $0 \le v_\varepsilon \le 1$, $v_\varepsilon \to 1$ in $L^1(I)$ and pointwise a.e., and*

$$\limsup_{\varepsilon\to 0} \int_I \left[\frac{\varepsilon}{2}\left|v_\varepsilon'\right|^2 + \frac{1}{2\varepsilon}(v_\varepsilon - 1)^2\right] dx < \infty.$$

Then for arbitrary $0 < \eta < 1$ there exists an open set $H_\eta \subset I$ satisfying:
1. the set $I \setminus H_\eta$ is a collection of finitely many points in I;
2. for every set K compactly contained in H_η, we have $K \subset B^\eta_\varepsilon$ for $\varepsilon > 0$ small enough, where

$$B^\eta_\varepsilon := \left\{x \in I : v^2_\varepsilon(x) \ge \eta\right\}. \tag{3.1}$$

Proposition 3.2 is adapted from [8], page 1020-1021 (see Lemma A.3).

Proposition 3.3. (Γ-lim inf) *For $u \in L^1_\omega(I)$, let*

$$E^-_\omega(u) := \inf\Big\{\liminf_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : \\ (u_\varepsilon, v_\varepsilon) \in W^{1,2}_\omega(I) \times W^{1,2}(I), u_\varepsilon \to u \text{ in } L^1_\omega,\ v_\varepsilon \to 1 \text{ in } L^1,\ 0 \le v_\varepsilon \le 1\Big\}.$$

We have

$$E^-_\omega(u) \ge E_\omega(u).$$

Proof. If $E^-_\omega(u) = +\infty$ then there is nothing to prove. Assume that $M := E^-_\omega(u) < \infty$. Choose u_ε and v_ε admissible for $E^-_\omega(u)$ such that

$$\lim_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) = E^-_\omega(u) < \infty,$$

and note that $v_\varepsilon \to 1$ in $L^1(I)$. Since $\inf_{x\in\Omega} \omega(x) \ge 1$, we have

$$\liminf_{\varepsilon\to 0} E_{1,\varepsilon}(u_\varepsilon, v_\varepsilon) \le \liminf_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) < +\infty,$$

and by [8] we obtain that

$$u \in GSBV(I) \text{ and } \mathcal{H}^0(S_u) < +\infty. \tag{3.2}$$

Let $\bar\varepsilon > 0$ be sufficiently small so that, for all $0 < \varepsilon < \bar\varepsilon$,

$$E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \le M + 1.$$

We claim, separately, that

$$\int_I \left|u'\right|^2 \omega\, dx \le \liminf_{\varepsilon\to 0} \int_I \left|u_\varepsilon'\right|^2 v^2_\varepsilon\, \omega\, dx < +\infty, \tag{3.3}$$

and

$$\sum_{x\in S_u}\omega(x) \le \liminf_{\varepsilon\to 0}\int_I \left[\frac{1}{2}\varepsilon\left|v_\varepsilon'\right|^2 + \frac{1}{2\varepsilon}(1-v_\varepsilon)^2\right]\omega\,dx < +\infty. \tag{3.4}$$

Note that (3.3), (3.4), and Lemma 2.10 will yield $u \in GSBV_\omega(I)$.

Up to the extraction of a (not relabeled) subsequence, we have $u_\varepsilon \to u$ and $v_\varepsilon \to 1$ *a.e.* in I with

$$\limsup_{\varepsilon\to 0}\int_I \left[\frac{1}{2}\varepsilon\left|v_\varepsilon'\right|^2 + \frac{1}{2\varepsilon}(1-v_\varepsilon)^2\right]\,dx \le \limsup_{\varepsilon\to 0}\int_I \left[\frac{1}{2}\varepsilon\left|v_\varepsilon'\right|^2 + \frac{1}{2\varepsilon}(1-v_\varepsilon)^2\right]\omega\,dx < +\infty.$$

Therefore, up to the extraction of a (not relabeled) subsequence, we can apply Proposition 3.2 and deduce that, for a fixed $\eta \in (1/2, 1)$, there exists an open set H_η such that the set $I \setminus H_\eta$ contains only a finite number of points, and for every compact subset $K \subset\subset H_\eta$, K is contained in B_ε^η for $0 < \varepsilon < \varepsilon(K)$, where B_ε^η is defined in (3.1). We have

$$\begin{aligned}\int_K \left|u'\right|^2\omega\,dx &\le \liminf_{\varepsilon\to 0}\int_K \left|u_\varepsilon'\right|^2\omega\,dx\\ &\le \frac{1}{\eta}\liminf_{\varepsilon\to 0}\int_K v_\varepsilon^2\left|u_\varepsilon'\right|^2\omega\,dx \le \frac{1}{\eta}\liminf_{\varepsilon\to 0}\int_I v_\varepsilon^2\left|u_\varepsilon'\right|^2\omega\,dx,\end{aligned} \tag{3.5}$$

where we used Lemma 2.9 in the first inequality. By letting $K \nearrow H_\eta$ on the left hand side of (3.5) first and then $\eta \nearrow 1$ on the right hand side, we proved that

$$\int_I \left|u'\right|^2\omega\,dx \le \liminf_{\varepsilon\to 0}\int_I v_\varepsilon^2\left|u_\varepsilon'\right|^2\omega\,dx, \tag{3.6}$$

where we used the fact that $|I \setminus H_\eta| = 0$.

We claim that $S_u \subset I \setminus H_\eta$. Indeed, if there is $x_0 \in S_u \cap H_\eta$, since H_η is open there exists an open interval I_0' containing x_0 and compactly contained in H_η such that for $0 < \varepsilon < \varepsilon_0'$

$$\int_{I_0'}\left|u_\varepsilon'\right|^2 dx \le \int_{I_0'}\left|u_\varepsilon'\right|^2\omega\,dx \le \frac{1}{\eta}\int_I v_\varepsilon^2\left|u_\varepsilon'\right|^2\omega\,dx \le 2(M+1).$$

Thus $u \in W^{1,2}(I_0')$, and hence is continuous at x_0, which contradicts the fact that $x_0 \in S_u$.

Let $t \in S_u$, and for simplicity assume that $t = 0$. We claim that there exist $\left\{t_n^1\right\}_{n=1}^\infty$, $\left\{t_n^2\right\}_{n=1}^\infty$, and $\left\{s_n\right\}_{n=1}^\infty$ such that $-1 < t_n^1 < s_n < t_n^2 < 1$,

$$\lim_{n\to\infty} t_n^1 = \lim_{n\to\infty} t_n^2 = \lim_{n\to\infty} s_n = 0,$$

and, up to the extraction of a subsequence of $\left\{v_\varepsilon\right\}_{\varepsilon>0}$,

$$\lim_{n\to\infty} v_{\varepsilon(n)}(t_n^1) = \lim_{n\to\infty} v_{\varepsilon(n)}(t_n^2) = 1, \text{ and } \lim_{n\to\infty} v_{\varepsilon(n)}(s_n) = 0. \tag{3.7}$$

Because $I \setminus H_\eta$ is discrete and $0 \in I \setminus H_\eta$, we may choose $\delta_0 > 0$ small enough such that

$$(-2\delta_0, 2\delta_0) \cap (I \setminus H_\eta) = \{0\}.$$

We claim that

$$\limsup_{\delta\to 0^+}\limsup_{\varepsilon\to 0^+}\inf_{x\in I_\delta} v_\varepsilon(x) = 0, \tag{3.8}$$

where $I_\delta := (-\delta, \delta)$. Assume that

$$\limsup_{\delta\to 0^+} \limsup_{\varepsilon\to 0^+} \inf_{x\in I_\delta} v_\varepsilon(x) =: \alpha > 0.$$

Then there exists $0 < \delta_\alpha < \delta_0$ such that

$$\limsup_{\varepsilon\to 0^+} \inf_{x\in I_{\delta_\alpha}} v_\varepsilon(x) \geq \frac{2}{3}\alpha > 0.$$

Up to the extraction of a subsequence of $\{v_\varepsilon\}_{\varepsilon>0}$, there exists $\varepsilon_0^{\delta_\alpha} > 0$ such that

$$\inf_{x\in I_{\delta_\alpha}} v_\varepsilon(x) \geq \frac{1}{2}\alpha > 0,$$

for all $0 < \varepsilon < \varepsilon_0^{\delta_\alpha}$, and we have

$$\begin{aligned}
&\int_{I_{\delta_\alpha}} |u'|^2\, dx \leq \int_{I_{\delta_\alpha}} |u'|^2\, \omega\, dx \\
&\leq \liminf_{\varepsilon\to 0} \int_{I_{\delta_\alpha}} |u_\varepsilon'|^2\, \omega\, dx \leq \liminf_{\varepsilon\to 0} \frac{2}{\alpha} \int_{I_{\delta_\alpha}} |u_\varepsilon'|^2\, v_\varepsilon^2\, \omega\, dx \leq \liminf_{\varepsilon\to 0} \frac{2}{\alpha} \int_I |u_\varepsilon'|^2\, v_\varepsilon^2\, \omega\, dx < \frac{2}{\alpha}(M+1).
\end{aligned}$$

Hence $u \in W^{1,2}(I_{\delta_\alpha})$ and so u is continuous at $0 \in S_u$, and we reduce a contradiction. Therefore, in view of (3.8) we may find $\delta_n \to 0^+$, $\varepsilon(n) \to 0^+$, and $s_n \in (-\delta_n, \delta_n)$ such that

$$\lim_{n\to\infty} s_n = 0 \text{ and } \lim_{n\to\infty} v_{\varepsilon(n)}(s_n) = 0.$$

We claim that for all $\tau \in (0, 1/2)$,

$$\lim_{n\to\infty} \left[\inf_{x\in(s_n-\tau,s_n)} (1 - v_{\varepsilon(n)}(x)) + \inf_{y\in(s_n,s_n+\tau)} (1 - v_{\varepsilon(n)}(x)) \right] = 0. \tag{3.9}$$

To reach a contradiction, assume that there exists $\tau \in (0, 1/2)$ such that

$$\limsup_{n\to\infty} \left[\inf_{x\in(s_n-\tau,s_n)} (1 - v_{\varepsilon(n)}(x)) + \inf_{x\in(s_n,s_n+\tau)} (1 - v_{\varepsilon(n)}(x)) \right] =: \beta > 0.$$

Without loss of generality, suppose that

$$\limsup_{n\to\infty} \inf_{x\in(s_n-\tau,s_n)} (1 - v_{\varepsilon(n)}(x)) \geq \frac{1}{2}\beta > 0.$$

Then

$$\liminf_{n\to\infty} \sup_{x\in(s_n-\tau,s_n)} v_{\varepsilon(n)}(x) \leq 1 - \frac{1}{2}\beta,$$

which implies that

$$\sup_{x\in(s_{n_k}-\tau,s_{n_k})} v_{\varepsilon(n_k)}(x) \leq 1 - \frac{1}{3}\beta \tag{3.10}$$

for a subsequence $\{\varepsilon(n_k)\}_{k=1}^\infty \subset \{\varepsilon(n)\}_{n=1}^\infty$. However, (3.10) contradicts the fact that $v_{\varepsilon(n_k)}(x) \to 1$ a.e. since for k large enough so that $|s_{n_k}| < \tau/4$ it holds

$$(s_{n_k} - \tau, s_{n_k}) \supset \left(-\frac{3}{4}\tau, -\frac{\tau}{4}\right).$$

Therefore, in view of (3.9) we may find $t_m^1 \in (s_{n(m)} - 1/m, s_{n(m)})$ and $t_m^2 \in (s_{n(m)}, s_{n(m)} + 1/m)$ such that

$$\lim_{n\to\infty} t_m^1 = \lim_{n\to\infty} t_m^2 = 0 \text{ and } \lim_{n\to\infty} v_{\varepsilon(n(m))}(t_m^1) = \lim_{n\to\infty} v_{\varepsilon(n(m))}(t_m^2) = 1.$$

We next show that

$$\liminf_{m\to\infty} \int_{t_m^1}^{s_{n(m)}} \left[\frac{1}{2}\varepsilon(n(m)) \left|(v_{\varepsilon(n(m))})'\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] dx \geq \frac{1}{2}.$$

Indeed, we have

$$\begin{aligned}
&\liminf_{m\to\infty} \int_{t_m^1}^{s_{n(m)}} \left[\frac{1}{2}\varepsilon(n(m)) \left|(v_{\varepsilon(n(m))})'\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] dx \\
&\geq \liminf_{m\to\infty} \int_{t_m^1}^{s_{n(m)}} (1 - v_{\varepsilon(n(m))}) \left|v'_{\varepsilon(n(m))}\right| dx \geq \liminf_{m\to\infty} \left| \int_{t_m^1}^{s_{n(m)}} (1 - v_{\varepsilon(n(m))}) v'_{\varepsilon(n(m))} dx \right| \\
&= \liminf_{m\to\infty} \frac{1}{2} \left| \int_{t_m^1}^{s_{n(m)}} \frac{d}{dt}(1 - v_{\varepsilon(n(m))})^2 dx \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left[(1 - v_{\varepsilon(n(m))}(s_{n(m)}))^2 - (1 - v_{\varepsilon(n(m))}(t_m^1))^2\right] = \frac{1}{2},
\end{aligned}$$

where we used (3.7). Similarly, we obtain

$$\liminf_{m\to\infty} \int_{s_{n(m)}}^{t_m^2} \left[\frac{1}{2}\varepsilon(n(m)) \left|(v_{\varepsilon(n(m))})'\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] dx \geq \frac{1}{2}.$$

We observe that, since ω is positive,

$$\begin{aligned}
&\int_{t_m^1}^{t_m^2} \left[\frac{1}{2}\varepsilon(n(m)) \left|v'_{\varepsilon(n(m))}\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] \omega(x)\, dx \\
&\geq \left(\inf_{r\in(t_m^1, t_m^2)} \omega(r)\right) \cdot \left\{ \int_{t_m^1}^{s_{n(m)}} \left[\frac{1}{2}\varepsilon(n(m)) \left|v'_{\varepsilon(n(m))}\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] dx \right. \qquad (3.11) \\
&\quad \left. + \int_{s_{n(m)}}^{t_m^2} \left[\frac{1}{2}\varepsilon(n(m)) \left|(v_{\varepsilon(n(m))})'\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] dx \right\},
\end{aligned}$$

and so

$$\begin{aligned}
&\liminf_{m\to\infty} \int_{t_m^1}^{t_m^2} \left[\frac{1}{2}\varepsilon(n(m)) \left|v'_{\varepsilon(n(m))}\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] \omega(x)\, dx \\
&\geq \left(\liminf_{m\to\infty} \inf_{r\in(t_m^1, t_m^2)} \omega(r)\right) \liminf_{n\to\infty} \left\{ \int_{t_m^1}^{s_{n(m)}} \left[\frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2 + \frac{\varepsilon}{2} \left|(v_{\varepsilon(n(m))})'\right|^2\right] dx \right. \\
&\quad \left. + \int_{s_{n(m)}}^{t_m^2} \left[\frac{1}{2}\varepsilon(n(m)) \left|v'_{\varepsilon(n(m))}\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] dx \right\} \\
&\geq \left(\frac{1}{2} + \frac{1}{2}\right) \omega(0) = \omega(0),
\end{aligned}$$

where we used the fact that ω is continuous at 0.

Finally, since $S_u \subset I \setminus H_\eta$, by (3.2) we have that S_u is a finite collection of points, and we may repeat the above argument for all $t \in S_u$ by partitioning I into non-overlaping intervals where there is at most one point of S_u, to deduce that

$$\liminf_{\varepsilon\to 0} \int_I \left[\frac{1}{2}\varepsilon\, |v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(1-v_\varepsilon)^2\right]\omega(x)\,dx \geq \sum_{x\in S_u}\omega(x). \tag{3.12}$$

In view of (3.6) and (3.12), we conclude that

$$\liminf_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \geq E_\omega(u).$$

□

Proposition 3.4. (Γ-lim sup) *For $u \in L^1_\omega(I)\cap L^\infty(I)$, let*

$$E^+_\omega(u) := \inf\Big\{\limsup_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : (u_\varepsilon, v_\varepsilon) \in W^{1,2}_\omega(I)\times W^{1,2}(I), u_\varepsilon \to u \text{ in } L^1_\omega,\ v_\varepsilon \to 1 \text{ in } L^1,\ 0 \leq v_\varepsilon \leq 1\Big\}.$$

We have

$$E^+_\omega(u) \leq E_\omega(u). \tag{3.13}$$

Proof. Without loss of generality, assume that $E_\omega(u) < \infty$. Then by Lemma 2.10 we have $u \in GSBV_\omega(I)$ and $\mathcal{H}^0(S_u) < \infty$. To prove (3.13), we show that there exist $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,2}_\omega(I)$ and $\{v_\varepsilon\}_{\varepsilon>0} \subset W^{1,2}(I)$ such that $u_\varepsilon \to u$ in L^1_ω, $v_\varepsilon \to 1$ in L^1, $0 \leq v_\varepsilon \leq 1$, and

$$\limsup_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \leq E_\omega(u). \tag{3.14}$$

Step 1: Assume that $S_u = \{0\}$.

Fix $\eta > 0$, and let $T > 0$ and $v_0 \in W^{1,2}(0,T)$ be such that

$$0 \leq v_0 \leq 1 \quad \text{and} \quad \int_0^T \left[(1-v_0)^2 + |v_0'|^2\right] dx \leq 1+\eta, \tag{3.15}$$

with $v_0(0) = 0$ and $v_0(T) = 1$.

For $\xi_\varepsilon = o(\varepsilon)$ we define

$$v_\varepsilon(x) := \begin{cases} 0 & \text{if } |x| \leq \xi_\varepsilon, \\ v_0\left(\frac{|x|-\xi_\varepsilon}{\varepsilon}\right) & \text{if } \xi_\varepsilon < |x| < \xi_\varepsilon + \varepsilon T, \\ 1 & \text{if } |x| \geq \xi_\varepsilon + \varepsilon T. \end{cases} \tag{3.16}$$

Since $\|v_\varepsilon\|_{L^\infty(I)} \leq 1$, by Lebesgue Dominated Convergence Theorem we have $v_\varepsilon \to 1$ in L^1. Let

$$u_\varepsilon(x) := \begin{cases} u(x) & \text{if } |x| \geq \frac{1}{2}\xi_\varepsilon, \\ \text{affine from } u\left(-\frac{1}{2}\xi_\varepsilon\right) \text{ to } u\left(\frac{1}{2}\xi_\varepsilon\right) & \text{if } |x| < \frac{1}{2}\xi_\varepsilon. \end{cases} \tag{3.17}$$

and we observe that (recall in assumption we have $u \in L^\infty(I)$)

$$\|u_\varepsilon\|_{L^\infty(I)} \leq \|u\|_{L^\infty(I)},$$

and

$$\int_I \|u\|_{L^\infty(I)}\,\omega\,dx < \infty.$$

Therefore, by Lebesgue Dominated Convergence Theorem we deduce that $u_\varepsilon \to u$ in L^1_ω. Moreover, by (3.16) and (3.17) we observe that

$$v_\varepsilon^2\,|u_\varepsilon'|^2 = \begin{cases} v_\varepsilon^2\,|u'|^2 & \text{if } x \geq |\xi_\varepsilon|\,, \\ 0 & \text{if } x < |\xi_\varepsilon|\,, \end{cases}$$

and so $v_\varepsilon^2\,|u_\varepsilon'|^2 \leq |u'|^2$. Since $E_\omega(u) < \infty$ we have $u' \in L^2_\omega(I)$, by Lebesgue Dominated Convergence Theorem we obtain

$$\lim_{\varepsilon\to 0} \int_I v_\varepsilon^2\,|u_\varepsilon'|^2 \omega\,dx = \int_I |u'|^2\,\omega\,dx.$$

Next, since ω is positive we have

$$\begin{aligned}
&\int_I \left[\frac{\varepsilon}{2}\,|v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon-1)^2\right]\omega(x)\,dx \\
&= \int_{-\xi_\varepsilon-\varepsilon T}^{-\xi_\varepsilon} \left[\frac{\varepsilon}{2}\,|v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon-1)^2\right]\omega(x)\,dx + \int_{\xi_\varepsilon}^{\xi_\varepsilon+\varepsilon T} \left[\frac{\varepsilon}{2}\,|v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon-1)^2\right]\omega(x)\,dx + \frac{1}{2\varepsilon}\int_{-\xi_\varepsilon}^{\xi_\varepsilon}\omega(x)dx \\
&\leq \left(\sup_{t\in(-\xi_\varepsilon-\varepsilon T,\xi_\varepsilon+\varepsilon T)} \omega(t)\right)\cdot\left\{\int_{-\xi_\varepsilon-\varepsilon T}^{-\xi_\varepsilon} \left[\frac{\varepsilon}{2}\,|v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon-1)^2\right]\,dx\right. \\
&\quad \left. + \int_{\xi_\varepsilon}^{\xi_\varepsilon+\varepsilon T} \left[\frac{\varepsilon}{2}\,|v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon-1)^2\right]\,dx\right\} + \frac{\xi_\varepsilon}{\varepsilon}\,\|\omega\|_{L^\infty}\,.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_I \left[\frac{\varepsilon}{2}\,|v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon-1)^2\right]\omega(x)\,dx \\
&\qquad \leq \limsup_{\varepsilon\to 0}\left(\sup_{t\in(-\xi_\varepsilon-\varepsilon T,\xi_\varepsilon+\varepsilon T)} \omega(t)\right)\cdot \\
&\qquad\quad \limsup_{\varepsilon\to 0}\left\{\int_{-\xi_\varepsilon-\varepsilon T}^{-\xi_\varepsilon} \left[\frac{\varepsilon}{2}\,|v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon-1)^2\right]\,dx + \int_{\xi_\varepsilon}^{\xi_\varepsilon+\varepsilon T} \left[\frac{\varepsilon}{2}\,|v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon-1)^2\right]\,dx\right\} \\
&\qquad \leq \omega(0)(1+\eta),
\end{aligned}$$

where we used (3.15).

We conclude that

$$\limsup_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \leq \int_I |u'|^2\,\omega\,dx + \omega(0)(1+\eta),$$

and (3.14) follows by the arbitrariness of η.

Step 2: In the general case in which S_u is finite, we obtain u_ε by repeating the construction in Step 1 (see (3.17)) in small non-overlapping intervals centered at each point in S_u. To obtain v_ε, we repeat the construction (3.16) in those intervals and extend by 1 in the complement of the union of those intervals. Hence, by Step 1 we have

$$\limsup_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \le \int_I |u'|^2\,\omega\,dx + (1+\eta)\sum_{x\in S_u}\omega(x),$$

and again (3.14) follows by letting $\eta \to 0^+$. □

Proof of Theorem 3.1. The lim inf inequality follows from Proposition 3.3. For the lim sup inequality, we note that for any given $u \in GSBV_\omega$ such that $E_\omega(u) < +\infty$, by Lebesgue Monotone Convergence Theorem we have that

$$E_\omega(u) = \lim_{K\to\infty} E_\omega(K \wedge u \vee -K),$$

and hence a diagonal argument together with Proposition 3.4 conclude the proof. □

3.2. **The Case $\omega \in \mathcal{W}(I) \cap SBV(I)$.**

Consider the functionals

$$E_{\omega,\varepsilon}(u,v) := \int_I |u'|^2\, v^2\omega\,dx + \int_I \left[\frac{\varepsilon}{2}|v'|^2 + \frac{1}{2\varepsilon}(v-1)^2\right]\omega\,dx$$

for $(u,v) \in W^{1,2}_\omega(I)\times W^{1,2}(I)$, and for $u \in GSBV_\omega(I)$ let

$$E_\omega(u) := \int_I |u'|^2\,\omega\,dx + \sum_{x\in S_u}\omega^-(x).$$

We note that if $\omega \in \mathcal{W}(I)\cap SBV(I)$ and ω is continuous in a neighborhood of S_u, for $u \in GSBV_\omega(I)$, then

$$\sum_{x\in S_u}\omega^-(x) = \sum_{x\in S_u}\omega(x)$$

and Theorem 3.1 still holds.

Here we study the case in which ω is no longer continuous on a neighborhood of S_u. We recall that $\omega \in SBV(I)$ implies that $\omega \in L^\infty(I)$ and by definition of $\omega \in \mathcal{W}(I)$, we have $\mathcal{H}^0(S_\omega) < \infty$. Also, we note that ω^- is defined $\mathcal{H}^0$-a.e, hence everywhere in I.

Theorem 3.5. *Let $\mathcal{E}_\varepsilon\colon L^1_\omega(I)\times L^1(I) \to [0,+\infty]$ be defined by*

$$\mathcal{E}_{\omega,\varepsilon}(u,v) := \begin{cases} E_{\omega,\varepsilon}(u,v) & \text{if } (u,v)\in W^{1,2}_\omega(I)\times W^{1,2}(I),\ 0\le v\le 1,\\ +\infty & \text{otherwise.}\end{cases}$$

Then the functionals $\mathcal{E}_{\omega,\varepsilon}$ Γ-converge, with respect to the $L^1_\omega \times L^1$ topology, to the functional

$$\mathcal{E}_\omega(u,v) := \begin{cases} E_\omega(u) & \text{if } u\in GSBV_\omega(I) \text{ and } v = 1 \text{ a.e.},\\ +\infty & \text{otherwise.}\end{cases}$$

The proof of Theorem 3.5 will be split into two propositions.

Proposition 3.6. (Γ-lim inf) *For $u \in L^1_\omega(I)$, let*

$$E^-_\omega(u) := \inf\Big\{\liminf_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon,v_\varepsilon) : (u_\varepsilon,v_\varepsilon)\in W^{1,2}_\omega(I)\times W^{1,2}(I),\ u_\varepsilon\to u \text{ in } L^1_\omega, v_\varepsilon \to 1 \text{ in } L^1,\ 0\le v_\varepsilon\le 1\Big\}.$$

We have

$$E_\omega^-(u) \geq E_\omega(u).s_{n(m)}$$

Proof. Without lose of generality, assume that $E_\omega^-(u) < +\infty$. We use the same arguments of the proof of Proposition 3.3 until (3.11). In particular, (3.2) and (3.3) still hold, that is

$$\mathcal{H}^0(S_u) < +\infty \text{ and } \int_I |u'|^2 \,\omega\, dx \leq \liminf_{\varepsilon\to 0} \int_I |u_\varepsilon'|^2 \, v_\varepsilon^2 \,\omega\, dx.$$

Invoking (3.11), we have

$$\begin{aligned}
&\liminf_{m\to\infty} \int_{t_m^1}^{t_m^2} \left[\frac{1}{2}\varepsilon(n(m)) \left|v_{\varepsilon(n(m))}'\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] \omega(x)\, dx \\
&\geq \left(\liminf_{m\to\infty} \operatorname*{ess\,inf}_{r\in(t_m^1,t_m^2)} \omega(r)\right) \cdot \liminf_{n\to\infty} \left\{ \int_{t_m^1}^{s_{n(m)}} \left[\frac{1}{2}\varepsilon(n(m)) \left|(v_{\varepsilon(n(m))})'\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] dx \right. \\
&\qquad \left. + \int_{s_{n(m)}}^{t_m^2} \left[\frac{1}{2}\varepsilon(n(m)) \left|(v_{\varepsilon(n(m))})'\right|^2 + \frac{1}{2\varepsilon(n(m))}(1 - v_{\varepsilon(n(m))})^2\right] dx \right\} \\
&\geq \omega^-(0) \left(\frac{1}{2} + \frac{1}{2}\right) = \omega^-(0),
\end{aligned}$$

where the last step is justified by (2.5).

Since S_u is finite, we may repeat the above argument for all $t \in S_u$ by partitioning I into finitely many non-overlapping intervals where there is at most one point of S_u, to conclude that

$$\liminf_{\varepsilon\to 0} \int_I \left[\frac{1}{2}\varepsilon\, |v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(1 - v_\varepsilon)^2\right] \omega(x)\, dx \geq \sum_{x\in S_u} \omega^-(x),$$

as desired. □

The construction of the recovery sequence uses a reflection argument nearby points of $S_\omega \cap S_u$.

Proposition 3.7. (Γ-lim sup) *For $u \in L_\omega^1(I) \cap L^\infty(I)$, let*

$$\begin{aligned}
E_\omega^+(u) := \inf \Big\{ &\limsup_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : \\
&(u_\varepsilon, v_\varepsilon) \in W_\omega^{1,2}(I) \times W^{1,2}(I), u_\varepsilon \to u \text{ in } L_\omega^1,\, v_\varepsilon \to 1 \text{ in } L^1,\, 0 \leq v_\varepsilon \leq 1 \Big\}.
\end{aligned}$$

We have

$$E_\omega^+(u) \leq E_\omega(u). \tag{3.18}$$

Proof. To prove (3.18), we only need to explicitly construct a sequence $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset W_\omega^{1,2}(I) \times W^{1,2}(I)$ such that $u_\varepsilon \to u$ in L_ω^1, $v_\varepsilon \to 1$ in L^1, $0 \leq v_\varepsilon \leq 1$, and

$$\limsup_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \leq E_\omega(u). \tag{3.19}$$

Step 1: Assume that $\{0\} = S_u \subset S_\omega$.

Recall that we always identify ω with its approximation representative $\bar{\omega}$, and by (2.6) we may assume that (the converse situation may be dealt with similarly)

$$\lim_{t\nearrow 0^-} \omega(t) = \omega^-(0) \text{ and } \lim_{t\searrow 0^+} \omega(t) = \omega^+(0).$$

Fix $\eta > 0$. For $\varepsilon > 0$ small enough, and with $\xi_\varepsilon = o(\varepsilon)$, as in (3.15), (3.16) let

$$\tilde{v}_\varepsilon(x) := \begin{cases} 0 & \text{if } |x| \le \xi_\varepsilon \\ v_0\left(\frac{|x|-\xi_\varepsilon}{\varepsilon}\right) & \text{if } \xi_\varepsilon < |x| < \xi_\varepsilon + \varepsilon T \\ 1 & \text{if } |x| \ge \xi_\varepsilon + \varepsilon T, \end{cases}$$

and define

$$v_\varepsilon(x) := \tilde{v}_\varepsilon(x + 2\xi_\varepsilon + \varepsilon T).$$

Note that from (3.16) $v_\varepsilon \to 1$ a.e., and since $0 \le v_\varepsilon \le 1$, by Lebesgue Dominated Convergence Theorem we have $v_\varepsilon \to v$ in L^1. We also note that

$$\frac{\varepsilon}{2}\left|v_\varepsilon'(x)\right|^2 + \frac{1}{2\varepsilon}(1 - v_\varepsilon(x))^2 = 0 \tag{3.20}$$

if $x \in (-1, -3\xi_\varepsilon - 2\varepsilon T) \cup (-\xi_\varepsilon, 1)$, and if $x \in (-3\xi_\varepsilon - \varepsilon T, -\xi_\varepsilon - \varepsilon T)$ then

$$v_\varepsilon(x) = 0. \tag{3.21}$$

Set

$$\tilde{u}_\varepsilon(x) := \begin{cases} u(x) & \text{if } x \in (-1, -2\xi_\varepsilon - \varepsilon T) \cup (0, 1), \\ u(-x) & \text{if } x \in [-2\xi_\varepsilon - \varepsilon T, 0]. \end{cases}$$

Observe that $\tilde{u}_\varepsilon(x)$ is continuous at 0 since $\tilde{u}_\varepsilon^+(0) = \tilde{u}_\varepsilon^-(0) = u^+(0)$ by the definition of $\tilde{u}_\varepsilon(x)$, and $\tilde{u}_\varepsilon$ may only jump at $t = -2\xi_\varepsilon - \varepsilon T$ but not at $t = 0$ where u jumps.

We define the recovery sequence

$$u_\varepsilon(x) := \begin{cases} \tilde{u}_\varepsilon(x) & \text{if } x \in I \setminus [-2.5\xi_\varepsilon - \varepsilon T, -1.5\xi_\varepsilon - \varepsilon T], \\ \text{affine from } \tilde{u}_\varepsilon(-2.5\xi_\varepsilon - \varepsilon T) \text{ to } \tilde{u}_\varepsilon(-1.5\xi_\varepsilon - \varepsilon T) & \text{if } x \in [-2.5\xi_\varepsilon - \varepsilon T, -1.5\xi_\varepsilon - \varepsilon T]. \end{cases}$$

We claim that

$$\lim_{\varepsilon\to 0} \int_I |u_\varepsilon - u|\, \omega\, dx = 0 \tag{3.22}$$

and

$$\limsup_{\varepsilon\to 0} \int_I \left|u_\varepsilon'\right|^2 v_\varepsilon^2\, \omega\, dx \le \int_I \left|u'\right|^2 \omega\, dx. \tag{3.23}$$

To show (3.22), we observe that

$$\lim_{\varepsilon\to 0} \int_I |u_\varepsilon - u|\, \omega\, dx \le \lim_{\varepsilon\to 0} \int_{-2.5\xi_\varepsilon - \varepsilon T}^0 |u_\varepsilon - u|\, \omega\, dx \le \lim_{\varepsilon\to 0} 2\, \|u\|_{L^\infty} \|\omega\|_{L^\infty} (2.5\xi_\varepsilon + \varepsilon T) = 0.$$

We next prove (3.23). By (3.20) we have

$$\int_I \left|u_\varepsilon'\right|^2 v_\varepsilon^2\, \omega\, dx \le \int_I \left|u'\right|^2 \omega\, dx + \|\omega\|_{L^\infty} \int_{-\xi_\varepsilon - \varepsilon T}^0 \left|u'(-x)\right|^2 dx,$$

and so

$$\limsup_{\varepsilon\to 0} \int_I |u'_\varepsilon|^2 \, v_\varepsilon^2 \, \omega \, dx \le \int_I |u'|^2 \, \omega \, dx,$$

since $u' \in L^2_\omega(I)$, and we conclude that $u' \in L^2(I)$.

On the other hand, by (3.20) and (3.21),

$$\begin{aligned}
&\int_I \left[\frac{\varepsilon}{2} |v'_\varepsilon|^2 + \frac{1}{2\varepsilon}(v_\varepsilon - 1)^2\right] \omega(x) \, dx \\
&= \int_{-3\xi_\varepsilon - 2\varepsilon T}^{-\xi_\varepsilon} \left[\frac{\varepsilon}{2} |v'_\varepsilon|^2 + \frac{1}{2\varepsilon}(v_\varepsilon - 1)^2\right] \omega(x) \, dx \\
&\le \left(\operatorname*{ess\,sup}_{t\in(-3\xi_\varepsilon - 2\varepsilon T, -\xi_\varepsilon)} \omega(t) \right) \int_{-3\xi_\varepsilon - 2\varepsilon T}^{-\xi_\varepsilon} \left[\frac{\varepsilon}{2} |v'_\varepsilon|^2 + \frac{1}{2\varepsilon}(v_\varepsilon - 1)^2\right] \, dx \\
&= \left(\operatorname*{ess\,sup}_{t\in(-3\xi_\varepsilon - 2\varepsilon T, -\xi_\varepsilon)} \omega(t) \right) \int_{-\xi_\varepsilon - \varepsilon T}^{\xi_\varepsilon + \varepsilon T} \left[\frac{\varepsilon}{2} |\tilde{v}'_\varepsilon|^2 + \frac{1}{2\varepsilon}(\tilde{v}_\varepsilon - 1)^2\right] \, dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_I \left[\frac{\varepsilon}{2} |v'_\varepsilon|^2 + \frac{1}{2\varepsilon}(v_\varepsilon - 1)^2\right] \omega(x) \, dx \\
&\le \limsup_{\varepsilon\to 0} \left(\operatorname*{ess\,sup}_{t\in(-3\xi_\varepsilon - 2\varepsilon T, -\xi_\varepsilon)} \omega(t) \right) \left\{ \limsup_{\varepsilon\to 0} \int_{-\xi_\varepsilon - \varepsilon T}^{\xi_\varepsilon + \varepsilon T} \left[\frac{\varepsilon}{2} |\tilde{v}'_\varepsilon|^2 + \frac{1}{2\varepsilon}(\tilde{v}_\varepsilon - 1)^2\right] \, dx \right\} \\
&\le \omega^-(0)(1+\eta),
\end{aligned}$$

where at the last inequality we used the definition of $\tilde{v}_\varepsilon$, (3.15), and (2.6).

We conclude that

$$\limsup_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \le \int_I |u'|^2 \, \omega \, dx + \omega^-(0)(1+\eta),$$

and (3.19) follows due to the arbitrariness of η.

Step 2: In the general case, we recall that S_u is finite. We may obtain u_ε and v_ε by repeating the construction in Step 1 in small non-overlapping intervals centered at every point of $S_u \cap S_\omega$, and by repeating the construction in Step 1 in Lemma 3.4 in those non-overlaping intervals centered at points of $S_u \setminus S_\omega$. Hence, we have

$$\limsup_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) \le \int_I |u'|^2 \, \omega \, dx + (1+\eta) \sum_{x\in S_u} \omega^-(x),$$

and (3.19) follows due to the arbitrariness of η. □

Proof of Theorem 3.5. The proof follows that of Theorem 3.1, using Proposition 3.6 and Proposition 3.7, in place of Proposition 3.3 and 3.4, respectively. □

4. The Multi-Dimensional Case

4.1. **One-Dimensional Restrictions and Slicing Properties.**

Let $\mathcal{S}^{N-1}$ be the unit sphere in $\mathbb{R}^N$ and let $\nu \in \mathcal{S}^{N-1}$ be a fixed direction. We set

$$\begin{cases} \Pi_\nu := \left\{x \in \mathbb{R}^N : \langle x, \nu\rangle = 0\right\}; \\ \Omega^1_{x,\nu} := \{t \in \mathbb{R} : x + t\nu \in \Omega\} \quad \text{for } x \in \Pi_\nu; \\ \Omega_{x,\nu} := \{y = x + t\nu : t \in \mathbb{R}\} \cap \Omega; \\ \Omega_\nu := \{x \in \Pi_\nu : \Omega_{x,\nu} \neq \varnothing\}. \end{cases} \tag{4.1}$$

We also define the 1-d restriction function $u_{x,\nu}$ of the function u as

$$u_{x,\nu}(t) := u(x + t\nu),\ x \in \Omega_\nu,\ t \in \Omega^1_{x,\nu}.$$

We recall the result below from [8], Theorem 3.3.

Theorem 4.1. *Let $\nu \in \mathcal{S}^{N-1}$ be given, and assume that $u \in W^{1,2}(\Omega)$. Then, for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\nu$, $u_{x,\nu}$ belongs to $W^{1,2}(\Omega_{x,\nu})$ and*

$$u'_{x,\nu}(t) = \langle \nabla u(x + t\nu), \nu\rangle .$$

Lemma 4.2. *Let $\omega \in \mathcal{W}(\Omega)$ and $u \in W^{1,p}_\omega(\Omega)$, for $p \in [1, \infty)$, be given. If $\nu \in \mathcal{S}^{N-1}$ and $v \in W^{1,p}(\Omega)$ is nonnegative, then*

$$\int_\Omega |\nabla u|^p\, v^p\, \omega\, dx \geq \int_{\Omega_\nu} \int_{\Omega^1_{x,\nu}} \left|u'_{x,\nu}(t)\right|^p v^p_{x,\nu}(t)\, \omega_{x,\nu}(t)\, dtdx.$$

Proof. Since $\operatorname{ess\,inf}_\Omega \omega \geq 1$, we have $W^{1,p}_\omega(\Omega) \subset W^{1,p}(\Omega)$. Given $\nu \in \mathcal{S}^{N-1}$ and a nonnegative function $v \in W^{1,p}(\Omega)$, by Fubini's Theorem and Theorem 4.1 we have

$$\begin{aligned} \int_\Omega |\nabla u|^p\, v^p\, \omega\, dx &= \int_{\Omega_\nu} \int_{\Omega^1_{x,\nu}} |\nabla u|^p\, v^p\, \omega\, dt\, d\mathcal{H}^{N-1}(x) \\ &\geq \int_{\Omega_\nu} \int_{\Omega^1_{x,\nu}} |\langle \nabla u(x + tv), \nu\rangle|^p\, v^p_{x,\nu}(t)\, \omega_{x,\nu}(t)\, dtd\mathcal{H}^{N-1}(x) \\ &= \int_{\Omega_\nu} \int_{\Omega^1_{x,\nu}} \left|u'_{x,\nu}(t)\right|^p v^p_{x,\nu}(t)\, \omega_{x,\nu}(t)\, dtd\mathcal{H}^{N-1}(x), \end{aligned}$$

where we used the fact that

$$\left|u'_{x,\nu}(t)\right| = |\langle \nabla u(x + t\nu), \nu\rangle| \leq |\nabla u(x + t\nu)|$$

$\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\nu$. □

Proposition 4.3. *Let $\nu \in \mathcal{S}^{N-1}$ be a fixed direction, $\Gamma \subset \mathbb{R}^N$ be such that $\mathcal{H}^{N-1}(\Gamma) < \infty$, and $\mathbb{P}_\nu\colon \mathbb{R}^N \to \Pi_\nu$ be a projection operator, where by (4.1) $\Pi_\nu \subset \mathbb{R}^N$ is a hyperplane in $\mathbb{R}^{N-1}$. Then*

$$\mathcal{H}^{N-1}(\mathbb{P}_\nu(\Gamma)) \leq \mathcal{H}^{N-1}(\Gamma), \tag{4.2}$$

and for $\mathcal{H}^{N-1}$-a.e. $x \in \Pi_\nu$,

$$\mathcal{H}^0(\Omega_{x,\nu} \cap \Gamma) < +\infty. \tag{4.3}$$

Proof. Note that (4.2) follows immediately from Theorem 7.5 in [40] since $\mathbb{P}_\nu$ is a Lipschitz map with Lipschitz constant less or equal to one. To show (4.3), we apply co-area formula (see [4], Theorem 2.93) with $\mathbb{P}_\nu$ and again since $\mathbb{P}_\nu$ is a Lipschitz map with Lipschitz constant less or equal to one, we are done. □

Set $x = (x', x_N) \in \mathbb{R}^N$, where

$$x' \in \mathbb{R}^{N-1} \text{ denotes the first } N-1 \text{ component of } x \in \mathbb{R}^N, \tag{4.4}$$

and given $u\colon \mathbb{R}^{N-1} \to \mathbb{R}$ and $G \subset \mathbb{R}^{N-1}$, we define the graph of u over G as

$$F(u;G) := \left\{(x', x_N) \in \mathbb{R}^N : x' \in G,\, x_N = u(x')\right\}.$$

If u is Lipschitz, then we call $F(u;G)$ a Lipschitz -$(N-1)$-graph.

Lemma 4.4. *Let $\Gamma \subset \mathbb{R}^N$ be a $\mathcal{H}^{N-1}$-rectifiable set, and let $\mathbb{P}_{x,\nu_\Gamma}\colon \mathbb{R}^N \to T_{x,\nu_\Gamma}$ be a projection operator for $x \in \Gamma$. Then*

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma \cap Q_{\nu_\Gamma}(x_0,r)))}{r^{N-1}} = 1 \tag{4.5}$$

for $\mathcal{H}^{N-1}$-a.e. $x_0 \in \Gamma$.

Proof. By Proposition 4.3 we have

$$\limsup_{r\to 0} \frac{\mathcal{H}^{N-1}(\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma \cap Q_{\nu_\Gamma}(x_0,r)))}{r^{N-1}} \le \limsup_{r\to 0} \frac{\mathcal{H}^{N-1}(\Gamma \cap Q_{\nu_\Gamma}(x_0,r))}{r^{N-1}} = 1 \tag{4.6}$$

for a.e. $x_0 \in \Gamma$. By Theorem 2.76 in [4] we may write

$$\Gamma = \Gamma_0 \cup \bigcup_{i=1}^{\infty} \Gamma_i$$

as a disjoint union with $\mathcal{H}^{N-1}(\Gamma_0) = 0$, $\Gamma_i = (N_i, l_i(N_i))$ where $l_i : \mathbb{R}^{N-1} \to \mathbb{R}$ is of class C^1 and $N_i \subset \mathbb{R}^{N-1}$.

Let $x_0 \in \Gamma_{i_0}$ for some $i_0 \in \mathbb{N}$ and, without loss of generality, let $(-\nabla l_{i_0}(x_0'), 1) = \nu_\Gamma(x_0)$, with x_0 a point of density one in Γ_0 (see Exercise 10.6 in [39]). Up to a rotation and a translation, we may assume that $\nabla l_{i_0}(x_0') = (0,0,\ldots,0) \in \mathbb{R}^{N-1}$, $x_0 = (0,0,\ldots,0)$, and $\mathbb{P}_{x_0,\nu_\Gamma}\colon \Gamma_{i_0} \to \mathbb{R}^{N-1} \times \{0\}$. Therefore, for $r > 0$ small enough,

$$\Gamma_{i_0} \cap Q_{\nu_\Gamma}(x_0,r) = (\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma_{i_0} \cap Q_{\nu_\Gamma}(x_0,r))\,,\, l_{i_0}((\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma_{i_0} \cap Q_{\nu_\Gamma}(x_0,r)))')),$$

and by Theorem 9.1 in [40] we obtain that,

$$\mathcal{H}^{N-1}(\Gamma_{i_0} \cap Q_{\nu_\Gamma}(x_0,r)) = \int_{\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma_{i_0} \cap Q_{\nu_\Gamma}(x_0,r))} \sqrt{1 + |\nabla l_{i_0}(x')|^2}\, d\mathcal{H}^{N-1}(x').$$

Since l_{i_0} is of class C^1 and $\nabla l_{i_0}(x_0) = 0$, for $\varepsilon > 0$ choose $r_\varepsilon > 0$ such that $|\nabla l_{i_0}(x)| < \varepsilon$ for all $0 < r < r_\varepsilon$. Therefore, we have that

$$\begin{aligned}
\mathcal{H}^{N-1}(\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma \cap Q_{\nu_\Gamma}(x_0,r))) &\ge \mathcal{H}^{N-1}(\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma_{i_0} \cap Q_{\nu_\Gamma}(x_0,r))) \\
&\ge \frac{1}{\sqrt{1+\varepsilon^2}} \int_{\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma_{i_0} \cap Q_{\nu_\Gamma}(x_0,r))} \sqrt{1 + |\nabla l_{i_0}(x')|^2}\, dx' \\
&= \frac{1}{\sqrt{1+\varepsilon^2}} \mathcal{H}^{N-1}(\Gamma_{i_0} \cap Q_{\nu_\Gamma}(x_0,r)).
\end{aligned}$$

We obtain

$$\liminf_{r\to 0} \frac{\mathcal{H}^{N-1}(\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma \cap Q_{\nu_\Gamma}(x_0,r)))}{r^{N-1}} \geq \liminf_{r\to 0} \frac{1}{\sqrt{1+\varepsilon^2}} \frac{\mathcal{H}^{N-1}(\Gamma_{i_0} \cap Q_{\nu_\Gamma}(x_0,r))}{r^{N-1}} = \frac{1}{\sqrt{1+\varepsilon^2}}.$$

By the arbitrariness of $\varepsilon > 0$, we deduce that

$$\liminf_{r\to 0} \frac{\mathcal{H}^{N-1}(\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma \cap Q_{\nu_\Gamma}(x_0,r)))}{r^{N-1}} \geq 1,$$

and, in view of (4.6), we conclude that

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(\mathbb{P}_{x_0,\nu_\Gamma}(\Gamma \cap Q_{\nu_\Gamma}(x_0,r)))}{r^{N-1}} = 1.$$

□

Lemma 4.5. *Let $Q := (-1,1)^N$ and let $\Gamma \subset Q$ be a $\mathcal{H}^{N-1}$-rectifiable set such that $\mathcal{H}^{N-1}(\Gamma) < \infty$ and*

$$\mathcal{H}^0(\Gamma \cap (\{x'\} \times (-1,1))) \geq 1 \tag{4.7}$$

for $\mathcal{H}^{N-1}$-a.e. $x' \in (-1,1)^{N-1}$. Then there exists a $\mathcal{H}^{N-1}$-measurable subset $\Gamma' \subset \Gamma$ such that

$$\mathcal{H}^0(\Gamma' \cap (\{x'\} \times (-1,1))) = 1. \tag{4.8}$$

for $\mathcal{H}^{N-1}$-a.e. $x' \in (-1,1)^{N-1}$.

Proof. By Lemma 4.3 we have

$$\mathcal{H}^0(\Gamma' \cap (\{x'\} \times (-1,1))) < +\infty$$

for $\mathcal{H}^{N-1}$-a.e. $x' \in (-1,1)^{N-1}$. Thus, for $\mathcal{H}^{N-1}$-a.e. $x' \in (-1,1)^{N-1}$, the set

$$\Gamma_{x'} := \Gamma \cap (\{x'\} \times (-1,1))$$

is a finite collection of singletons, hence closed, and by (4.7) is non-empty. Applying Corollary 1.1[1] in [28], page 237, we obtain a $\mathcal{H}^{N-1}$ measurable subset $\Gamma' \subset \Gamma$ which satisfies (4.8). □

Lemma 4.6. *Let $\tau > 0$ and $\eta > 0$ be given. Let $u \in SBV(\Omega)$ and assume that $\mathcal{H}^{N-1}(S_u) < \infty$. The following statements hold:*

1. *there exist a set $S \subset S_u$ with $\mathcal{H}^{N-1}(S_u \setminus S) < \eta$, and a countable collection $\mathcal{Q}$ of mutually disjoint open cubes centered on elements of S_u such that*

$$\bigcup_{Q\in\mathcal{Q}} Q \subset \Omega,$$

and

$$\mathcal{H}^{N-1}\left(S \setminus \bigcup_{Q\in\mathcal{Q}} Q\right) = 0;$$

2. *for every $Q \in \mathcal{Q}$ there exists a direction vector $\nu_Q \in \mathcal{S}^{N-1}$ such that*

$$\mathcal{H}^0(S \cap Q_{x,\nu_Q}) = 1,$$

for $\mathcal{H}^{N-1}$ a.e. $x \in Q \cap S$;

[1] when applying Corollary 1.1, Ω is $(-1,1)^{N-1}$, B is $(-1,1)$, and C_x is $\Gamma_{x'}$, and we construct Γ' by using $\bar{u}(x)$ which is obtained Corollary 1.1

3. *for every $Q \in \mathcal{Q}$, $S \cap Q$ is contained in a Lipschitz $(N-1)$- graph Γ_Q with Lipschitz constant less than τ.*

Proof. Let $\tau, \eta > 0$ be given. By Theorem 2.76 in [4], there exist countably many Lipschitz $(N-1)$-graphs $\Gamma_i \subset \mathbb{R}^N$ such that (up to a rotation and a translation)

$$\Gamma_i = \{(x', x_N) : \ x' \in N_i, \ x_N = l_i(x')\}$$

with $N_i \subset \mathbb{R}^{N-1}$, l_i: $\mathbb{R}^{N-1} \to \mathbb{R}$ of class C^1, $|\nabla l_i| < \tau$ for all $i \in \mathbb{N}$, and

$$\mathcal{H}^{N-1}\left(S_u \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right) = 0. \tag{4.9}$$

Without lose of generality, we assume that

$$\mathcal{H}^{N-1}(\Gamma_i \cap \Gamma_{i'}) = 0 \text{ if } i \neq i' \in \mathbb{N}, \text{ and } \mathcal{H}^{N-1}(\Gamma_i) > 0. \tag{4.10}$$

We denote by $\mathcal{P}$ the collection of Lipschitz $(N-1)$-graphs Γ_i in (4.9)-(4.10). By (4.10), for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ there exists only one $\Gamma \in \mathcal{P}$ such that $x \in \Gamma$, and we denote such Γ by Γ_x and we write

$$\Gamma_x = \left\{(y', y_N) : \ y' \in N_x \subset \mathbb{R}^{N-1}, \ y_N = l_x(y')\right\}.$$

For simplicity of notation, in what follows we will abbreviate $\nu_{\Gamma_x}(x) = \nu_{S_u}(x)$ by $\nu(x)$, $Q_{\nu_{S_u}}(x,r)$ by $Q(x,r)$, and $T_{x,\nu_{S_u}}$ by T_x.

We also note that $\mathcal{H}^{N-1}(\Gamma \cap S_u) < \mathcal{H}^{N-1}(S_u) < \infty$ for each $\Gamma \in \mathcal{P}$. Then $\mathcal{H}^{N-1}$- a.e. x has density 1 in $\Gamma_x \cap S_u$ (see Theorem 2.63 in [4]). Denote by S_1 the set of points such that S_u has density 1 at x and

$$\lim_{r \to 0} \frac{\mathcal{H}^{N-1}(S_u \cap \Gamma_x \cap Q_{\nu_{\Gamma_x}}(x,r))}{r^{N-1}} = 1. \tag{4.11}$$

Then $\mathcal{H}^{N-1}(S_u \setminus S_1) = 0$.

Define

$$f_r(x) := \frac{\mathcal{H}^{N-1}(S_1 \cap Q(x,r))}{r^{N-1}}.$$

Since $f_r(x) \to 1$ as $r \to 0^+$ for $x \in S_1$, by Egoroff's Theorem there exists a set $S_2 \subset S_1$ such that $\mathcal{H}^{N-1}(S_1 \setminus S_2) < \eta/4$ and $f_r \to 1$ uniformly on S_2. Find $r_1 > 0$ such that

$$\frac{\mathcal{H}^{N-1}(S_1 \cap Q(x,r))}{r^{N-1}} \geq \frac{1}{2},$$

i.e.,

$$\mathcal{H}^{N-1}(S_1 \cap Q(x,r)) \geq \frac{1}{2} r^{N-1} \tag{4.12}$$

for all $0 < r < r_1$ and $x \in S_2$. Since $S_2 \subset S_1$, S_2 is also $\mathcal{H}^{N-1}$-rectifiable and so $\mathcal{H}^{N-1}$ a.e. $x \in S_2$ has density one. Without loss of generality, we assume that every point in S_2 has density one and satisfies (4.5) in Lemma 4.4.

Let $x_0 \in S_2$ be given and recall (4.1). We define

$$\begin{aligned} T_b(x_0,r) &:= \left\{ x \in Q(x_0,r) \cap T_{x_0} :\ \mathcal{H}^0([Q(x_0,r)]_{x,\nu(x_0)} \cap S_2) \geq 2 \right\}, \\ T_g(x_0,r) &:= \left\{ x \in Q(x_0,r) \cap T_{x_0} :\ \mathcal{H}^0([Q(x_0,r)]_{x,\nu(x_0)} \cap S_2) = 1 \right\}, \\ S_b(x_0,r) &:= \bigcup_{x \in T_b(x_0,r)} \left(S_2 \cap [Q(x_0,r)]_{x,\nu(x_0)} \right), \\ S_g(x_0,r) &:= \bigcup_{x \in T_g(x_0,r)} \left(S_2 \cap [Q(x_0,r)]_{x,\nu(x_0)} \right). \end{aligned} \tag{4.13}$$

Note that

$$T_b(x_0,r) \cap T_g(x_0,r) = \emptyset \text{ and } S_b(x_0,r) \cap S_g(x_0,r) = \emptyset, \tag{4.14}$$

and by Proposition 4.3 we have

$$\mathcal{H}^{N-1}(S_g(x_0,r)) \geq \mathcal{H}^{N-1}(T_g(x_0,r)). \tag{4.15}$$

We claim that

$$\mathcal{H}^{N-1}(S_b(x_0,r)) \geq 2\mathcal{H}^{N-1}(T_b(x_0,r)). \tag{4.16}$$

By Lemma 4.5 there exists a measurable selection $S_b^1 \subset S_b(x_0,r)$ such that

$$\mathcal{H}^{N-1}(S_b^1(x_0,r) \cap [Q(x_0,r)]_{x,\nu(x_0)}) = 1$$

for $\mathcal{H}^{N-1}$-a.e. $x \in T_b(x_0,r)$. We define

$$S_b^2(x_0,r) := S_b(x_0,r) \setminus S_b^1(x_0,r).$$

By the definition of $S_b(x_0,r)$ in (4.13), we have

$$\mathcal{H}^0([Q(x_0,r)]_{x,\nu(x_0)} \cap S_b^1(x_0,r)) \geq 1 \text{ and } \mathcal{H}^0([Q(x_0,r)]_{x,\nu(x_0)} \cap S_b^2(x_0,r)) \geq 1$$

for all $x \in T_b(x_0,r)$. We observe that

$$\mathcal{H}^{N-1}(S_b(x_0,r)) = \mathcal{H}^{N-1}(S_b^1(x_0,r)) + \mathcal{H}^{N-1}(S_b^2(x_0,r)) \geq 2\mathcal{H}^{N-1}(T_b(x_0,r))$$

by Proposition 4.3 and we deduce (4.16).

We next show that

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(S_b(x_0,r))}{r^{N-1}} = 0. \tag{4.17}$$

Indeed, since T_{x_0} is the tangent hyperplane to S_2 at x_0,

$$T_b(x_0,r) \cup T_g(x_0,r) = \mathbb{P}_{x_0,\nu_{S_u}}(S_2 \cap Q(x_0,r)),$$

and by Lemma 4.4 it follows that

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(T_b(x_0,r) \cup T_g(x_0,r))}{r^{N-1}} = 1. \tag{4.18}$$

On the other hand, in view of (4.14), (4.15), and (4.16), we deduce that

$$\begin{aligned} \mathcal{H}^{N-1}(S_b(x_0,r) \cup S_g(x_0,r)) &= \mathcal{H}^{N-1}(S_b(x_0,r)) + \mathcal{H}^{N-1}(S_g(x_0,r)) \\ &\geq 2\mathcal{H}^{N-1}(T_b(x_0,r)) + \mathcal{H}^{N-1}(T_g(x_0,r)). \end{aligned}$$

That is,

$$\mathcal{H}^{N-1}(T_b(x_0,r)) \leq \mathcal{H}^{N-1}(S_b(x_0,r) \cup S_g(x_0,r)) - \left[\mathcal{H}^{N-1}(T_b(x_0,r)) + \mathcal{H}^{N-1}(T_g(x_0,r))\right]$$

$$= \mathcal{H}^{N-1}(S_b(x_0,r)\cup S_g(x_0,r)) - \mathcal{H}^{N-1}(T_b(x_0,r)\cup T_g(x_0,r)). \tag{4.19}$$

Since $x_0 \in S_2$ has density 1, we have

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(S_b(x_0,r)\cup S_g(x_0,r))}{r^{N-1}} = \lim_{r\to 0} \frac{\mathcal{H}^{N-1}(S_2\cap Q(x_0,r))}{r^{N-1}} = 1. \tag{4.20}$$

In view of (4.18), (4.19), and (4.20), we conclude that

$$\limsup_{r\to 0} \frac{\mathcal{H}^{N-1}(T_b(x_0,r))}{r^{N-1}}$$
$$\leq \lim_{r\to 0} \frac{\mathcal{H}^{N-1}(S_b(x_0,r)\cup S_g(x_0,r))}{r^{N-1}} - \lim_{r\to 0} \frac{\mathcal{H}^{N-1}(T_b(x_0,r)\cup T_g(x_0,r))}{r^{N-1}} = 0,$$

which implies that

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(T_b(x_0,r))}{r^{N-1}} = 0.$$

This, together with (4.14) and (4.18), yields

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(T_g(x_0,r))}{r^{N-1}} = 1,$$

and so by (4.15) we have

$$\liminf_{r\to 0} \frac{\mathcal{H}^{N-1}(S_g(x_0,r))}{r^{N-1}} \geq \lim_{r\to 0} \frac{\mathcal{H}^{N-1}(T_g(x_0,r))}{r^{N-1}} = 1,$$

while by (4.20)

$$\limsup_{r\to 0} \frac{\mathcal{H}^{N-1}(S_g(x_0,r))}{r^{N-1}} \leq \lim_{r\to 0} \frac{\mathcal{H}^{N-1}(S_b(x_0,r)\cup S_g(x_0,r))}{r^{N-1}} = 1,$$

and we conclude that

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}(S_g(x_0,r))}{r^{N-1}} = 1.$$

Now, also in view of (4.14) and (4.20), we deduce (4.17).

We define, for $x \in S_2$,

$$g_r(x) := \frac{\mathcal{H}^{N-1}(S_b(x,r))}{r^{N-1}}.$$

By (4.17) we have $\lim_{r\to 0} g_r(x) = 0$ for all $x \in S_2$, therefore by Egoroff's Theorem there exists a set $S_3 \subset S_2$ such that

$$\mathcal{H}^{N-1}(S_2 \setminus S_3) < \frac{\eta}{4}$$

and $g_r \to 0$ uniformly on S_3. Choose $0 < r_2 < r_1$ such that

$$\frac{\mathcal{H}^{N-1}(S_b(x,r))}{r^{N-1}} < \frac{\eta}{16}\frac{1}{\mathcal{H}^{N-1}(S_u)} \tag{4.21}$$

for all $x \in S_3$ and $0 < r < r_2$. We claim that, for $x \in S_3$ and the corresponding $\Gamma_x \in \mathcal{P}$,

$$\lim_{r\to 0} \frac{\mathcal{H}^{N-1}\left(S_g(x,r)\setminus [S_u\cap\Gamma_x\cap Q(x,r)]\right)}{r^{N-1}} = 0. \tag{4.22}$$

Suppose that

$$0 < \limsup_{r\to 0} \frac{\mathcal{H}^{N-1}\left(S_g(x,r)\setminus [S_u\cap\Gamma_x\cap Q(x,r)]\right)}{r^{N-1}} =: \delta.$$

By (4.11), and the fact that $\Gamma_x \subset S_u$, we have that

$$\begin{aligned} 1 &= \lim_{r\to 0} \frac{\mathcal{H}^{N-1}(S_u \cap Q(x,r))}{r^{N-1}} \\ &= \lim_{r\to 0} \frac{\mathcal{H}^{N-1}\left(([S_u \cap Q(x,r)] \setminus [S_u \cap \Gamma_x \cap Q(x,r)]) \cup [S_u \cap \Gamma_x \cap Q(x,r)]\right)}{r^{N-1}} \\ &\geq \limsup_{r\to 0} \frac{\mathcal{H}^{N-1}\left([S_g(x,r)] \setminus [S_u \cap \Gamma_x \cap Q(x,r)]\right)}{r^{N-1}} \\ &\quad + \lim_{r\to 0} \frac{\mathcal{H}^{N-1}[S_u \cap \Gamma_x \cap Q(x,r)]}{r^{N-1}} \\ &= \delta + 1 > 1, \end{aligned}$$

which is a contradiction.

We define, for $x \in S_3$,

$$h_r(x) := \frac{\mathcal{H}^{N-1}\left(S_g(x,r) \setminus [S_u \cap \Gamma_x \cap Q(x,r)]\right)}{r^{N-1}}.$$

By (4.22) $\lim_{r\to 0} h_r(x) = 0$ for all $x \in S_3$, therefore by Egoroff's Theorem there exists a set of $S_4 \subset S_3$ such that

$$\mathcal{H}^{N-1}(S_3 \setminus S_4) < \frac{\eta}{4},$$

and $h_r \to 0$ uniformly on S_4. Choose $0 < r_3 < r_2$ such that

$$\frac{\mathcal{H}^{N-1}\left(S_g(x,r) \setminus [S_u \cap \Gamma_x \cap Q(x,r)]\right)}{r^{N-1}} < \frac{\eta}{16} \frac{1}{\mathcal{H}^{N-1}(S_u)} \tag{4.23}$$

for all $x \in S_4$ and $0 < r < r_3$, and let

$$\mathcal{Q}' := \{Q(x,r) : \ x \in S_4,\, 0 < r < r_3\}.$$

By Besicovitch's Covering Theorem we may extract a countable collection $\mathcal{Q}$ of mutually disjoint cubes from $\mathcal{Q}'$ such that

$$\bigcup_{Q\in\mathcal{Q}} Q \subset \Omega \text{ and } \mathcal{H}^{N-1}\left(S_4 \setminus \left(\bigcup_{Q\in\mathcal{Q}} Q\right)\right) = 0.$$

Define

$$S := S_4 \setminus \left[\left(\bigcup_{Q\in\mathcal{Q}} S_b(x_Q, r_Q)\right) \cup \left(\bigcup_{Q\in\mathcal{Q}} \left[S_g(x_Q, r_Q) \setminus \left(S_u \cap \Gamma_{x_Q} \cap Q\right)\right]\right)\right], \tag{4.24}$$

where x_Q is the center of cube Q and r_Q is the side length of Q. Note that the set S satisfies properties 2 and 3 in the statement of Lemma 4.6. Finally, we show that

$$\mathcal{H}^{N-1}(S_u \setminus S) < \eta.$$

Indeed, in view of (4.21) and (4.23), and using the fact that the cubes $Q \in \mathcal{Q}$ are mutually disjoint, we have

$$\mathcal{H}^{N-1}\left(\bigcup_{Q\in\mathcal{Q}} S_b(x_Q, r_Q)\right) = \sum_{Q\in\mathcal{Q}} \mathcal{H}^{N-1}(S_b(x_Q, r_Q)) \leq \frac{\eta}{16\mathcal{H}^{N-1}(S_u)} \sum_{Q\in\mathcal{Q}} r_Q^{N-1}, \tag{4.25}$$

and

$$\mathcal{H}^{N-1}\left(\bigcup_{Q\in\mathcal{Q}}\left[S_g(x_Q,r_Q)\setminus\big(S_u\cap\Gamma_{x_Q}\cap Q\big)\right]\right)$$
$$=\sum_{Q\in\mathcal{Q}}\mathcal{H}^{N-1}(S_g(x_Q,r_Q)\setminus\big(S_u\cap\Gamma_{x_Q}\cap Q\big))\le\frac{\eta}{16\mathcal{H}^{N-1}(S_u)}\sum_{Q\in\mathcal{Q}}r_Q^{N-1}. \quad (4.26)$$

By (4.12) we obtain

$$\sum_{Q\in\mathcal{Q}}\frac{1}{2}r_Q^{N-1}\le\sum_{Q\in\mathcal{Q}}\mathcal{H}^{N-1}(S_1\cap Q)=\mathcal{H}^{N-1}\left(\bigcup_{Q\in\mathcal{Q}}S_u\cap Q\right)\le\mathcal{H}^{N-1}(S_u). \quad (4.27)$$

Using (4.25), (4.26), and (4.27), we deduce that

$$\mathcal{H}^{N-1}\left(\bigcup_{Q\in\mathcal{Q}}S_b(x_Q,r_Q)\right)\le\frac{\eta}{8},$$

and

$$\mathcal{H}^{N-1}\left(\bigcup_{Q\in\mathcal{Q}}\left[S_g(x_Q,r_Q)\setminus\big(S_u\cap\Gamma_{x_Q}\cap Q\big)\right]\right)\le\frac{\eta}{8},$$

and so by (4.24) we get

$$\mathcal{H}^{N-1}(S_4\setminus S)\le\frac{\eta}{4}.$$

Since $S\subset S_4\subset S_3\subset S_2\subset S_1\subset S_u$, we conclude that

$$\begin{aligned}&\mathcal{H}^{N-1}(S_u\setminus S)\\ \le&\mathcal{H}^{N-1}(S_u\setminus S_1)+\mathcal{H}^{N-1}(S_1\setminus S_2)+\mathcal{H}^{N-1}(S_2\setminus S_3)+\mathcal{H}^{N-1}(S_3\setminus S_4)+\mathcal{H}^{N-1}(S_4\setminus S)\\ \le&\frac{\eta}{4}+\frac{\eta}{4}+\frac{\eta}{4}+\frac{\eta}{4}=\eta.\end{aligned}$$

□

Lemma 4.7. *Let $\omega\in C(\Omega)$ be nonnegative, let $\Gamma\subset\Omega$ be a $\mathcal{H}^{N-1}$-rectifiable set, and let $\tau\in(0,1)$ be given. Then for $\mathcal{H}^{N-1}$-a.e. $x_0\in\Gamma$, there exists $r_0:=r_0(x_0)>0$ such that for each $0<r<r_0$ there exist $t_0\in(-\tau r/4,\tau r/4)$ and $0<t_{0,r}=t_{0,r}(t_0,\tau,x_0,r)<|t_0|$ such that*

$$\sup_{0<t\le t_{0,r}}\frac{1}{|I(t_0,t)|}\int_{I(t_0,t)}\int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}(l)}\omega(x)\,d\mathcal{H}^{N-1}dl$$
$$\le\int_{Q_{\nu_\Gamma}(x_0,r)\cap\Gamma}\omega(x)\,d\mathcal{H}^{N-1}+(1+\omega(x_0))O(\tau)r^{N-1},$$

where $I(t_0,t):=(t_0-t,t_0+t)$, $T_{x_0,\nu_\delta}(l):=T_{x_0,\nu_\Gamma}+l\nu_\Gamma$.

Proof. Fix $x_0\in\Gamma$ with density 1 and let $\tau>0$ be given. There exists $r_1>0$ such that

$$\frac{1}{1+\tau^2}\le\frac{\mathcal{H}^{N-1}(\Gamma\cap Q_{\nu_\Gamma}(x_0,r))}{r^{N-1}}\le1+\tau^2, \quad (4.28)$$

for all $0 < r < r_1$. Since by continuity of ω we have that

$$\lim_{r\to 0} \fint_{Q_{\nu_\Gamma}(x_0,r)} |\omega(x) - \omega(x_0)|\, dx = 0,$$

and

$$\lim_{r\to 0} \fint_{Q_{\nu_\Gamma}(x_0,r)\cap\Gamma} |\omega(x) - \omega(x_0)|\, d\mathcal{H}^{N-1} = 0,$$

we may choose $0 < r_2 < r_1$ such that for all $0 < r < r_2$

$$\fint_{Q_{\nu_\Gamma}(x_0,r)} |\omega(x) - \omega(x_0)|\, dx \le \tau^2,$$

and

$$\int_{Q_{\nu_\Gamma}(x_0,r)\cap\Gamma} |\omega(x) - \omega(x_0)|\, d\mathcal{H}^{N-1} \le \frac{\tau}{1+\tau^2}\mathcal{H}^{N-1}(Q_{\nu_\Gamma}(x_0,r)\cap\Gamma) \le O(\tau) r^{N-1}, \tag{4.29}$$

where we used (4.28).

Therefore

$$\int_{-\tau r/4}^{\tau r/4}\int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}(t)} |\omega(x) - \omega(x_0)|\, d\mathcal{H}^{N-1} dt \le \int_{Q_{\nu_\Gamma}(x_0,r)} |\omega(x) - \omega(x_0)|\, dx \le \tau^2 r^N,$$

and by the Mean Value Theorem there exists a set $A \subset (-\tau r/4, \tau r/4)$ with positive 1 dimensional Lebesgue measure such that for every $t \in A$,

$$\int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}(t)} |\omega(x) - \omega(x_0)|\, d\mathcal{H}^{N-1} \le 2\tau r^{N-1}. \tag{4.30}$$

If $t_0 \in A$ then we have, by the continuity of ω,

$$\lim_{t\to 0} \frac{1}{|I(t_0,t)|}\int_{I(t_0,t)}\int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}(l)} \omega(x) d\mathcal{H}^{N-1} dl = \int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}(t_0)} \omega(x) d\mathcal{H}^{N-1},$$

hence there exists $t_{0,r} > 0$, depending on r, t_0, τ, and x_0, such that $I(t_0, t_{0,r}) \subset (-\tau r/2, \tau r/2)$ and

$$\begin{aligned}\sup_{0<t\le t_{0,r}} \frac{1}{|I(t_0,t)|}\int_{I(t_0,t)}\int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}(l)} &\omega(x) d\mathcal{H}^{N-1} dl \\ &\le \int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}(t_0)} \omega(x) d\mathcal{H}^{N-1} + O(\tau) r^{N-1}. \end{aligned}\tag{4.31}$$

Moreover, since

$$\mathcal{H}^{N-1}\left[Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}(t_0)\right] = \mathcal{H}^{N-1}\left[Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}\right],$$

we have

$$\begin{aligned}\int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}(t_0)} \omega(x_0) d\mathcal{H}^{N-1} &= \int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma}} \omega(x_0) d\mathcal{H}^{N-1} \\ &= \omega(x_0) r^{N-1} \le (1+\tau^2)\int_{Q_{\nu_\Gamma}(x_0,r)\cap\Gamma} \omega(x_0) d\mathcal{H}^{N-1} \\ &\le \int_{Q_{\nu_\Gamma}(x_0,r)\cap\Gamma} \omega(x_0) d\mathcal{H}^{N-1} + O(\tau) r^{N-1},\end{aligned}$$

where in the last inequality we used (4.28), the non-negativeness of ω.

By (4.31), (4.30), in this order, for every $r \le r_2$ there exist $t_0 \in (-\tau r/4, \tau r/4)$ and $0 < t_{0,r} < |t_0|$, depending on t_0, τ, x_0 and r, such that

$$\begin{aligned}\sup_{0<t\le t_{0,r}} \frac{1}{|I(t_0,t)|} \int_{I(t_0,t)} \int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma(x_0)}(l)} & \omega(x)\, d\mathcal{H}^{N-1} dl \\ &\le \omega(x_0)\mathcal{H}^{N-1}\left(Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma(x_0)}\right) + O(\tau)r^{N-1} = \omega(x_0)r^{N-1} + O(\tau)r^{N-1} \\ &\le \omega(x_0)(1+\tau^2)\mathcal{H}^{N-1}\left(\Gamma\cap Q_{\nu_\Gamma}(x_0,r)\right) + O(\tau)r^{N-1}\end{aligned}$$

where we used (4.28) in the last inequality. Finally, by (4.29) we conclude that

$$\begin{aligned}\sup_{0<t\le t_{0,r}} \frac{1}{|I(t_0,t)|} \int_{I(t_0,t)} \int_{Q_{\nu_\Gamma}(x_0,r)\cap T_{x_0,\nu_\Gamma(x_0)}(l)} & \omega(x)\, d\mathcal{H}^{N-1} dl \\ &\le \int_{Q_{\nu_\Gamma}(x_0,r)\cap\Gamma} \omega(x)\, d\mathcal{H}^{N-1} + (1+\omega(x_0))O(\tau)r^{N-1},\end{aligned}$$

as desired. □

Proposition 4.8. *Let $\omega \in C(\Omega)$ be nonnegative, let $\Gamma \subset \Omega$ be a $\mathcal{H}^{N-1}$-rectifiable set with $\mathcal{H}^{N-1}(\Gamma) < +\infty$, and let $\tau \in (0,1)$ be given. Then there exist a set $S \subset \Omega$ and a countable family of disjoint cubes $\mathcal{F} = \{Q_{\nu_\Gamma}(x_n, r_n)\}_{n=1}^\infty$ with $r_n \le \tau$, for all $n \in \mathbb{N}$, such that the following hold:*

1. *$\mathcal{H}^{N-1}(\Gamma \setminus S) < \tau$, $S \subset \bigcup_{n=1}^\infty Q_{\nu_\Gamma}(x_n, r_n)$;*
2. *$\mathcal{H}^{N-1}\left(S \cap Q_{\nu_\Gamma}(x_n, r)\right) \le (1+\tau^2)r^{N-1}$ for all $0 < r < r_n$;*
3. *$S \cap Q_{\nu_\Gamma}(x_n, r_n) \subset R_{\tau/2,\nu_\Gamma}(x_n, r_n)$;*
4. *if $0 < \kappa < 1$ then for every $n \in \mathbb{N}$ there exist $t_n^\kappa \in (-\kappa r_n/4, \kappa r_n/4)$ and $0 < t^\kappa_{x_n,r_n} < |t_n^\kappa|$, depending on τ, x_n, and κr_n, such that*

$$\begin{aligned}\sup_{0<t\le t^\kappa_{x_n,r_n}} \frac{1}{|I(t_n^\kappa,t)|} \int_{I(t_n^\kappa,t)} \int_{Q_{\nu_\Gamma}(x_n,\kappa r_n)\cap T_{x_n,\nu_\Gamma}(l)} & \omega(x) d\mathcal{H}^{N-1} dl \\ &\le \int_{\Gamma\cap Q_{\nu_\Gamma}(x_n,\kappa r_n)} \omega(x) d\mathcal{H}^{N-1} + (1+\omega(x_n))O(\tau)(\kappa r_n)^{N-1}, \quad (4.32)\end{aligned}$$

where $I(t_n^\kappa, t) := (t_n^\kappa - t, t_n^\kappa + t)$.

Proof. Let $\tau \in (0,1)$ and $\kappa \in (0,1)$ be given. Since $\mathcal{H}^{N-1}(\Gamma) < \infty$, there exists $S_1 \subset \Gamma$ such that $\mathcal{H}^{N-1}(\Gamma \setminus S_1) < \tau/3$, S_1 is compact and contained in a finite union of $(N-1)$-Lipschitz graphs Γ_i, $i = 1, \dots, M$, with Lipschitz constants less than $\tau/(2\sqrt{N})$.

Moreover, since $\mathcal{H}^{N-1}$ a.e. $x \in S_1$ a point of density one, by Egorov's Theorem, we may find $S_2 \subset S_1$ such that $\mathcal{H}^{N-1}(S_1 \setminus S_2) < \tau/3$ and there exists $r_1 > 0$ such that for all $0 < r < r_1$ and $x \in S_2$,

$$\mathcal{H}^{N-1}\left(S_1 \cap Q_{\nu_\Gamma}(x,r)\right) \le (1+\tau^2)r^{N-1}.$$

Let $L_i := S_2 \cap \Gamma_i$ and without lose of generality we assume that L_i are mutually disjoint. Let $L_i' \subset L_i$ be such that

$$\mathcal{H}^{N-1}(L_i \setminus L_i') < \frac{\tau}{3}\frac{1}{2^i} \text{ and } d_{ij} := \operatorname{dist}(L_i', L_j') > 0$$

for $i \neq j$. We observe that

$$\mathcal{H}^{N-1}\left(S_2 \setminus \bigcup_{i=1}^{M} L_i'\right) < \frac{\tau}{3} \text{ and } d := \min_{i\neq j}\{d_{ij}\} > 0.$$

Define

$$S := \bigcup_{i=1}^{M} L_i'.$$

We claim that there exists $0 < r_2 < \min\left\{\tau^2, d/2, r_1\right\}$ such that for every $0 < r < r_2$ and every x, $y \in S$ with $|x-y| < \sqrt{N}r$ we have

$$S \cap Q_{\nu_\Gamma}(x,r) \subset R_{\tau/2,\nu_\Gamma}(x,r),$$

where we are using the notation introduced in Notation 2.3. Indeed, to verify this inclusion, we write (up to a rotation)

$$S \cap Q_{\nu_\Gamma}(x,r) = \{(y', l_x(y')) : \ y \in T_{x,\nu_\Gamma} \cap Q_{\nu_\Gamma}(x,r)\} \subset \Gamma_x$$

where y' [2] is defined in (4.4) and $\|\nabla l_x\|_{L^\infty} < \tau/(2\sqrt{N})$. Assuming, without loss of generality, that $x = 0$ and $l_x(0) = 0$, we have for all $y \in T_{0,\nu_\Gamma} \cap Q_{\nu_\Gamma}(0,r)$

$$|l_0(y)| \leq \|\nabla l_0\|_{L^\infty} \leq \frac{1}{2}\tau r$$

because for every $y \in S \cap Q_{\nu_\Gamma}(0,r)$ we have $|y| < \sqrt{N}r$.

Next, for $\mathcal{H}^{N-1}$-a.e. $x \in S$ we may find $r_2(x) > 0$ such that $Q_{\nu_\Gamma}(x,r_3) \subset \Omega$ and $\kappa r_2(x) \leq r_0(x)$ where $r_0(x)$ is determined in Lemma 4.7. Let $\bar{r}_0(x) := \min\{r_1, r_2(x)\}$. The collection

$$\mathcal{F}' := \{Q_{\nu_\Gamma}(x,r) : \ x \in S, \ r < \bar{r}_0(x)\}$$

is a fine cover for S, and so by Besicovitch's Covering Theorem we may obtain a countable subcollection $\mathcal{F} \subset \mathcal{F}'$ with pairwise disjoint cubes such that

$$S \subset \bigcup_{Q_{\nu_\Gamma}(x_n,r_n)\in\mathcal{F}} Q_{\nu_\Gamma}(x_n,r_n).$$

For each $Q_{\nu_\Gamma}(x_n,r_n) \in \mathcal{F}$ we apply Lemma 4.7 to obtain $t_n^\kappa \in (-\kappa r_n/4, \kappa r_n/4)$ and $t_{x_n,r_n}^\kappa > 0$, depending on $t_{x_n}^\kappa$, τ, κr_n, and x_n, such that (4.32) hold.

Finally, we observe that

$$\mathcal{H}^{N-1}(\Gamma\setminus S) \leq \mathcal{H}^{N-1}(\Gamma\setminus S_1) + \mathcal{H}^{N-1}(S_1\setminus S_2) + \mathcal{H}^{N-1}(S_2\setminus S) \leq \tau,$$

and this completes the proof. □

Proposition 4.9. *Let $\omega \in C(\Omega)$ be nonnegative, let $\Gamma \subset \Omega$ be $\mathcal{H}^{N-1}$-rectifiable with $\mathcal{H}^{N-1}(\Gamma) < +\infty$, and let $\tau \in (0,1)$ be given. There exists a $\mathcal{H}^{N-1}$-rectifiable set $S \subset \Gamma$ and a countable family of disjoint cubes $\mathcal{F} = \{Q_{\nu_\Gamma}(x_n,r_n)\}_{n=1}^\infty$ with $r_n < \tau$ such that the following hold:*

1.

$$\mathcal{H}^{N-1}(\Gamma\setminus S) < \tau,\ S \subset \bigcup_{n=1}^{\infty} Q_{\nu_\Gamma}(x_n,r_n),\ \text{ and } S \cap Q_{\nu_\Gamma}(x_n,r_n) \subset R_{\tau/2,\nu_\Gamma}(x_n,r_n); \tag{4.33}$$

[2] Yes, we assume $\nu_\Gamma(x) = e_N$, and this is why we say "up to a rotation" above

2.

$$\mathcal{H}^{N-1}\left(S \cap Q_{\nu_\Gamma}(x_n, r_n)\right) \leq (1+\tau^2) r_n^{N-1}; \tag{4.34}$$

3. for $n \neq m$

$$\operatorname{dist}(Q_{\nu_\Gamma}(x_n, r_n), Q_{\nu_\Gamma}(x_m, r_m)) > 0; \tag{4.35}$$

4.

$$\sum_{n=1}^{+\infty} r_n^{N-1} \leq 4\mathcal{H}^{N-1}(\Gamma) \tag{4.36}$$

5. for each $n \in \mathbb{N}$ there exist $t_n \in (-\tau r_n/4, \tau r_n/4)$ and $0 < t_{x_n,r_n} < |t_n|$, depending on τ, r_n, and x_n, such that $T_{x_n,\nu_\Gamma}(t_n \pm t_{x_n,r_n}) \subset R_{\tau/2,\nu_\Gamma}(x_n, r_n)$ and

$$\sup_{0<t\leq t_{x_n,r_n}} \frac{1}{|I(t_n,t)|} \int_{I(t_n,t)} \int_{Q_{\nu_\Gamma}(x_n,r_n)\cap T_{x_n,\nu_\Gamma}(l)} \omega(x) d\mathcal{H}^{N-1} dl \leq \int_{S\cap Q_{\nu_\Gamma}(x_n,r_n)} \omega d\mathcal{H}^{N-1} + (1+\omega(x_n))\tau {r_n}^{N-1}, \tag{4.37}$$

where $I(t_n, t) := (t_n - t, t_n + t)$.

Proof. We apply items 1, 2, and 3 in Proposition 4.8 to obtain a countable collection $\{Q_{\nu_\Gamma}(x_n, r'_n)\}_{n=1}^{\infty}$ and a set $S' \subset \Gamma$ such that

$$\mathcal{H}^{N-1}(\Gamma \setminus S') < \frac{\tau}{2},\ S' \subset \bigcup_{n=1}^{\infty} Q_{\nu_\Gamma}(x_n, r'_n),\ S' \cap Q_{\nu_\Gamma}(x_n, r'_n) \subset R_{\tau/2,\nu_\Gamma}(x_n, r'_n),$$

and

$$\mathcal{H}^{N-1}\left(S \cap Q_{\nu_\Gamma}(x_n, r)\right) \leq (1+\tau^2) r^{N-1}$$

for all $0 < r < r'_n$. Find $0 < \kappa < 1$ such that

$$\mathcal{H}^{N-1}\left(S' \setminus \bigcup_{n=1}^{\infty} Q_{\nu_\Gamma}(x_n, \kappa r'_n)\right) < \frac{\tau}{2},$$

and let

$$S := S' \cap \left(\bigcup_{n=1}^{\infty} Q_{\nu_\Gamma}(x_n, \kappa r'_n)\right).$$

Then

$$S \subset \bigcup_{n=1}^{\infty} Q_{\nu_\Gamma}(x_n, \kappa r'_n)$$

and

$$\mathcal{H}^{N-1}(\Gamma \setminus S) \leq \mathcal{H}^{N-1}(\Gamma \setminus S') + \mathcal{H}^{N-1}(S' \setminus S) \leq \frac{\tau}{2} + \frac{\tau}{2} = \tau.$$

Note that the collection $\{Q_{\nu_\Gamma}(x_n, \kappa r'_n)\}_{n=1}^{\infty}$ satisfies (4.35). Next, we apply item 4 in Proposition 4.8 with such $\kappa > 0$ to find t_n^κ, t_{x_n,r'_n}^κ such that (4.32) holds. It suffices to set $r_n := \kappa r'_n$, $t_n := t_n^\kappa$, and $t_{x_n,r_n} := t_{x_n,r'_n}^\kappa$.

□

4.2. **The Case** $\omega \in \mathcal{W}(\Omega) \cap C(\Omega)$**.**

Consider the functionals

$$E_{\omega,\varepsilon}(u,v) := \int_\Omega v^2 |\nabla u|^2 \omega\, dx + \int_\Omega \left[\varepsilon |\nabla v|^2 + \frac{1}{4\varepsilon}(v-1)^2\right] \omega\, dx$$

for $(u,v) \in W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega)$, and let

$$E_\omega(u) := \int_\Omega |\nabla u|^2 \omega\, dx + \int_{S_u} \omega(x)\, d\mathcal{H}^{N-1},$$

be defined for $u \in GSBV_\omega(\Omega)$.

Theorem 4.10. *Let $\omega \in \mathcal{W}(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$ be given. Let $\mathcal{E}_{\omega,\varepsilon}\colon L^1_\omega(\Omega) \times L^1(\Omega) \to [0,+\infty]$ be defined by*

$$\mathcal{E}_{\omega,\varepsilon}(u,v) := \begin{cases} E_{\omega,\varepsilon}(u,v) & \text{if } (u,v) \in W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega),\ 0 \le v \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the functionals $\mathcal{E}_{\omega,\varepsilon}$ Γ-converge, with respect to the $L^1_\omega \times L^1$ topology, to the functional

$$\mathcal{E}_\omega(u,v) := \begin{cases} E_\omega(u) & \text{if } u \in GSBV_\omega(\Omega) \text{ and } v = 1 \text{ a.e.}, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 4.10 will be proved in two propositions.

Proposition 4.11. (Γ-lim inf) *For $\omega \in \mathcal{W}(\Omega) \cap C(\Omega)$ and $u \in L^1_\omega(\Omega)$, let*

$$E^-_\omega(u) := \inf\Big\{\liminf_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : \\ (u_\varepsilon, v_\varepsilon) \in W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega),\ u_\varepsilon \to u \text{ in } L^1_\omega,\ v_\varepsilon \to 1 \text{ in } L^1,\ 0 \le v_\varepsilon \le 1\Big\}.$$

We have

$$E^-_\omega(u) \ge E_\omega(u).$$

Proof. Without loss of generality, we assume that $M := E^-_\omega(u) < \infty$. Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega)$ be such that

$$u_\varepsilon \to u \text{ in } L^1_\omega,\ \ v_\varepsilon \to 1 \text{ in } L^1(\Omega),\ \text{ and } \lim_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) = E^-_\omega(u) < \infty.$$

Since $\inf_{x\in\Omega} \omega(x) \ge 1$, we have

$$\liminf_{\varepsilon\to 0} E_{1,\varepsilon}(u_\varepsilon, v_\varepsilon) \le \liminf_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) < \infty,$$

and by [8] we deduce that

$$u \in GSBV(\Omega) \text{ and } \mathcal{H}^{N-1}(S_u) < \infty.$$

We prove separately that

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2 v_\varepsilon\, \omega\, dx \ge \int_\Omega |\nabla u|^2 \omega\, dx, \tag{4.38}$$

and

$$\liminf_{\varepsilon\to 0} \int_\Omega \left(\varepsilon |\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon}(1-v_\varepsilon)^2\right) \omega\, dx \ge \int_{S_u} \omega(x) d\mathcal{H}^{N-1}. \tag{4.39}$$

Let A be an open subset of Ω. Fix $\nu \in \mathcal{S}^{N-1}$, and define $A_{x,\nu}$, $A^1_{x,\nu}$, and A_ν as in (4.1). For $K \in \mathbb{R}^+$, set $u_K := K \wedge u \vee -K$, and observe that, by Fubini's Theorem,

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_A |\nabla u_\varepsilon|^2 v_\varepsilon^2\,\omega\,dx &\geq \liminf_{\varepsilon\to 0}\int_{A_\nu}\int_{A^1_{x,\nu}} \left|(u_\varepsilon)'_{x,\nu}\right|^2 (v_\varepsilon)^2_{x,\nu}\,\omega_{x,\nu}\,dt\,d\mathcal{H}^{N-1}(x)\\ &\geq \int_{A_\nu}\liminf_{\varepsilon\to 0}\int_{A^1_{x,\nu}} \left|(u_\varepsilon)'_{x,\nu}\right|^2 (v_\varepsilon)^2_{x,\nu}\,\omega_{x,\nu}\,dt\,d\mathcal{H}^{N-1}(x)\\ &\geq \int_{A_\nu}\int_{A^1_{x,\nu}} \left|u'_{x,\nu}\right|^2 \omega_{x,\nu}\,dt\,d\mathcal{H}^{N-1}(x)\\ &\geq \int_{A_\nu}\int_{A^1_{x,\nu}} \left|(u_K)'_{x,\nu}\right|^2 \omega_{x,\nu}\,dt\,d\mathcal{H}^{N-1}(x),\end{aligned}$$

where in the first inequality we used Lemma 4.2, in the second inequality we used Fatou's Lemma, and in the third inequality we used (3.3). Since $u_K \in L^\infty(\Omega)\cap SBV_\omega(\Omega) \subset L^\infty(\Omega)\cap SBV(\Omega)$, we may apply Theorem 2.3 in [8] to u_K to obtain

$$\liminf_{\varepsilon\to 0}\int_A |\nabla u_\varepsilon|^2 v_\varepsilon^2\,\omega\,dx \geq \int_{A_\nu}\int_{A^1_{x,\nu}} \left|(u_K)'_{x,\nu}\right|^2 \omega_{x,\nu}\,dt\,d\mathcal{H}^{N-1}(x) \geq \int_A |\langle\nabla u_K(x),\nu\rangle|^2\,\omega\,dx.$$

Letting $K\to\infty$ and using Lebesgue Monotone Convergence Theorem we have

$$\liminf_{\varepsilon\to 0}\int_A |\nabla u_\varepsilon|^2 v_\varepsilon^2\,\omega\,dx \geq \int_A |\langle\nabla u(x),\nu\rangle|^2\,\omega\,dx. \tag{4.40}$$

Let

$$\phi_n(x) := |\langle\nabla u(x),\nu_n\rangle|^2\,\omega \text{ for } \mathcal{L}^N\text{-a.e. } x\in\Omega,$$

where $\{\nu_n\}_{n=1}^\infty$ is a dense subset of $\mathcal{S}^{N-1}$, and let

$$\mu(A) := \liminf_{\varepsilon\to 0}\int_A |\nabla u_\varepsilon|^2 v_\varepsilon^2\,\omega\,dx.$$

Then μ is a positive function, super-additivity on open sets A, B, with disjoint closures, since

$$\begin{aligned}\mu(A\cup B) = \liminf_{\varepsilon\to 0}\int_{A\cup B} |\nabla u_\varepsilon|^2 v_\varepsilon^2\,\omega\,dx &= \liminf_{\varepsilon\to 0}\left(\int_A |\nabla u_\varepsilon|^2 v_\varepsilon^2\,\omega\,dx + \int_B |\nabla u_\varepsilon|^2 v_\varepsilon^2\,\omega\,dx\right)\\ &\geq \liminf_{\varepsilon\to 0}\int_A |\nabla u_\varepsilon|^2 v_\varepsilon^2\,\omega\,dx + \liminf_{\varepsilon\to 0}\int_B |\nabla u_\varepsilon|^2 v_\varepsilon^2\,\omega\,dx = \mu(A)+\mu(B).\end{aligned}$$

Hence by Lemma 15.2 in [17], together with (4.40), we conclude (4.38).

Now we prove (4.39). Assume first that $\omega \in L^\infty(\Omega)$. For any open set $A\subset\Omega$ and $\nu\in\mathcal{S}^{N-1}$, by Fubini's Theorem and Fatou's Lemma we have

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_A \left(\varepsilon\left|\nabla v_\varepsilon\right|^2+\frac{1}{4\varepsilon}(1-v_\varepsilon)^2\right)\omega\,dx\\
&\geq \liminf_{\varepsilon\to 0}\int_{A_\nu}\int_{A^1_{x,\nu}}\left[\varepsilon\left|(v_\varepsilon)'_{x,\nu}\right|^2+\frac{1}{4\varepsilon}(1-(v_\varepsilon)_{x,\nu})^2\right]\omega_{x,\nu}\,dtd\mathcal{H}^{N-1}(x)\\
&\geq \int_{A_\nu}\liminf_{\varepsilon\to 0}\int_{A^1_{x,\nu}}\left[\varepsilon\left|(v_\varepsilon)'_{x,\nu}\right|^2+\frac{1}{4\varepsilon}(1-(v_\varepsilon)_{x,\nu})^2\right]\omega_{x,\nu}\,dtd\mathcal{H}^{N-1}(x)\\
&\geq \int_{A_\nu}\left[\sum_{t\in S_{u_{x,\nu}}\cap A^1_{x,\nu}}\omega_{x,\nu}(t)\right]d\mathcal{H}^{N-1}(x),
\end{aligned}\tag{4.41}$$

where the last inequality follows from (3.12).

Next, given arbitrary $\tau>0$ and $\eta>0$ we choose a set $S\subset S_u$ and a collection $\mathcal{Q}$ of mutually disjoint cubes according to Lemma 4.6 with respect to S_u. Fix one such cube $Q_{\nu_S}(x_0,r_0)\in\mathcal{Q}$. By Lemma 4.6 we have

$$\mathcal{H}^{N-1}([Q_{\nu_S}(x_0,r_0)]_{x,\nu_S}\cap S)=1$$

for $\mathcal{H}^{N-1}$-a.e. $x\in Q_{\nu_S}(x_0,r_0)\cap S$, and $Q_{\nu_S}(x_0,r_0)\cap S\subset\Gamma_{x_0}$ such that, up to a rotation and a translation,

$$\Gamma_{x_0}=\{(y',l_{x_0}(y')):\ y\in T_{x_0,\nu_S}\cap Q_{\nu_S}(x_0,r_0)\}\ \text{and}\ \|\nabla l_{x_0}\|_{L^\infty}<\tau,\tag{4.42}$$

where y' denotes the first $N-1$ components of $y\in T_{x_0,\nu_S}\cap Q_{\nu_S}(x_0,r_0)$.

In (4.41) set $A=Q_{\nu_S}(x_0,r_0)$ and $\nu=\nu_S(x_0)$ and, using the same notation as in the proof of Lemma 4.6, we obtain

$$\begin{aligned}
&\int_{[Q_{\nu_S}(x_0,r_0)]_{\nu_S(x_0)}}\left(\sum_{t\in S_{u_{x,\nu_S(x_0)}}\cap[Q_{\nu_S}(x_0,r_0)]_{x,\nu_S(x_0)}}\omega_{x,\nu_S(x_0)}(t)\right)d\mathcal{H}^{N-1}(x)\\
&\geq\int_{T_g(x_0,r_0)}\left(\sum_{t\in S_{u_{x,\nu_S(x_0)}}\cap[Q_{\nu_S}(x_0,r_0)]_{x,\nu_S(x_0)}\cap S}\omega_{x,\nu_S(x_0)}(t)\right)d\mathcal{H}^{N-1}(x)\\
&=\int_{T_g(x_0,r_0)}\omega(x)\,d\mathcal{H}^{N-1}(x)=\int_{T_g(x_0,r_0)}\omega(x',l_{x_0}(x'))d\mathcal{L}^{N-1}(x'),
\end{aligned}\tag{4.43}$$

where the first inequality is due to the positivity of ω and the last equality is because $Q_{\nu_S}(x_0,r_0)\cap S\subset\Gamma_{x_0}$ which is defined in (4.42).

Next, by Theorem 9.1 in [39] and since $\omega\in C(\Omega)$, we have that

$$\begin{aligned}
\int_{Q_{\nu_S}(x_0,r_0)\cap S}\omega\,d\mathcal{H}^{N-1}&=\int_{T_{x_0,\nu_S}\cap Q_{\nu_S}(x_0,r_0)}\omega(x',l_{x_0}(x'))\sqrt{1+|\nabla l_{x_0}(x')|^2}dx'\\
&\leq\sqrt{1+\tau^2}\int_{T_{x_0,\nu_S}\cap Q_{\nu_S}(x_0,r_0)}\omega(x',l_{x_0}(x'))dx',
\end{aligned}$$

which, together with (4.43), yields

$$\int_{[Q_{\nu_S}(x_0,r_0)]_{\nu_S(x_0)}} \left(\sum_{t \in S_{u_{x,\nu_S(x_0)}} \cap [Q_{\nu_S}(x_0,r_0)]_{x,\nu_S(x_0)}} \omega_{x,\nu_S(x_0)}(t) \right) d\mathcal{H}^{N-1}(x) \\ \geq \frac{1}{\sqrt{1+\tau^2}} \int_{Q_{\nu_S}(x_0,r_0)\cap S} \omega \, d\mathcal{H}^{N-1}. \quad (4.44)$$

Since cubes in $\mathcal{Q}$ are pairwise disjoint and $\mathcal{H}^{N-1}(S \setminus \cup_{Q\in\mathcal{Q}} Q) = 0$, by (4.41), (4.43), and (4.44) we have

$$\begin{aligned} \liminf_{\varepsilon\to 0} \int_{\cup_{Q\in\mathcal{Q}} Q} & \left[\varepsilon \left|\nabla v_\varepsilon\right|^2 + \frac{1}{4\varepsilon}(v_\varepsilon - 1)^2 \right] \omega \, dx \\ &\geq \sum_{Q\in\mathcal{Q}} \liminf_{\varepsilon\to 0} \int_Q \left[\varepsilon \left|\nabla v_\varepsilon\right|^2 + \frac{1}{4\varepsilon}(v_\varepsilon - 1)^2 \right] \omega \, dx \\ &\geq \frac{1}{\sqrt{1+\tau^2}} \sum_{Q\in\mathcal{Q}} \int_{S\cap Q} \omega \, d\mathcal{H}^{N-1} = \frac{1}{\sqrt{1+\tau^2}} \int_S \omega \, d\mathcal{H}^{N-1} \\ &\geq \frac{1}{\sqrt{1+\tau^2}} \left(\int_{S_u} \omega \, d\mathcal{H}^{N-1} - \|\omega\|_{L^\infty} \eta \right). \end{aligned}$$

Therefore

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \int_\Omega \left[\varepsilon \left|\nabla v_\varepsilon\right|^2 + \frac{1}{4\varepsilon}(v_\varepsilon - 1)^2 \right] \omega \, dx \\ &\geq \liminf_{\varepsilon\to 0} \int_{\cup_{Q\in\mathcal{Q}} Q} \left[\varepsilon \left|\nabla v_\varepsilon\right|^2 + \frac{1}{4\varepsilon}(v_\varepsilon - 1)^2 \right] \omega \, dx \geq \frac{1}{\sqrt{1+\tau^2}} \left(\int_{S_u} \omega(x) d\mathcal{H}^{N-1} - \|\omega\|_{L^\infty} \eta \right), \end{aligned}$$

and (4.39) follows from the arbitrariness of η and τ, and the fact that η and τ are independent.

We now remove the assumption that $\omega \in L^\infty$. Define for each $k > 0$,

$$\omega_k(x) := \begin{cases} \omega & \text{if } \omega \leq k, \\ k & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \liminf_{\varepsilon\to 0} \int_\Omega & \left[\varepsilon \left|\nabla v_\varepsilon\right|^2 + \frac{1}{4\varepsilon}(v_\varepsilon - 1)^2 \right] \omega \, dx \\ &\geq \liminf_{\varepsilon\to 0} \int_\Omega \left[\varepsilon \left|\nabla v_\varepsilon\right|^2 + \frac{1}{4\varepsilon}(v_\varepsilon - 1)^2 \right] \omega_k \, dx \geq \int_{S_u} \omega_k(x) d\mathcal{H}^{N-1}, \end{aligned}$$

and we conclude

$$\liminf_{\varepsilon\to 0} \int_\Omega \left[\varepsilon \left|\nabla v_\varepsilon\right|^2 + \frac{1}{4\varepsilon}(v_\varepsilon - 1)^2 \right] \omega \, dx \geq \int_{S_u} \omega(x) d\mathcal{H}^{N-1}$$

by letting $k \nearrow \infty$ and using Lebesgue Monotone Convergence Theorem. □

Proposition 4.12. (Γ-lim sup) *For* $\omega \in \mathcal{W}(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$ *and* $u \in L^1_\omega(\Omega) \cap L^\infty(\Omega)$, *let*

$$E^+_\omega(u) := \inf \Big\{ \limsup_{\varepsilon \to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : \\ (u_\varepsilon, v_\varepsilon) \in W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega), u_\varepsilon \to u \text{ in } L^1_\omega,\ v_\varepsilon \to 1 \text{ in } L^1,\ 0 \le v_\varepsilon \le 1 \Big\}.$$

We have

$$E^+_\omega(u) \le E_\omega(u). \tag{4.45}$$

Proof. If $E_\omega(u) = \infty$ then there is nothing to prove. Assume that $E_\omega(u) < +\infty$ so that by Lemma 2.10 we have that $u \in GSBV_\omega(\Omega)$ and $\mathcal{H}^{N-1}(S_u) < \infty$. By assumption $u \in L^\infty(\Omega)$, thus $u \in SBV_\omega(\Omega)$.

Let $\tau \in (0, 2/9)$ be given. Apply Proposition 4.9 to ω and $\Gamma = S_u$ to obtain a set $S_\tau \subset S_u$, a countable collection $\mathcal{F}_\tau = \{Q_{\nu_{S_u}}(x_n, r_n)\}_{n=1}^\infty$ of mutually disjoint cubes with $r_n < \tau$, and corresponding

$$t_n \in (-\tau r_n/4, \tau r_n/4) \tag{4.46}$$

and t_{x_n,r_n} so that items 1-5 in Proposition 4.9 hold. Extract a finite collection $\mathcal{T}_\tau = \{Q_{\nu_{S_u}}(x_n, r_n)\}_{n=1}^{M_\tau}$ from $\mathcal{F}_\tau$ with $M_\tau > 0$ large enough such that

$$\mathcal{H}^{N-1}\left[S_\tau \setminus \bigcup_{n=1}^{M_\tau} Q_{\nu_{S_u}}(x_n, r_n)\right] < \tau,$$

and we define

$$F_\tau := S_\tau \cap \left[\bigcup_{n=1}^{M_\tau} Q_{\nu_{S_u}}(x_n, r_n)\right], \tag{4.47}$$

which implies that

$$\mathcal{H}^{N-1}(S_u \setminus F_\tau) \le \mathcal{H}^{N-1}(S_u \setminus S_\tau) + \mathcal{H}^{N-1}(S_\tau \setminus F_\tau) < 2\tau. \tag{4.48}$$

Let U_n be the part of $Q_{\nu_{S_u}}(x_n, r_n)$ which lies between $T_{x_n,\nu_{S_u}}(\pm\tau r_n)$, U_n^+ be the part above $T_{x_n,\nu_{S_u}}(\tau r_n)$ and U_n^- be the part below $T_{x_n,\nu_{S_u}}(-\tau r_n)$. Moreover, let $U_{t_n}^+$ be the part of U_n which lies above $T_{x_n,\nu_{S_u}}(t_n)$, and $U_{t_n}^-$ be the part below $T_{x_n,\nu_{S_u}}(t_n)$.

We claim that if $x \in U_{t_n}^\pm$,

$$x \pm 2\,\mathrm{dist}\left(x, T_{x_n,\nu_{S_u}}(\pm\tau r_n)\right) \nu_{S_u}(x_n) \in U_n^\pm \subset Q_{\nu_{S_u}}(x_n, r_n) \setminus R_{\tau/2,\nu_{S_u}}(x_n, r_n). \tag{4.49}$$

Let $x \in U_{t_n}^+$ (the case in which $x \in U_{t_n}^-$ can be handled similarly), we need to prove that

$$\tau r_n < \mathrm{dist}\left(x + 2\,\mathrm{dist}(x, T_{x_n,\nu_{S_u}}(\tau r_n))\nu_{S_u}(x_n),\ T_{x_n,\nu_{S_u}}\right) < \frac{r_n}{2}.$$

Note that

$$\mathrm{dist}\left(x, T_{x_n,\nu_{S_u}}(\tau r_n)\right) = \tau r_n - (x - \mathbb{P}_{x_n,\nu_{S_u}}(x))\nu_{S_u}(x_n),$$

and since $x \in U_{t_n}^+$, we have that

$$\left(x - \mathbb{P}_{x_n,\nu_{S_u}}(x)\right) \nu_{S_u}(x_n) \in (t_n, \tau r_n).$$

Hence, together with (4.46), we observe that

$$\tau r_n \le 2\,\mathrm{dist}\left(x, T_{x_n,\nu_{S_u}}(\tau r_n)\right) + \left(x - \mathbb{P}_{x_n,\nu_{S_u}}(x)\right) \nu_{S_u}(x_n) \tag{4.50}$$

$$= 2\tau r_n - (x - \mathbb{P}_{x_n,\nu_{S_u}}(x))\nu_{S_u}(x_n) \le 2\tau r_n - \left(-\frac{\tau r_n}{4}\right) = \frac{9}{4}\tau r_n < \frac{1}{2} r_n.$$

From the definition of projection operator $\mathbb{P}_{x_n,\nu_{S_u}}$ we have

$$\mathbb{P}_{x_n,\nu_{S_u}}\left[x + 2\,\mathrm{dist}\left(T_{x_n,\nu_{S_u}}(\tau r_n)\right)\nu_{S_u}(x_n)\right] = \mathbb{P}_{x_n,\nu_{S_u}}(x),$$

and hence

$$\begin{aligned} &\mathrm{dist}\left(x + 2\,\mathrm{dist}(x, T_{x_n,\nu_{S_u}}(\tau r_n))\nu_{S_u}(x_n),\, T_{x_n,\nu_{S_u}}\right) \\ &= \left(\left[x + 2\,\mathrm{dist}\left(x, T_{x_n,\nu_{S_u}}(\tau r_n)\right)\nu_{S_u}(x_n)\right]\right. \\ &\quad \left. - \mathbb{P}_{x_n,\nu_{S_u}}\left[x + 2\,\mathrm{dist}\left(T_{x_n,\nu_{S_u}}(\tau r_n)\right)\nu_{S_u}(x_n)\right]\right)\nu_{S_u}(x_n) \\ &= \left(2\,\mathrm{dist}\left(x, T_{x_n,\nu_{S_u}}(\tau r_n)\right)\nu_{S_u}(x_n)\right)\nu_{S_u}(x_n) \\ &\quad + \left(x - \mathbb{P}_{x_n,\nu_{S_u}}\left[x + 2\,\mathrm{dist}\left(T_{x_n,\nu_{S_u}}(\tau r_n)\right)\nu_{S_u}(x_n)\right]\right)\nu_{S_u}(x_n) \\ &= 2\,\mathrm{dist}\left(x, T_{x_n,\nu_{S_u}}(\tau r_n)\right) + \left(x - \mathbb{P}_{x_n,\nu_{S_u}}(x)\right)\nu_{S_u}(x_n), \end{aligned} \tag{4.51}$$

and by (4.50) we conclude (4.49).

We define $\bar{u}_\tau$ in $Q_{\nu_{S_u}}(x_n, r_n)$ as follows (see Figure 1):

$$\bar{u}_\tau(x) := \begin{cases} u(x) & \text{if } x \in U_n^+ \cup U_n^- \\ u\left(x + 2\,\mathrm{dist}(x, T_{x_n,\nu_{S_u}}(\tau r_n))\nu_{S_u}(x_n)\right) & \text{if } x \in U_{t_n}^+, \\ u\left(x - 2\,\mathrm{dist}(x, T_{x_n,\nu_{S_u}}(-\tau r_n))\nu_{S_u}(x_n)\right) & \text{if } x \in U_{t_n}^-, \end{cases} \tag{4.52}$$

and

$$\bar{u}_\tau(x) := u(x) \text{ if } x \in \Omega \setminus \left(\bigcup_{n=1}^{M_\tau} Q_{\nu_{S_u}}(x_n, r_n)\right).$$

We observe that, as $\tau \to 0$, and since $0 < r_n < \tau$,

$$\begin{aligned} &\mathcal{L}^N\left(\{x \in \Omega,\ u(x) \neq \bar{u}_\tau(x)\}\right) = \mathcal{L}^N\left(\bigcup_{n=1}^{M_\tau} U_{t_n}^+ \cup U_{t_n}^-\right) \\ &\le \sum_{n=1}^{M_\tau} \mathcal{L}^N\left(U_{t_n}^+ \cup U_{t_n}^-\right) = \sum_{n=1}^{M_\tau}\left(r_n^{N-1} 2\tau r_n\right) \le 2\tau^2 \sum_{n=1}^{M_\tau} r_n^{N-1} \le 8\tau^2 \mathcal{H}^{N-1}(S_u) \to 0, \end{aligned} \tag{4.53}$$

where the last inequality follows from (4.36). Moreover, using the same computation, we deduce that

$$\mathcal{L}^N\left(\bigcup_{n=1}^{M_\tau} Q_{\nu_{S_u}}(x_n, r_n)\right) \le \tau \sum_{n=1}^{M_\tau} r_n^{N-1} \le 4\tau \mathcal{H}^{N-1}(S_u) = O(\tau) \to 0. \tag{4.54}$$

Hence, in view of (4.53), we have

$$\bar{u}_\tau \to u \text{ and } \nabla \bar{u}_\tau \to \nabla u \text{ in measure,} \tag{4.55}$$

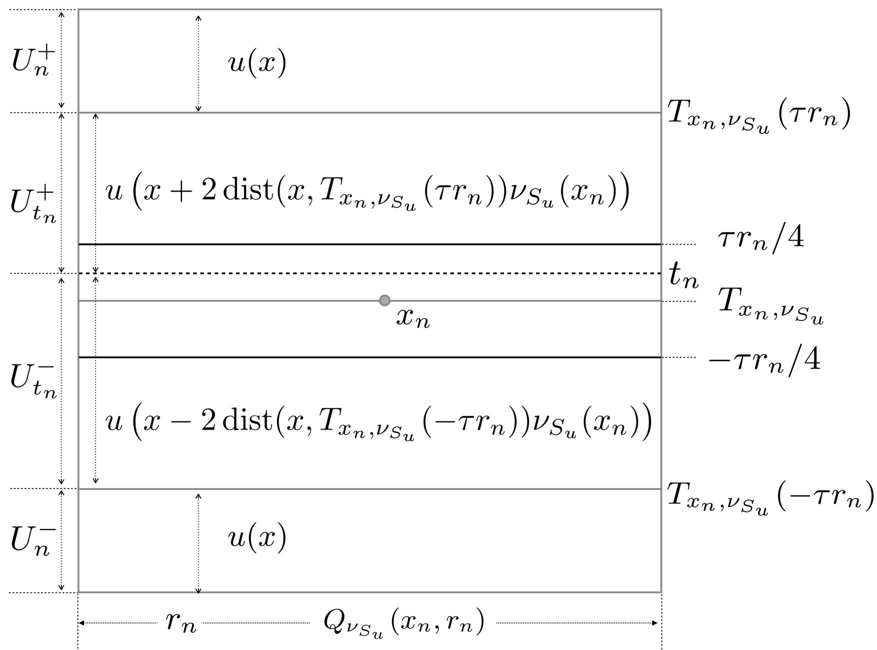

FIGURE 1. Construction of $\bar{u}_\tau(x)$ in (4.52)

and, since in $U_{t_n}^+ \cup U_{t_n}^-$ $\bar{u}_\tau$ is the reflection of u from $Q_{\nu_{S_u}}(x_n,r_n) \setminus U_{t_n}^+ \cup U_{t_n}^-$, we observe that

$$\begin{aligned}\int_\Omega |\nabla \bar{u}_\tau|^2 \,\omega\,dx &\le \int_{\Omega\setminus\{u(x)\neq\bar{u}_\tau(x)\}} |\nabla u|^2\,\omega\,dx + \|\omega\|_{L^\infty}\int_{\{u(x)\neq\bar{u}_\tau(x)\}} |\nabla\bar{u}_\tau|^2\,dx \\ &\le \int_{\Omega\setminus\{u(x)\neq\bar{u}_\tau(x)\}} |\nabla u|^2\,\omega\,dx + 2\,\|\omega\|_{L^\infty}\sum_{n=1}^{M_\tau}\int_{Q_{\nu_{S_u}}(x_n,r_n)} |\nabla u|^2\,dx \\ &= \int_{\Omega\setminus\{u(x)\neq\bar{u}_\tau(x)\}} |\nabla u|^2\,\omega\,dx + 2\,\|\omega\|_{L^\infty}\int_{\bigcup_{n=1}^{M_\tau} Q_{\nu_{S_u}}(x_n,r_n)} |\nabla u|^2\,dx \\ &\le \int_\Omega |\nabla u|^2\,\omega dx + O(\tau)\end{aligned} \tag{4.56}$$

where the last inequality follows from (4.54) and from the fact that because $E_1(u) \le E_\omega(u) < +\infty$, ∇u is L^2 integrable. Moreover, in view of (4.55) and by Lebesgue Dominated Convergence Theorem we conclude that

$$\lim_{\tau\to 0}\int_\Omega |\bar{u}_\tau - u|\,\omega\,dx \le \|\omega\|_{L^\infty}\lim_{\tau\to 0}\int_\Omega |\bar{u}_\tau - u|\,dx = 0 \tag{4.57}$$

because $\|\bar{u}_\tau\|_{L^\infty} \le \|u\|_{L^\infty} < +\infty$.

For simplicity of notation, in the rest of the proof of this lemma we shall abbreviate $Q_{\nu_{S_u}}(x_n,r_n)$ by Q_n and $T_{x_n,\nu_{S_u}}$ by T_{x_n}. Note that the jump set of $\bar{u}_\tau$ is contained by (recall item 4 in Proposition 4.9)

1.
$$\bigcup_{n=1}^{M_\tau} \left[T_{x_n}(t_n) \cap Q_n\right];$$

2.
$$\bigcup_{n=1}^{M_\tau} \partial Q_n \cap \overline{U_n};$$

3. $S_u \setminus F_\tau$, where F_τ is defined in (4.47).

The contributions to S_u from 2 and 3 are negligible. To be precise,

$$\begin{aligned}
&\mathcal{H}^{N-1}\left[(S_u \setminus F_\tau) \cup \left(\bigcup_{n=1}^{M_\tau} \partial Q_n \cap \overline{U_n}\right)\right] \\
&\leq \mathcal{H}^{N-1}(S_u \setminus F_\tau) + \sum_{n=1}^{M_\tau} \mathcal{H}^{N-1}(\partial Q_n \cap \overline{U_n}) \leq 2\tau + C\tau \sum_{n=1}^{\infty} r_n^{N-1}\tau \leq O(\tau).
\end{aligned}$$

where we used (4.33), (4.36), (4.48), and the fact that

$$\sum_{n=1}^{M_\tau} \mathcal{H}^{N-1}(\partial Q_n \cap \overline{U_n}) \leq 2\tau \sum_{n=1}^{M_\tau} r_n^{N-1} \leq 8\tau \mathcal{H}^{N-1}(S_u).$$

Hence, again by (4.36),

$$\mathcal{H}^{N-1}(S_{\bar{u}_\tau}) \leq \sum_{n=1}^{M_\tau} \mathcal{H}^{N-1}(T_{x_n} \cap Q_n) + O(\tau) \leq \sum_{n=0}^{\infty} r_n^{N-1} + O(\tau) < \infty.$$

By (4.35), let a_τ denote a quarter of the minimum distance between all cubes in $\mathcal{T}_\tau$. Let $\varepsilon > 0$ be such that

$$\varepsilon^2 + \sqrt{\varepsilon} << \frac{1}{4} \min\left\{\tau,\, a_\tau,\, t_{x_n,r_n} \text{ for } 1 \leq n \leq M_\tau\right\}. \tag{4.58}$$

Hence, by item 5 in Proposition 4.9 we have

$$\varepsilon^2 + \sqrt{\varepsilon} < t_{x_n,r_n} < |t_n| < \frac{1}{4}\tau r_n < r_n. \tag{4.59}$$

We set

$$u_{\tau,\varepsilon} := (1 - \varphi_\varepsilon)\bar{u}_\tau,$$

where φ_ε is such that

$$\varphi_\varepsilon \in C_c^\infty(\Omega; [0,1]), \qquad \varphi_\varepsilon \equiv 1 \text{ on } (\overline{S_{\bar{u}_\tau}})_{\varepsilon^2/4}, \qquad \text{and } \varphi_\varepsilon \equiv 0 \text{ in } \Omega \setminus (\overline{S_{\bar{u}_\tau}})_{\varepsilon^2/2}.$$

Since $\bar{u}_\tau \in W^{1,2}(\Omega \setminus \overline{S_{\bar{u}_\tau}})$, we have $\{u_{\tau,\varepsilon}\}_{\varepsilon>0} \subset W^{1,2}(\Omega)$ because $(1-\varphi_\varepsilon)(x) = 0$ if $x \in (\overline{S_{\bar{u}_\tau}})_{\varepsilon^2/4}$. Moreover, $\{u_{\tau,\varepsilon}\}_{\varepsilon>0} \subset W_\omega^{1,2}(\Omega)$ and, using Lebesgue Dominated Convergence Theorem and (4.57),

$$\lim_{\tau\to 0} \lim_{\varepsilon \to 0} \int_\Omega |u_{\tau,\varepsilon} - u|\, \omega = 0 \tag{4.60}$$

because $\omega \in L^\infty$, $u \in L^\infty$, and $\varphi_\varepsilon \to 0$ a.e.

Consider the sequence $\{v_{\tau,\varepsilon}\}_{\varepsilon>0} \subset W^{1,2}(\Omega)$ given by

$$v_{\tau,\varepsilon}(x) := \tilde{v}_\varepsilon \circ d_\tau(x)$$

where $d_\tau(x) := \operatorname{dist}(x, S_{\bar{u}_\tau})$ and $\tilde{v}_\varepsilon$ is defined by

$$\tilde{v}_\varepsilon(t) := \begin{cases} 0 & \text{if } t \le \varepsilon^2, \\ 1 - e^{-\frac{1}{2\sqrt{\varepsilon}}} & \text{if } t > \sqrt{\varepsilon} + \varepsilon^2, \end{cases} \tag{4.61}$$

and for $\varepsilon^2 \le t \le \sqrt{\varepsilon} + \varepsilon^2$ we define $\tilde{v}_\varepsilon$ as the solution of the differential equation

$$\tilde{v}'_\varepsilon(t) = \frac{1}{2\varepsilon}(1 - \tilde{v}_\varepsilon(t)). \tag{4.62}$$

with initial condition $\tilde{v}_\varepsilon(\varepsilon^2) = 0$. An explicit computation shows that

$$\tilde{v}_\varepsilon(t) = -e^{-\frac{1}{2}\frac{t-\varepsilon^2}{\varepsilon}} + 1 \tag{4.63}$$

for $\varepsilon^2 \le t \le \sqrt{\varepsilon} + \varepsilon^2$ and $\tilde{v}_\varepsilon(\sqrt{\varepsilon} + \varepsilon^2) = 1 - \exp(-1/2\sqrt{\varepsilon})$, and we remark that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} e^{-\frac{1}{2\sqrt{\varepsilon}}} = 0, \tag{4.64}$$

and

$$-\frac{d}{dt}\left(\frac{1}{2}(1 - \tilde{v}_\varepsilon(t))^2\right) = (1 - \tilde{v}_\varepsilon(t))\, \tilde{v}'_\varepsilon(t) \ge 0. \tag{4.65}$$

Next, since $|\nabla d_\tau| = 1$ a.e. (see [30], Section 3.2.34), we have $\{v_{\tau,\varepsilon}\}_{\varepsilon>0} \subset W^{1,2}(\Omega)$, $0 \le v_{\tau,\varepsilon} \le 1 - \exp(-1/2\sqrt{\varepsilon})$, and

$$v_{\tau,\varepsilon} \to 1 \text{ in } L^1 \text{ as } \varepsilon \to 0 \tag{4.66}$$

by Lebesgue Dominated Convergence Theorem since $v_{\tau,\varepsilon} \to 1$ a.e. by (4.63). By (4.56) and since if $\varphi_\varepsilon(x) \ne 0$ then $d_\tau(x) < \varepsilon^2/2$ and so $v_{\tau,\varepsilon}(x) = 0$,

$$\int_\Omega |\nabla u_{\tau,\varepsilon}|^2 v_{\tau,\varepsilon}^2\, \omega\, dx \le \int_\Omega |\nabla \bar{u}_\tau|^2\, \omega\, dx \le \int_\Omega |\nabla u|^2\, \omega\, dx + O(\tau). \tag{4.67}$$

Next we prove that

$$\int_\Omega \left[\varepsilon |\nabla v_{\tau,\varepsilon}|^2 + \frac{1}{4\varepsilon}(1 - v_{\tau,\varepsilon})^2\right] \omega\, dx \le \int_{S_u} \omega\, d\mathcal{H}^{N-1} + O(\varepsilon) + O(\tau). \tag{4.68}$$

Define

$$L_n := T_{x_n} \cap Q_n, \qquad L_n(\varepsilon) := (T_{x_n} \cap Q_n)_\varepsilon\,,$$

and observe that, using Fubini's Theorem,

$$\begin{aligned} &\int_{L_n(\varepsilon^2+\sqrt{\varepsilon})} \left[\varepsilon |\nabla v_{\tau,\varepsilon}|^2 + \frac{1}{4\varepsilon}(1 - v_{\tau,\varepsilon})^2\right] \omega\, dx \\ &= \int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} \left[\varepsilon |\tilde{v}'_\varepsilon(l)|^2 + \frac{1}{4\varepsilon}(1 - \tilde{v}_\varepsilon(l))^2\right] \int_{\{d_\tau(y)=l\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(y)\, d\mathcal{H}^{N-1}(y)\, dl \\ &\quad + \frac{1}{4\varepsilon} \int_{L_n(\varepsilon^2)} \omega(x) dx, \end{aligned}$$

where the latter term in the right hand side is of the order $O(\varepsilon)$. Next, in view of (4.62), using integration by parts, we have that

$$\begin{aligned}\int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} &\left[\varepsilon\left|\tilde{v}_\varepsilon'(l)\right|^2 + \frac{1}{4\varepsilon}(1-\tilde{v}_\varepsilon(l))^2\right] \int_{\{d_\tau(y)=l\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(y)\,d\mathcal{H}^{N-1}(y)\,dl \\ &= \int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} \frac{1}{2\varepsilon}(1-\tilde{v}_\varepsilon(l))^2 \int_{\{d_\tau(y)=l\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(y)\,d\mathcal{H}^{N-1}(y)\,dl \\ &= -\int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} \frac{1}{2\varepsilon}\frac{d}{dt}\left[(1-\tilde{v}_\varepsilon(l))^2\right] \int_{\{d_\tau(y)\le l\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(y)\,dy\,dl \\ &\quad + \mathcal{A}_\omega^n(\varepsilon^2+\sqrt{\varepsilon}) - \mathcal{A}_\omega^n(\varepsilon^2),\end{aligned} \tag{4.69}$$

where

$$\mathcal{A}_\omega^n(t) := \frac{1}{2\varepsilon}(1-\tilde{v}_\varepsilon(t))^2 \int_{\{d_\tau(x)\le t\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(y)\,dy.$$

By (4.59) and (4.64) we have

$$\begin{aligned}\mathcal{A}_\omega^n(\varepsilon^2+\sqrt{\varepsilon}) &= \frac{1}{2\varepsilon}e^{-\frac{1}{2\sqrt{\varepsilon}}} \int_{\left\{d_\tau(x)\le\varepsilon^2+\sqrt{\varepsilon}\right\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(x)\,dx \\ &\le \frac{1}{2\varepsilon}e^{-\frac{1}{2\sqrt{\varepsilon}}}\,\|\omega\|_{L^\infty}\,\mathcal{L}^N(L_n(\varepsilon^2+\sqrt{\varepsilon})) = \frac{1}{2\varepsilon}e^{-\frac{1}{2\sqrt{\varepsilon}}}\,\|\omega\|_{L^\infty}\,\mathcal{L}^N((T_{x_n}\cap Q_n)_{\varepsilon^2+\sqrt{\varepsilon}}) \\ &\le \frac{1}{2\varepsilon}e^{-\frac{1}{2\sqrt{\varepsilon}}}\,\|\omega\|_{L^\infty}\left[2(\varepsilon^2+\sqrt{\varepsilon})[r_n+(\varepsilon^2+\sqrt{\varepsilon})]^{N-1}\right] \le O(\varepsilon)r_n^{N-1}.\end{aligned} \tag{4.70}$$

We write

$$\begin{aligned}-\int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} &\frac{1}{2\varepsilon}\frac{d}{dt}\left[(1-\tilde{v}_\varepsilon(l))^2\right] \int_{\{d_\tau(y)\le l\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(y)\,dy\,dl \\ &= \int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} 2l\left(-\frac{1}{2\varepsilon}\frac{d}{dt}\left[(1-\tilde{v}_\varepsilon(l))^2\right]\right)\left[\frac{1}{2l}\int_{\{d_\tau(y)\le l\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(x)\,dx\right] dl.\end{aligned}$$

Recalling the notation from Proposition 4.9 and the fact that $\omega(x_n) \le \|\omega\|_{L^\infty}$, we have

$$\begin{aligned}\frac{1}{2l}\int_{\{d_\tau(y)\le l\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(x)\,dx &\le \sup_{t\le\varepsilon^2+\sqrt{\varepsilon}}\left(\frac{1}{|I(t_n,t)|}\int_{I(t_n,t)}\int_{Q(x_n,r_n)\cap T_{x_n}(l)} \omega(x)d\mathcal{H}^{N-1}dl\right) \\ &\le \int_{S_\tau\cap Q(x_n,r_n)} \omega(x)\,d\mathcal{H}^{N-1} + O(\tau)r_n^{N-1}.\end{aligned}$$

where by (4.58) we could use (4.37) in the last inequality. Therefore, by (4.65)

$$\begin{aligned}&-\int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} \frac{1}{2\varepsilon}\frac{d}{dt}\left[(1-\tilde{v}_\varepsilon(l))^2\right] \int_{\{d_\tau(y)\le l\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(x)\,dx\,dl \\ &\le 2\left(\int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} -\frac{1}{2\varepsilon}\frac{d}{dt}\left[(1-\tilde{v}_\varepsilon(l))^2\right] l\,dl\right)\left(\int_{S_\tau\cap Q(x_n,r_n)} \omega(x)\,d\mathcal{H}^{N-1} + O(\tau)r_n^{N-1}\right).\end{aligned} \tag{4.71}$$

A new integration by parts and by using (4.63) yields

$$\begin{aligned}
&\int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} -\frac{1}{2\varepsilon}\frac{d}{dt}\left[(1-\tilde{v}_\varepsilon(l))^2\right] l\, dl \\
&= \int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} \frac{1}{2\varepsilon}(1-\tilde{v}_\varepsilon(l))^2 dl - \frac{1}{2\varepsilon}(\varepsilon^2+\sqrt{\varepsilon})(1-\tilde{v}_\varepsilon(\varepsilon^2+\sqrt{\varepsilon}))^2 + \frac{\varepsilon^2}{2\varepsilon}(1-\tilde{v}_\varepsilon(\varepsilon^2))^2 \\
&\leq \int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} 2\varepsilon\, |\tilde{v}'_\varepsilon(l)|^2\, dl + \frac{\varepsilon^2}{2\varepsilon}(1-\tilde{v}_\varepsilon(\varepsilon^2))^2 = \frac{1}{2}\left(1-e^{-\frac{1}{\sqrt{\varepsilon}}}\right) + \frac{\varepsilon}{2}(1-\tilde{v}_\varepsilon(\varepsilon^2))^2 \\
&\leq \frac{1}{2} + \frac{1}{2}\varepsilon,
\end{aligned}$$

which, together with (4.71) and (4.34), gives

$$\begin{aligned}
&-\int_{\varepsilon^2}^{\varepsilon^2+\sqrt{\varepsilon}} \frac{1}{2\varepsilon}\frac{d}{dt}\left[(1-\tilde{v}_\varepsilon(l))^2\right]\int_{\{d_\tau(y)\leq l\}\cap L_n(\varepsilon^2+\sqrt{\varepsilon})} \omega(x)\, dx\, dl \qquad (4.72)\\
&\leq \int_{S_\tau\cap Q(x_n,r_n)} \omega(x)\, d\mathcal{H}^{N-1} + O(\tau)r_n^{N-1} + \varepsilon\, \|\omega\|_{L^\infty}\, \mathcal{H}^{N-1}(S_\tau\cap Q(x_n,r_n)) + \varepsilon O(\tau) r_n^{N-1} \\
&\leq \int_{S_\tau\cap Q(x_n,r_n)} \omega(x)\, d\mathcal{H}^{N-1} + O(\tau)r_n^{N-1} + O(\varepsilon)O(\tau)r_n^{N-1}.
\end{aligned}$$

Hence, in view of (4.69), (4.70), (4.72), and since $A_\omega(\varepsilon^2)\geq 0$, we obtain that

$$\begin{aligned}
&\int_{L_n(\varepsilon^2+\sqrt{\varepsilon})} \left[\varepsilon\, |\nabla v_{\tau,\varepsilon}|^2 + \frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right]\omega\, dx \\
&\qquad\leq \int_{S_\tau\cap Q(x_n,r_n)} \omega(x) d\mathcal{H}^{N-1} + O(\tau)r_n^{N-1} + O(\varepsilon)O(\tau)r_n^{N-1} + O(\varepsilon)r_n^{N-1}. \qquad (4.73)
\end{aligned}$$

Next we define

$$L_0 := (S_u\setminus F_\tau)\cup\left(\bigcup_{n=1}^{M_\tau}\partial Q_n\cap\overline{U}_n\right) \qquad \text{and} \qquad L_0(\varepsilon) := \left[(S_u\setminus F_\tau)\cup\left(\bigcup_{n=1}^{M_\tau}\partial Q_n\cap\overline{U_n}\right)\right]_\varepsilon.$$

Since $\omega\in L^\infty(\Omega)$, we have

$$\begin{aligned}
&\int_{L_0(\varepsilon^2+\sqrt{\varepsilon})} \left[\varepsilon\, |\nabla v_{\tau,\varepsilon}|^2 + \frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right]\omega\, dx \\
&\qquad\qquad\leq \|\omega\|_{L^\infty} \int_{L_0(\varepsilon^2+\sqrt{\varepsilon})} \left[\varepsilon\, |\nabla v_{\tau,\varepsilon}|^2 + \frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right] dx,
\end{aligned}$$

and we note that the term

$$\int_{L_0(\varepsilon^2+\sqrt{\varepsilon})} \left[\varepsilon\, |\nabla v_{\tau,\varepsilon}|^2 + \frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right] dx$$

is the recovery sequence constructed in [8], page 1034, Added in Proof. Therefore, recalling that by assumption that $u\in SBV_\omega(\Omega)\cap L^\infty(\Omega)\subset SBV(\Omega)\cap L^\infty(\Omega)$ and invoking Proposition 5.1 and 5.3 in [8] and calculation within, we conclude that

$$\limsup_{\varepsilon\to 0}\int_{L_0(\varepsilon^2+\sqrt{\varepsilon})} \left[\varepsilon\, |\nabla v_{\tau,\varepsilon}|^2 + \frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right] dx \leq \limsup_{\varepsilon\to 0}\frac{\mathcal{H}^{N-1}\left(x\in\Omega:\ \operatorname{dist}(x,L_0)<\varepsilon\right)}{2\varepsilon}$$

$$\leq \mathcal{H}^{N-1}(L_0).$$

Thus,

$$\begin{aligned}\int_{L_0(\varepsilon^2+\sqrt{\varepsilon})}\left[\varepsilon\left|\nabla v_{\tau,\varepsilon}\right|^2+\frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right]\omega\,dx &\leq \|\omega\|_{L^\infty}\left(\mathcal{H}^{N-1}(L_0)+O(\varepsilon)\right)\\ &\leq O(\tau)+O(\varepsilon).\end{aligned} \tag{4.74}$$

Furthermore, by (4.61)

$$\int_{\Omega\setminus(S_{\bar{u}_\tau})_{\varepsilon^2+\sqrt{\varepsilon}}}\left[\varepsilon\left|\nabla v_{\tau,\varepsilon}\right|^2+\frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right]\omega\,dx \leq \frac{1}{4\varepsilon}e^{-\frac{1}{\sqrt{\varepsilon}}}\|\omega\|_{L^\infty}\mathcal{L}^N(\Omega)\leq O(\varepsilon), \tag{4.75}$$

where in the last inequality we used (4.64).

Since cubes in $\mathcal{T}_\tau$ are pairwise disjoint, in view of (4.73), (4.74), and (4.75) we have that

$$\begin{aligned}&\int_\Omega\left[\varepsilon\left|\nabla v_{\tau,\varepsilon}\right|^2+\frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right]\omega\,dx\\ =&\int_{(S_{\bar{u}_\tau})_{\varepsilon^2+\sqrt{\varepsilon}}}\left[\varepsilon\left|\nabla v_{\tau,\varepsilon}\right|^2+\frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right]\omega\,dx\\ &+\int_{\Omega\setminus(S_{\bar{u}_\tau})_{\varepsilon^2+\sqrt{\varepsilon}}}\left[\varepsilon\left|\nabla v_{\tau,\varepsilon}\right|^2+\frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right]\omega\,dx\\ \leq&\int_{L_0(\varepsilon^2+\sqrt{\varepsilon})}\left[\varepsilon\left|\nabla v_{\tau,\varepsilon}\right|^2+\frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right]\omega\,dx\\ &+\sum_{n=1}^{M_\tau}\int_{L_n(\varepsilon^2+\sqrt{\varepsilon})}\left[\varepsilon\left|\nabla v_{\tau,\varepsilon}\right|^2+\frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon})^2\right]\omega\,dx+O(\varepsilon)\\ \leq& O(\varepsilon)+O(\tau)+\sum_{n=1}^{M_\tau}\left(\int_{S_\tau\cap Q(x_n,r_n)}\omega(x)d\mathcal{H}^{N-1}+\left[O(\varepsilon)+O(\tau)+O(\varepsilon)O(\tau)\right]r_n^{N-1}\right)\\ \leq&\int_{\bigcup_{n=1}^{M_\tau}(S_\tau\cap Q(x_n,r_n))}\omega(x)\,d\mathcal{H}^{N-1}+\left[O(\varepsilon)+O(\tau)+O(\varepsilon)O(\tau)\right]\sum_{n=1}^{M_\tau}r_n^{N-1}+O(\tau)+O(\varepsilon)\\ \leq&\int_{S_u}\omega(x)\,d\mathcal{H}^{N-1}+O(\varepsilon)+O(\tau)+O(\varepsilon)O(\tau),\end{aligned}$$

where in the last inequality we used (4.36), and this concludes the proof of (4.68). Hence, also in view of (4.67) and (4.68), for each $\tau>0$, we may choose $\varepsilon(\tau)$ such that

$$\int_\Omega\left|\nabla u_{\tau,\varepsilon(\tau)}\right|^2 v_{\tau,\varepsilon(\tau)}^2\,\omega\,dx\leq\int_\Omega\left|\nabla u\right|^2\omega\,dx+O(\tau),$$

and

$$\int_\Omega\left[\varepsilon\left|\nabla v_{\tau,\varepsilon(\tau)}\right|^2+\frac{1}{4\varepsilon}(1-v_{\tau,\varepsilon(\tau)})^2\right]\omega\,dx\leq\int_{S_u}\omega(x)d\mathcal{H}^{N-1}+O(\tau),$$

and we thus constructed a recovery sequence $\{(u_\tau,v_\tau)\}_{\tau>0}$ given by

$$u_\tau:=u_{\tau,\varepsilon(\tau)}\text{ and }v_\tau:=v_{\tau,\varepsilon(\tau)}$$

which satisfies (4.45) and by (4.60) and (4.66) we have

$$\left\|u_{\tau,\varepsilon(\tau)} - u\right\|_{L^1_\omega} < \tau \text{ and } \left\|v_{\tau,\varepsilon(\tau)} - v\right\|_{L^1} < \tau.$$

Hence, we proved Proposition 4.12. □

Proof of Theorem 4.10. The lim inf inequality follows from Lemma 4.11. On the other hand, for any given $u \in GSBV_\omega$ such that $E_\omega(u) < \infty$, we have, by Lebesgue Monotone Convergence Theorem,

$$E_\omega(u) = \lim_{K\to\infty} E_\omega(K \wedge u \vee -K),$$

and a diagonal argument, together with Proposition 4.12, concludes the proof. □

4.3. **The Case** $\omega \in \mathcal{W}(\Omega) \cap SBV(\Omega)$**.**

Consider the functionals

$$E_{\omega,\varepsilon}(u,v) := \int_\Omega v^2 |\nabla u|^2 \omega\, dx + \int_\Omega \left[\varepsilon |\nabla v|^2 + \frac{1}{4\varepsilon}(v-1)^2\right] \omega\, dx,$$

for $(u,v) \in W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega)$, and

$$E_\omega(u) := \int_\Omega |\nabla u|^2 \omega\, dx + \int_{S_u} \omega^-(x)\, d\mathcal{H}^{N-1}$$

defined for $u \in GSBV_\omega(\Omega)$.

Theorem 4.13. *Let $\omega \in \mathcal{W}(\Omega) \cap SBV(\Omega) \cap L^\infty(\Omega)$ be given. Let $\mathcal{E}_{\omega,\varepsilon}\colon L^1_\omega(\Omega) \times L^1(\Omega) \to [0,+\infty]$ be defined by*

$$\mathcal{E}_{\omega,\varepsilon}(u,v) := \begin{cases} E_{\omega,\varepsilon}(u,v) & \text{if } (u,v) \in W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega),\ 0 \le v \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

Then the functionals $\mathcal{E}_{\omega,\varepsilon}$ Γ-converge, with respect to the $L^1_\omega \times L^1$ topology, to the functional

$$\mathcal{E}_\omega(u,v) := \begin{cases} E_\omega(u) & \text{if } u \in GSBV_\omega(\Omega) \text{ and } v = 1 \text{ a.e.}, \\ +\infty & \text{otherwise.} \end{cases}$$

We start by proving the Γ-lim inf.

Proposition 4.14. (Γ-lim inf) *For $\omega \in \mathcal{W}(\Omega) \cap SBV(\Omega)$ and $u \in L^1_\omega(\Omega)$, let*

$$E^-_\omega(u) := \inf \Big\{ \liminf_{\varepsilon\to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : \\ (u_\varepsilon, v_\varepsilon) \in W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega), u_\varepsilon \to u, v_\varepsilon \to 1 \text{ in } L^1_\omega \times L^1,\ 0 \le v_\varepsilon \le 1 \Big\}.$$

We have

$$E^-_\omega(u) \ge E_\omega(u).$$

Proof. Without lose of generality, we assume that $E^-_\omega(u) < +\infty$. The proof of this lemma uses the same arguments of the proof of Proposition 4.11 until the beginning of (4.41), and we obtain

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2 v_\varepsilon^2 \omega\, dx \ge \int_\Omega |\nabla u|^2 \omega\, dx.$$

By applying Proposition 3.6 to the last inequality of (4.41), we have

$$\liminf_{\varepsilon\to 0}\int_A\left(\varepsilon\,|\nabla v_\varepsilon|^2+\frac{1}{4\varepsilon}(1-v_\varepsilon)^2\right)\omega\,dx\geq\int_{A_\nu}\sum_{t\in S_{u_{x,\nu}}\cap A_{x,\nu}}\omega^-_{x,\nu}(t)dx.$$

The rest of the proof follows that of Proposition 4.11 with $\omega^-_{x,\nu}$ in place of $\omega_{x,\nu}$ and taking into consideration of the fact that $\omega^-_{x,\nu}(t)=\omega^-(x+t\nu)$ (see Remark 3.109 in [4]). □

The next lemma is the SBV version of Lemma 4.7. We recall that $I(t_0,t):=(t_0-t,t_0+t)$.

Proposition 4.15. *let $\tau\in(0,1/4)$ be given, and let $\omega\in SBV(\Omega)\cap L^\infty(\Omega)$ be nonnegative. Then for $\mathcal{H}^{N-1}$ a.e. $x_0\in S_\omega$ a point of density one, there exists $r_0:=r_0(x_0)>0$ such that for each $0<r<r_0$ there exist $t_0\in(2\tau r,4\tau r)$ and $0<t_{0,r}=t_{0,r}(t_0,\tau,x_0,r)<t_0$ such that*

$$\begin{aligned}\sup_{0<t\leq t_{0,r}}\frac{1}{|I(t_0,t)|}\int_{I(t_0,t)}\int_{Q^\pm_{\nu_{S_\omega}}(x_0,r)\cap T_{x_0,\nu_{S_\omega}}(\pm l)}&\omega(x)d\mathcal{H}^{N-1}(x)dl\\&\leq\int_{S_\omega\cap Q^\pm_{\nu_{S_\omega}}(x_0,r)}\omega^\pm(x)\,d\mathcal{H}^{N-1}+O(\tau)r^{N-1}.\end{aligned}\tag{4.76}$$

Proof. For simplicity of notation, in what follows we abbreviate $Q_{\nu_{S_\omega}}(x_0,r)$ as $Q(x_0,r)$ and $T_{x_0,\nu_{S_\omega}}$ as T_{x_0}.

Since $\mathcal{H}^{N-1}(S_\omega)<\infty$, and so $\mu:=\mathcal{H}^{N-1}\lfloor S_\omega$ is a nonnegative radon measure, and since $\omega^-\in L^1(\Omega,\mu)$, it follows that for $\mathcal{H}^{N-1}$ a.e. $x_0\in S_\omega$

$$\lim_{r\to 0}\fint_{Q(x_0,r)\cap S_\omega}\left|\omega^-(x)-\omega^-(x_0)\right|d\mathcal{H}^{N-1}(x)=0.\tag{4.77}$$

Choose one such $x_0\in S_\omega$, also a point of density 1 of S_ω, and let $\tau>0$ be given. Select $r_1>0$ such that for all $0<r<r_1$,

$$\frac{1}{1+\tau^2}\leq\frac{\mathcal{H}^{N-1}(S_\omega\cap Q(x_0,r))}{r^{N-1}}\leq 1+\tau^2.\tag{4.78}$$

Let $0<r_2<r_1$ be such that, in view of (4.77),

$$\int_{Q(x_0,r)\cap S_\omega}\left|\omega^-(x)-\omega^-(x_0)\right|d\mathcal{H}^{N-1}\leq\tau^2r^{N-1}\tag{4.79}$$

for all $0<r<r_2$, and we observe that

$$\begin{aligned}\omega^-(x_0)\mathcal{H}^{N-1}\left[Q(x_0,r)\cap T_{x_0}(-t_0)\right]&=\omega^-(x_0)\,r^{N-1}\\&\leq(1+\tau^2)\,\omega^-(x_0)\mathcal{H}^{N-1}\left[Q(x_0,r)\cap S_\omega\right].\end{aligned}\tag{4.80}$$

Since by Theorem 2.4

$$\lim_{r\to 0}\fint_{Q^-(x_0,r)}\left|\omega(x)-\omega^-(x_0)\right|dx=0,$$

we may choose $0<r_3<r_2$ such that

$$\fint_{Q^-(x_0,r)}\left|\omega(x)-\omega^-(x_0)\right|dx\leq\tau^2,$$

for all $0 < r < r_3$, and so, since $3.5\tau r < r$, we have

$$\int_{2.5\tau r}^{3.5\tau r}\int_{Q^-(x_0,r)\cap T_{x_0}(-t)} \left|\omega(x)-\omega^-(x_0)\right| d\mathcal{H}^{N-1}(x)dt \le \int_{Q^-(x_0,r)} \left|\omega(x)-\omega^-(x_0)\right| dx \le \tau^2 r^N.$$

There exists a set $A \subset (2.5\tau r, 3.5\tau r)$ with positive 1 dimensional Lebesgue measure such that for every $t \in A$,

$$\int_{Q^-(x_0,r)\cap T_{x_0}(-t)} \left|\omega(x)-\omega^-(x_0)\right| d\mathcal{H}^{N-1}(x) \le \frac{\tau^2 r^N}{\tau r} = \tau r^{N-1}. \tag{4.81}$$

and choose $t_0 \in A$ a Lebesgue point for

$$l \in (-r/2, r/2) \longmapsto \int_{Q^-(x_0,r)\cap T_{x_0}(l)} \omega \, d\mathcal{H}^{N-1}(x)$$

so that

$$\lim_{t\to 0} \frac{1}{|I(t_0,t)|}\int_{I(t_0,t)}\int_{Q^-(x_0,r)\cap T_{x_0}(-l)} \omega(x) d\mathcal{H}^{N-1}(x)dl = \int_{Q^-(x_0,r)\cap T_{x_0}(-t_0)} \omega(x) d\mathcal{H}^{N-1}(x).$$

Hence, there exists $t_{0,r} > 0$, depending on t_0, τ, r, and x_0, such that $I(t_0, t_{0,r}) \subset (2.5\tau r, 3.5\tau r)$ and

$$\sup_{0<t\le t_{0,r}} \frac{1}{|I(t_0,t)|}\int_{I(t_0,t)}\int_{Q^-(x_0,r)\cap T_{x_0}(-l)} \omega(x) d\mathcal{H}^{N-1}(x)dl$$
$$\le \int_{Q^-(x_0,r)\cap T_{x_0}(-t_0)} \omega(x) d\mathcal{H}^{N-1} + \tau r^{N-1}. \tag{4.82}$$

In view of (4.82), (4.81), (4.80), and (4.79), in this order, we have that for every $0 < r < r_3$ there exist $t_0 \in (2.5\tau r, 3.5\tau r)$ and $0 < t_{0,r} < t_0$ such that

$$\begin{aligned}
&\sup_{0<t\le t_{0,r}} \frac{1}{|I(t_0,t)|}\int_{I(t_0,t)}\int_{Q^-(x_0,r)\cap T_{x_0}(-l)} \omega(x) d\mathcal{H}^{N-1}(x)dl \\
&\le \int_{Q^-(x_0,r)\cap T_{x_0}(-t_0)} \omega(x) d\mathcal{H}^{N-1} + \tau r^{N-1} \\
&\le \int_{Q^-(x_0,r)\cap T_{x_0}(-t_0)} \left|\omega(x)-\omega^-(x_0)\right| d\mathcal{H}^{N-1} \\
&\quad + \omega^-(x_0)\mathcal{H}^{N-1}\left[Q^-(x_0,r)\cap T_{x_0}(-t_0)\right] + \tau r^{N-1} \\
&\le O(\tau) r^{N-1} + (1+\tau^2)\,\omega^-(x_0)\mathcal{H}^{N-1}\left[Q(x_0,r)\cap S_\omega\right] \\
&\le O(\tau) r^{N-1} + (1+\tau^2)\int_{Q(x_0,r)\cap S_\omega} \omega^-(x) d\mathcal{H}^{N-1}.
\end{aligned}$$

Since $\omega \in L^\infty(\Omega)$, we have $\omega^- \in L^\infty(S_\omega)$ and thus, invoking (4.78),

$$\tau^2 \int_{Q(x_0,r)\cap S_\omega} \omega^-(x) d\mathcal{H}^{N-1} \le O(\tau)\left\|\omega\right\|_{L^\infty} \mathcal{H}^{N-1}\left[Q(x_0,r)\cap S_\omega\right] \le O(\tau) r^{N-1},$$

and we deduce the ω^- version of (4.76).

Similarly, we may refine t_0, $r_0 > 0$, and $0 < t_{0,r} < t_0$ such that

$$\sup_{0<t\le t_{0,r}} \frac{1}{|I(t_0,t)|}\int_{I(t_0,t)}\int_{Q^+(x_0,r)\cap T_{x_0}(l)} \omega(x) d\mathcal{H}^{N-1}dl \le \int_{Q(x_0,r)\cap S_\omega} \omega^+(x) d\mathcal{H}^{N-1} + O(\tau) r^{N-1}.$$

□

Proposition 4.16. *Let $\omega \in SBV(\Omega) \cap L^\infty(\Omega)$ be nonnegative and let $\tau \in (0, 1/4)$ be given. Then, there exist a set $S \subset S_\omega$ and a countable family of disjoint cubes $\mathcal{F} = \left\{Q_{\nu_{S_\omega}}(x_n, r_n)\right\}_{n=1}^\infty$, with $r_n < \tau$, such that the following hold:*

1. $\mathcal{H}^{N-1}(S_\omega \setminus S) < \tau$ *and* $S \subset \bigcup_{n=1}^\infty Q_{\nu_{S_\omega}}(x_n, r_n)$;
2. $$\operatorname{dist}(Q_{\nu_{S_\omega}}(x_n, r_n), Q_{\nu_{S_\omega}}(x_m, r_m)) > 0$$ *for* $n \neq m$;
3. $$\sum_{n=1}^\infty r_n^{N-1} \leq 4\mathcal{H}^{N-1}(S_\omega);$$
4. $S \cap Q_{\nu_{S_\omega}}(x_n, r_n) \subset R_{\tau/2, \nu_{S_\omega}}(x_n, r_n)$;
5. *for each* $n \in \mathbb{N}$, *there exists* $t_n \in (2.5\tau r_n, 3.5\tau r_n)$ *and* $0 < t_{x_n, r_n} < t_n$, *depending on* τ, r_n, *and* x_n, *such that* $$T_{x_n, \nu_{S_\omega}}(-t_n \pm t_{x_n, r_n}) \subset Q^-_{\nu_{S_\omega}}(x_n, r_n) \setminus R_{\tau/2, \nu_{S_\omega}}(x_n, r_n)$$ *and* $$\sup_{0 < t \leq t_{x_n, r_n}} \frac{1}{|I(t_n, t)|} \int_{I(t_n, t)} \int_{Q_{\nu_{S_\omega}}(x_n, r_n) \cap T_{x_n, \nu_\Gamma}(l)} \omega(x) d\mathcal{H}^{N-1} dl \leq \int_{S \cap Q_{\nu_{S_\omega}}(x_n, r_n)} \omega^-\, d\mathcal{H}^{N-1} + C\tau r_n^{N-1}, \quad (4.83)$$ *where* $I(t_n, t) := (-t_n - t, -t_n + t)$.

Proof. The proof of this proposition uses the same arguments of the proof of Proposition 4.8 and Proposition 4.9 where we apply Lemma 4.15 in place of Lemma 4.7. □

Proposition 4.17. (Γ-lim sup) *For $\omega \in \mathcal{W}(\Omega) \cap SBV(\Omega) \cap L^\infty(\Omega)$ and $u \in L^1_\omega(\Omega) \cap L^\infty(\Omega)$, let*

$$E^+_\omega(u) := \inf \Big\{ \limsup_{\varepsilon \to 0} E_{\omega,\varepsilon}(u_\varepsilon, v_\varepsilon) : (u_\varepsilon, v_\varepsilon) \in W^{1,2}_\omega(\Omega) \times W^{1,2}(\Omega), u_\varepsilon \to u \text{ in } L^1_\omega,\ v_\varepsilon \to 1 \text{ in } L^1,\ 0 \leq v_\varepsilon \leq 1 \Big\}.$$

We have

$$E^+_\omega(u) \leq E_\omega(u). \quad (4.84)$$

Proof. Step 1: Assume $\mathcal{H}^{N-1}((S_\omega \setminus S_u) \cup (S_u \setminus S_\omega)) = 0$, i.e., S_ω and S_u coincide $\mathcal{H}^{N-1}$ a.e.

If $E_\omega(u) = \infty$ then there is nothing to prove. If $E_\omega(u) < +\infty$ then by Lemma 2.10 we have that $u \in GSBV_\omega(\Omega)$ and $\mathcal{H}^{N-1}(S_u) < +\infty$.

Fix $\tau \in (0, 2/21)$. Applying Proposition 4.16 to ω we obtain a set $S_\tau \subset S_\omega$, a countable collection of mutually disjoint cubes $\mathcal{F}_\tau = \left\{Q_{\nu_{S_\omega}}(x_n, r_n)\right\}_{n=1}^\infty$, and corresponding

$$t_n \in (2.5\tau r_n, 3.5\tau r_n) \quad (4.85)$$

and t_{x_n,r_n} for which (4.83) holds. Extract a finite collection $\mathcal{T}_\tau = \left\{Q_{\nu_{S_\omega}}(x_n,r_n)\right\}_{n=1}^{M_\tau}$ from $\mathcal{F}_\tau$ with $M_\tau > 0$ large enough such that

$$\mathcal{H}^{N-1}\left[S_\tau \setminus \bigcup_{n=1}^{M_\tau} Q_{\nu_{S_\omega}}(x_n,r_n)\right] < \tau, \tag{4.86}$$

and we define

$$F_\tau := S_\tau \cap \left[\bigcup_{n=1}^{M_\tau} Q_{\nu_{S_\omega}}(x_n,r_n)\right]. \tag{4.87}$$

Let U_n be the part of $Q_{\nu_{S_\omega}}(x_n,r_n)$ which lies between $T_{x_n,\nu_{S_\omega}}(\pm t_n)$, U_n^+ be the part above $T_{x_n,\nu_{S_\omega}}(t_n)$, and U_n^- be the part below $T_{x_n,\nu_{S_\omega}}(-t_n)$.

We claim that if $x \in U_n$,

$$x + 2\,\mathrm{dist}(x, T_{x_n,\nu_{S_\omega}}(t_n))\nu_{S_\omega}(x_n) \in U_n^+. \tag{4.88}$$

Note that

$$\mathrm{dist}\left(x, T_{x_n,\nu_{S_\omega}}(t_n)\right) = t_n - \left(x - \mathbb{P}_{x_n,\nu_{S_\omega}}(x)\right)\nu_{S_\omega}(x_n),$$

and since $x \in U_n$, we have that

$$\left(x - \mathbb{P}_{x_n,\nu_{S_\omega}}(x)\right)\nu_{S_\omega}(x_n) \in (-t_n, t_n)$$

and

$$t_n \leq 2\,\mathrm{dist}\left(x, T_{x_n,\nu_{S_\omega}}(t_n)\right) + \left(x - \mathbb{P}_{x_n,\nu_{S_\omega}}(x)\right)\nu_{S_\omega}(x_n) \leq 3t_n \leq 10.5\tau r_n < \frac{1}{2}r_n.$$

Hence, following a similar computation in (4.51), we deduce (4.88).

Moreover, according to (4.85) and the definition of $R_{\tau/2,\nu_{S_\omega}}(x_n,r_n)$, we have that

$$\left(U_n^+ \cup U_n^-\right) \cap R_{\tau/2,\nu_{S_\omega}}(x_n,r_n) = \emptyset.$$

We define $\bar{u}_\tau$ as follows (see Figure 2):

$$\bar{u}_\tau(x) := \begin{cases} u(x) & \text{if } x \in U_n^+ \cup U_n^-, \\ u\left(x + 2\mathrm{dist}(x, T_{x_n,\nu_{S_\omega}}(t_n))\nu_{S_\omega}(x_n)\right) & \text{if } x \in U_n, \end{cases} \tag{4.89}$$

and

$$\bar{u}_\tau(x) := u(x) \text{ if } x \in \Omega \setminus \left(\bigcup_{n=1}^{M_\tau} Q_{\nu_{S_\omega}}(x_n,r_n)\right).$$

Note that the jump set of $\bar{u}_\tau$ is contained by

1. $$\bigcup_{n=1}^{M_\tau}\left[T_{x_n,\nu_{S_\omega}}(-t_n) \cap Q_{\nu_{S_\omega}}(x_n,r_n)\right];$$

2. $$\bigcup_{n=1}^{M_\tau} \partial\left(Q_{\nu_{S_\omega}}(x_n,r_n)\right) \cap \overline{U_n};$$

3. $S_u \setminus F_\tau$, where F_τ is defined in (4.87).

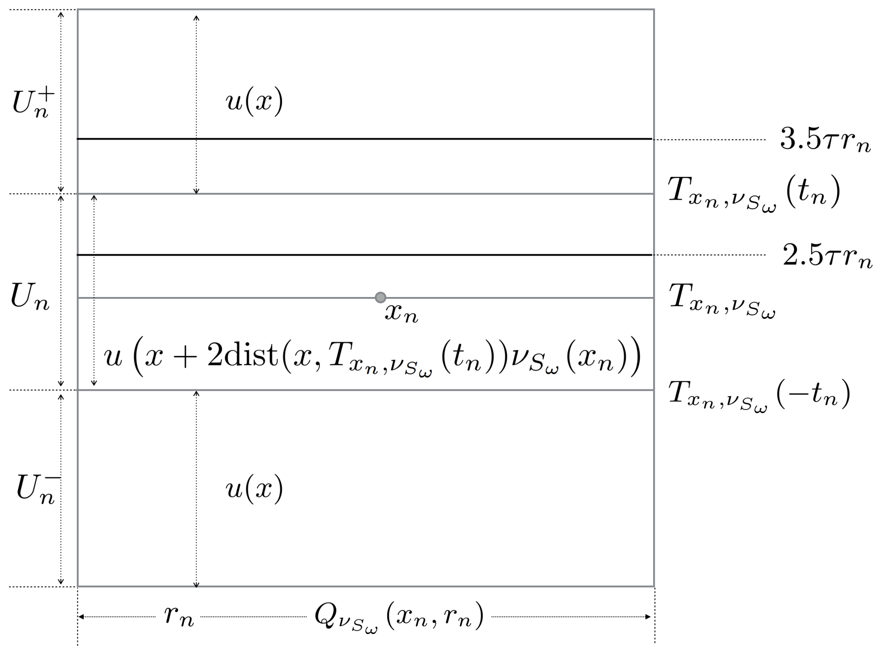

FIGURE 2. Construction of $\bar{u}_\tau(x)$ in (4.89)

The construction of $\{u_\varepsilon\}_{\varepsilon>0} \subset W^{1,2}_\omega(\Omega)$ and $\{v_\varepsilon\}_{\varepsilon>0} \subset W^{1,2}(\Omega)$ satisfying (4.84) is same as in the proof of Proposition 4.12, using (4.89) instead of (4.52), and at (4.71) we apply (4.83) instead of (4.37).

Step 2: Suppose that $\mathcal{H}^{N-1}((S_\omega \setminus S_u) \cup (S_u \setminus S_\omega)) > 0$. Note that we are only interested in the part $S_u \setminus S_\omega$ but not $S_\omega \setminus S_u$, because we only need to recover S_u.

We first apply Proposition 4.9 on S_u to obtain a countable family of disjoint cubes $\mathcal{F} = \left\{Q_{\nu_{S_u}}(x_n, r_n)\right\}_{n=1}^\infty$ such that (4.33)-(4.36) hold. Furthermore, extract a finite collection $\mathcal{T}_\tau$ from $\mathcal{F}$ such that (4.86) holds.

We define $\bar{u}_\tau$ inside each $Q_{\nu_{S_u}}(x_n, r_n) \in \mathcal{T}_\tau$ as follows (see Figure 3):

1. if $x_n \in \overline{S}_\omega$, we apply Proposition 4.15 to obtain item 5 in Proposition 4.16 for this $Q_{\nu_{S_u}}(x_n, r_n)$, and we define $\bar{u}_\tau$ in this cube in the way of (4.89);
2. if $x_n \in S_u \setminus \overline{S}_\omega$, we apply Lemma 4.7 to obtain item 5 in Proposition 4.9 for this $Q_{\nu_{S_u}}(x_n, r_n)$, and we define $\bar{u}_\tau$ in this cube in the way of (4.52).

For points x outside $\mathcal{T}_\tau$, we define $\bar{u}_\tau(x) := u(x)$.
Reasoning as in Proposition 4.12 and Proposition 4.17, we conclude (4.84). □

Proof of Theorem 4.13. The lim inf inequality follows from Proposition 4.14. On the other hand, for any given $u \in GSBV_\omega$ such that $E_\omega(u) < \infty$, we have, by Lebesgue Monotone Convergence

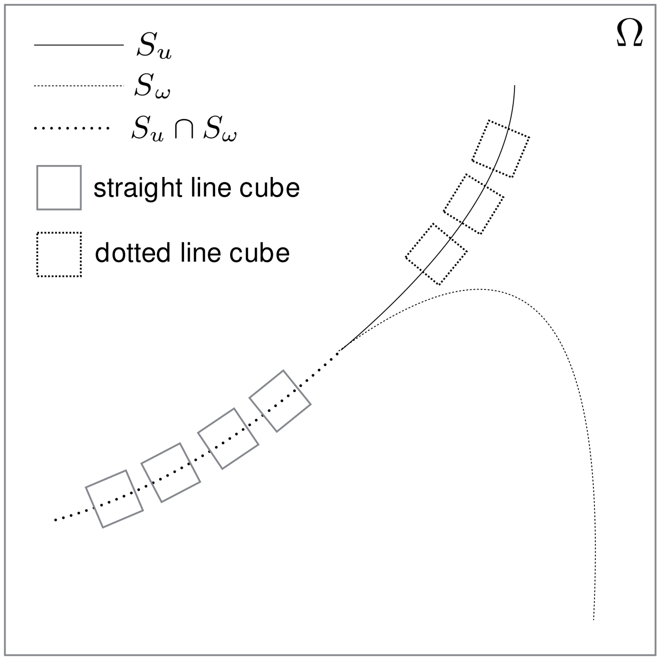

FIGURE 3. Applying (4.89) in dotted line cube and (4.52) in straight line cube.

Theorem,

$$E_\omega(u) = \lim_{K\to\infty} E_\omega(K \wedge u \vee -K),$$

and a diagonal argument, together with Proposition 4.17, yields the lim sup inequality for u. □

APPENDIX

Definition A.1 ([7], Definition 4.4.9)**.** *Let $\mathcal{X}$ be a metric space. We denote by $\mathcal{C}_\mathcal{X}$ the family of all nonempty closed subsets of X. Then*

$$d_\mathcal{H}(C, D) := \min\{1, h(C, D)\}, \quad C, D \in \mathcal{C}_\mathcal{X},$$

where

$$h(C, D) := \inf\{\delta \in [0, +\infty] : C \subset D_\delta \text{ and } D \subset C_\delta\},$$

is a metric on $\mathcal{C}_\mathcal{X}$, and is called the Hausdorff *distance between the set C and D (see* (2.4) *for definition of D_δ and C_δ).*

Consider $\mathcal{X}$ to be the interval $(0, 1)$ with the Euclidian distance. We remark that for two intervals $[a_1, b_1]$ and $[a_2, b_2]$ in $(0, 1)$,

$$d_\mathcal{H}([a_1, b_1], [a_2, b_2]) = \min\{1, \max\{|a_1 - a_2|, |b_1 - b_2|\}\}. \quad (A.1)$$

Indeed, the δ-neighborhood of $[a_1, b_1]$ is $[a_1 - \delta, b_1 + \delta]$, and contains $[a_2, b_2]$ if and only if

$$\delta \geq \max\{a_1 - a_2, b_2 - b_1\}.$$

Similarly, the δ-neighborhood of $[a_2, b_2]$ contains $[a_1, b_1]$ if and only if

$$\delta \geq \max\{a_2 - a_1, b_1 - b_2\},$$

and we conclude (A.1).

Lemma A.2. *Let $I_n := [a_n, b_n] \subset (-1,1)$. Then, up to the extraction of a subsequence,*

$$I_n \stackrel{\mathcal{H}}{\to} I_\infty \subset (-1,1),$$

where I_∞ is connected and closed in $(-1,1)$, and

$$\mathcal{L}^1(I_\infty) = \lim_{n\to\infty} \mathcal{L}^1(I_n).$$

Moreover, for arbitrary $K \subset\subset I_\infty$, K must be contained in I_n for n large enough.

Proof. Because $I_n \subset (-1,1)$, we have that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are bounded and so, up to the extraction of a subsequence, there exist

$$a_\infty := \lim_{n\to\infty} a_n \text{ and } b_\infty := \lim_{n\to\infty} b_n, \tag{A.2}$$

where $-1 \leq a_\infty \leq b_\infty \leq 1$. We define $I_\infty := [a_\infty, b_\infty]$ if $-1 < a_\infty \leq b_\infty < 1$, $I_\infty := (-1, b_\infty]$ if $a_\infty = -1$, and $I_\infty := [a_\infty, 1)$ if $b_\infty = 1$. Hence I_∞ is connected and closed in $(-1,1)$ (in the case in which $a_\infty = b_\infty = -1$, or $a_\infty = b_\infty = 1$, we have $I_\infty = \varnothing$ and it is still closed in $(-1,1)$).

Hence

$$\lim_{n\to\infty} d_{\mathcal{H}}(I_n, I_\infty) = \lim_{n\to\infty} \max\{|a_n - a_\infty|, |b_n - b_\infty|\} = 0,$$

and we have for $I_\infty \neq \varnothing$,

$$\mathcal{L}^1(I_\infty) = b_\infty - a_\infty = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \mathcal{L}^1(I_n),$$

as desired.

Next, if $K \subset\subset I_\infty$ then $K \subset (\alpha, \beta)$ for some α, β such that $a_\infty < \alpha < \beta < b_\infty$. By (A.2) choose N large enough such that for all $n \geq N$,

$$a_n < \alpha < \beta < b_n,$$

so that $K \subset I_n$ for all $n \geq N$. □

Lemma A.3. *Let $\{v_\varepsilon\}_{\varepsilon>0} \subset W^{1,2}(I)$ be such that $0 \leq v_\varepsilon \leq 1$, $v_\varepsilon \to 1$ in $L^1(I)$ and pointwise a.e., and*

$$\limsup_{\varepsilon\to 0} \int_I \left[\frac{\varepsilon}{2}|v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon - 1)^2\right] dx < \infty. \tag{A.3}$$

Then for arbitrary $0 < \eta < 1$ there exists an open set $H_\eta \subset I$ satisfying:

1. *the set $I \setminus H_\eta$ is a collection of finitely many points in I;*
2. *for every set K compactly contained in H_η, we have $K \subset B_\varepsilon^\eta$ for $\varepsilon > 0$ small enough, where*

$$B_\varepsilon^\eta := \left\{x \in I : v_\varepsilon^2(x) \geq \eta\right\}.$$

Proof. Choose a constant $M > 0$ such that

$$M \geq \limsup_{\varepsilon \to 0} \int_I \left[\frac{\varepsilon}{2} |v_\varepsilon'|^2 + \frac{1}{2\varepsilon}(v_\varepsilon - 1)^2 \right] dx \geq \limsup_{\varepsilon \to 0} \int_I |v_\varepsilon'| \, |1 - v_\varepsilon| \, dx = \limsup_{\varepsilon \to 0} \frac{1}{2} \int_I |c_\varepsilon'| \, dx,$$

where $c_\varepsilon(x) := (1 - v_\varepsilon(x))^2$. Note that by (A.3), $c_\varepsilon \to 0$ in $L^1(I)$. Fix σ, δ with

$$0 < \sigma < \delta < 1.$$

By the co-area formula we have, for $0 < \varepsilon < \varepsilon_0$ with ε_0 sufficiently small,

$$2M + 1 \geq \int_I |c_\varepsilon'(x)| \, dx = \int_{-\infty}^{\infty} \mathcal{H}^0(\{x : \, c_\varepsilon(x) = t\}) \, dt \geq \int_\sigma^\delta \mathcal{H}^0(\{x : \, c_\varepsilon(x) = t\}) \, dt.$$

Hence, for each $\varepsilon > 0$ there exist $\delta_\varepsilon \in (\sigma, \delta)$ such that

$$\frac{2M + 1}{\delta - \sigma} \geq \mathcal{H}^0(\{x : \, c_\varepsilon(x) = \delta_\varepsilon\}). \tag{A.4}$$

Define, for a fixed $r > 0$,

$$A_\varepsilon^r := \{x \in I : \, c_\varepsilon(x) \leq r\}.$$

Since $v_\varepsilon \in W^{1,2}(I)$, v_ε is continuous and so is c_ε, therefore $A_\varepsilon^{\delta_\varepsilon}$ is closed and has at most $(2M + 1)/(\delta - \sigma) + 1$ connected components because of (A.4) and in view of the continuity of c_ε. Note that the number $(2M + 1)/(\delta - \sigma)$ does not depend on $\varepsilon > 0$.

For $\varepsilon \in (0, \varepsilon_0)$ and $k \in \mathbb{N}$ depending only on $\delta - \sigma$ and M, we have

1. $A_\varepsilon^{\delta_\varepsilon} = \bigcup_{i=1}^k I_\varepsilon^i$, where each I_ε^i is a closed interval or $\varnothing$;
2. for all $i < j$, $\max\left\{x : \, x \in I_\varepsilon^i\right\} < \min\left\{x : \, x \in I_\varepsilon^j\right\}$.

By Lemma A.2, up to the extraction of a subsequence, for each $i \in \{1, 2, \ldots, k\}$ let I_0^i be the *Hausdorff* limit of the I_ε^i as $\varepsilon \to 0$, i.e., $I_\varepsilon^i \xrightarrow{\mathcal{H}} I_0^i$, with I_0^i is connected and closed in I, and for all $i < j$, $\max I_0^i \leq \min I_0^j$.

Set

$$T_\delta := \bigcup_{i=1}^k (I_0^i)^\circ \text{ and } T_{\delta,\varepsilon} := \bigcup_{i=1}^k (I_\varepsilon^i)^\circ, \tag{A.5}$$

where by $(\cdot)^\circ$ we denote the interior of a set. Since

$$I \setminus A_\varepsilon^{\delta_\varepsilon} \subset \{x \in I : \, c_\varepsilon(x) \geq \sigma\}$$

and $c_\varepsilon \to 0$ in $L^1(I)$, by Chebyshev's inequality we have

$$\lim_{\varepsilon \to 0} \mathcal{L}^1(T_{\delta,\varepsilon}) = \lim_{\varepsilon \to 0} \mathcal{L}^1(A_\varepsilon) = 2.$$

Moreover, since $T_{\delta,\varepsilon} \xrightarrow{\mathcal{H}} T_\delta$, by Lemma A.2 we have

$$\mathcal{L}^1(T_\delta) = \sum_{i=1}^k \mathcal{L}^1(I_0^i)^\circ = \sum_{i=1}^k \lim_{\varepsilon \to 0} \mathcal{L}^1(I_\varepsilon^i)^\circ = \lim_{\varepsilon \to 0} \sum_{i=1}^k \mathcal{L}^1(I_\varepsilon^i)^\circ = \lim_{\varepsilon \to 0} \mathcal{L}^1(T_{\delta,\varepsilon}) = 2.$$

Thus $|I \setminus T_\delta| = 0$. Moreover, since T_δ has at most k connected components, $I \setminus T_\delta$ is a finite collection of points in I.

Next, let $K \subset\subset T_\delta$ be a compact subset. We claim that K must be contained in $A_\varepsilon^{\delta_\varepsilon}$ for $\varepsilon > 0$ small enough. Recall I_0^i and I_ε^i from (A.5). Define $K_i := K \cap (I_0^i)^\circ$ for $i = 1, \ldots, k$. Then $K_i \subset\subset (I_o^i)^\circ$ for each i, and so by Lemma A.2 there exists $\varepsilon_i > 0$ such that for all $0 < \varepsilon < \varepsilon_i$, $K_i \subset I_\varepsilon^i$. Define

$$\varepsilon' := \min_{i \in \{1,\ldots,k\}} \{\varepsilon_i\}.$$

For $0 < \varepsilon < \varepsilon'$ we have $K_i \subset I_\varepsilon^i$, and so

$$K = \bigcup_{i=1}^k K_i \subset \bigcup_{i=1}^k I_\varepsilon^i = A_\varepsilon^{\delta_\varepsilon}.$$

Finally, given $\eta \in (0,1)$, set $\delta := \left(1 - \sqrt{\eta}\right)^2$ with $H_\eta := T_{(1-\sqrt{\eta})^2}$ and $B_\varepsilon^\eta := A_\varepsilon^{(1-\sqrt{\eta})^2}$, and properties 1 and 2 are satisfied. □

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References

[1] L. Ambrosio. A compactness theorem for a new class of functions of bounded variation. *Boll. Un. Mat. Ital. B (7)*, 3(4):857–881, 1989.

[2] L. Ambrosio. Variational problems in SBV and image segmentation. *Acta Appl. Math.*, 17(1):1–40, 1989.

[3] L. Ambrosio. Existence theory for a new class of variational problems. *Arch. Rational Mech. Anal.*, 111(4):291–322, 1990.

[4] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[5] L. Ambrosio and V. Magnani. Weak differentiability of BV functions on stratified groups. *Math. Z.*, 245(1):123–153, 2003.

[6] L. Ambrosio, M. Miranda, Jr., and D. Pallara. Special functions of bounded variation in doubling metric measure spaces. 14:1–45, 2004.

[7] L. Ambrosio and P. Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[8] L. Ambrosio and V. M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence. *Comm. Pure Appl. Math.*, 43(8):999–1036, 1990.

[9] H. Attouch. *Variational convergence for functions and operators*. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[10] G. Aubert and P. Kornprobst. *Mathematical problems in image processing*, volume 147 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2006. Partial differential equations and the calculus of variations, With a foreword by Olivier Faugeras.

[11] A. Baldi. Weighted BV functions. *Houston J. Math.*, 27(3):683–705, 2001.

[12] A. Baldi and B. Franchi. A Γ-convergence result for doubling metric measures and associated perimeters. *Calc. Var. Partial Differential Equations*, 16(3):283–298, 2003.

[13] A. Baldi and B. Franchi. Mumford-Shah-type functionals associated with doubling metric measures. *Proc. Roy. Soc. Edinburgh Sect. A*, 135(1):1–23, 2005.

[14] G. Bellettini, G. Bouchitté, and I. Fragalà. BV functions with respect to a measure and relaxation of metric integral functionals. *J. Convex Anal.*, 6(2):349–366, 1999.

[15] G. Bouchitte, G. Buttazzo, and P. Seppecher. Energies with respect to a measure and applications to low-dimensional structures. *Calc. Var. Partial Differential Equations*, 5(1):37–54, 1997.

[16] A. C. Bovik. *Handbook of image and video processing*. Academic press, 2010.

[17] A. Braides. *Γ-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[18] L. Capogna, D. Danielli, and N. Garofalo. The geometric Sobolev embedding for vector fields and the isoperimetric inequality. *Comm. Anal. Geom.*, 2(2):203–215, 1994.

[19] T. Chan, S. Esedoglu, F. Park, and A. Yip. Total variation image restoration: overview and recent developments. In *Handbook of mathematical models in computer vision*, pages 17–31. Springer, New York, 2006.

[20] Y. Chen, T. Pock, R. Ranftl, and H. Bischof. Revisiting loss-specific training of filter-based mrfs for image restoration. In *Pattern Recognition*, pages 271–281. Springer, 2013.

[21] Y. Chen, R. Ranftl, and T. Pock. Insights into analysis operator learning: From patch-based sparse models to higher order mrfs. *IEEE Transactions on Image Processing*, 23(3):1060–1072, March 2014.

[22] G. Dal Maso, J.-M. Morel, and S. Solimini. A variational method in image segmentation: existence and approximation results. *Acta Math.*, 168(1-2):89–151, 1992.

[23] G. David and S. Semmes. Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings. 122:101–111, 1990.

[24] E. De Giorgi, M. Carriero, and A. Leaci. Existence theorem for a minimum problem with free discontinuity set. *Arch. Rational Mech. Anal.*, 108(3):195–218, 1989.

[25] J. C. De Los Reyes, C.-B. Schönlieb, and T. Valkonen. The structure of optimal parameters for image restoration problems. *J. Math. Anal. Appl.*, 434(1):464–500, 2016.

[26] J. Domke. Generic methods for optimization-based modeling. In *AISTATS*, volume 22, pages 318–326, 2012.

[27] J. Domke. Learning graphical model parameters with approximate marginal inference. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 35(10):2454–2467, Oct 2013.

[28] I. Ekeland and R. Témam. *Convex analysis and variational problems*, volume 28 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, english edition, 1999. Translated from the French.

[29] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. CRC press, 2015.

[30] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[31] I. Fonseca and G. Leoni. *Modern methods in the calculus of variations: Sobolev spaces*. Springer Monographs in Mathematics. Unpublished yet, 2015.

[32] B. Franchi, R. Serapioni, and F. Serra Cassano. Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields. *Boll. Un. Mat. Ital. B (7)*, 11(1):83–117, 1997.

[33] B. Franchi, R. Serapioni, and F. Serra Cassano. Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups. *Comm. Anal. Geom.*, 11(5):909–944, 2003.

[34] G. Leoni. *A first course in Sobolev spaces*, volume 105 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[35] G. Leoni and R. Murray. Second-order Γ-limit for the Cahn-Hilliard functional. *Arch. Ration. Mech. Anal.*, 219(3):1383–1451, 2016.

[36] P. Lions, S. Osher, and L. Rudin. Denoising and deblurring images using constrained nonlinear partial differential equations. *Preprint, Cognitech, Inc*, pages 2800–28, 1993.

[37] P. Liu. The bi-level learning and uniqueness of optimal spatially dependent parameters for image reconstruction problem. *In preparation.*

[38] P. Liu. The weighted Γ-convergence of higher order singular perturbation models. *In preparation.*

[39] F. Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[40] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[41] D. Mumford and J. Shah. Boundary detection by minimizing functionals. In *IEEE Conference on Computer Vision and Pattern Recognition*, volume 17, pages 137–154. San Francisco, 1985.

[42] L. Rudin. Segmentation and restoration using local constraints. *Technical Repport*, 36, 1993.

[43] L. I. Rudin and S. Osher. Total variation based image restoration with free local constraints. In *Image Processing, 1994. Proceedings. ICIP-94., IEEE International Conference*, volume 1, pages 31–35 vol.1, Nov 1994.

[44] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Phys. D*, 60(1-4):259–268, 1992. Experimental mathematics: computational issues in nonlinear science (Los Alamos, NM, 1991).

[45] M. F. Tappen. Utilizing variational optimization to learn markov random fields. In *2007 IEEE Conference on Computer Vision and Pattern Recognition*, pages 1–8, June 2007.

[46] M. F. Tappen, C. Liu, E. H. Adelson, and W. T. Freeman. Learning gaussian conditional random fields for low-level vision. In *2007 IEEE Conference on Computer Vision and Pattern Recognition*, pages 1–8, June 2007.