THE NONRELATIVISTIC LIMIT OF RELATIVISTIC VLASOV-MAXWELL SYSTEM

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Abstract. We consider the one and one-half dimensional multi-species relativistic Vlasov-Maxwell system with non-decaying(in space) initial data. We prove its well-posedness and nonrelativistic limit as the speed of light $c \to \infty$. These results mainly rely on a delicate analysis of energy structure and application of estimates along the characteristic lines.
Keywords: non-decaying data; well-posedness; nonrelativistic limit.

1. Introduction

Consider the one and one-half dimensional relativistic Vlasov-Maxwell system:

$$\begin{cases} \partial_t f^\alpha + V_1^\alpha(p)\partial_x f^\alpha + e^\alpha\Big(E_1 + c^{-1}V_2^\alpha(p)B\Big)\partial_{p_1} f^\alpha + e^\alpha\Big(E_2 - c^{-1}V_1^\alpha(p)B\Big)\partial_{p_2} f^\alpha = 0,\\ \partial_x E_1 = 4\pi\rho, \quad \partial_t E_1 = -4\pi j_1,\\ \partial_t E_2 + c\partial_x B = -4\pi j_2, \quad \partial_t B + c\partial_x E_2 = 0,\\ \rho(t,x) = \sum_\alpha \Big(e^\alpha \int_{\mathbb{R}^2} f^\alpha(t,x,p)\mathrm{d}p\Big), \quad j(t,x) = \sum_\alpha \Big(e^\alpha \int_{\mathbb{R}^2} V^\alpha(p) f^\alpha(t,x,p)\mathrm{d}p\Big). \end{cases} \tag{1.1}$$

Here t is time, $x \in \mathbb{R}$ is position, $p \in \mathbb{R}^2$ is momentum, and $f^\alpha(t,x,p)$ is the number density in phase space of particles of charge e^α and mass m^α. The velocity of a particle is

$$V^\alpha(p) = \frac{p}{\sqrt{(m^\alpha)^2 + c^{-2}p^2}}, \tag{1.2}$$

where c is the speed of light. As defined in (1.1) ρ and j are respectively the charge and current densities. The induced electromagnetic field is given by

$$\vec{E} = (E_1, E_2, 0), \quad \vec{B} = (0,0,B). \tag{1.3}$$

As initial data for (1.1) we take

$$\begin{cases} f^\alpha(t,x,p) = f_0^\alpha(x,p),\\ E_1(0,x) = E_{1,0}(x),\\ E_2(0,x) = E_{2,0}(x),\\ B(0,x) = B_0(x) \end{cases} \tag{1.4}$$

to be given, where it is assumed that

$$\partial_x E_{1,0} = 4\pi \sum_\alpha \Big(e^\alpha \int_{\mathbb{R}^2} f_0^\alpha(x,p)\mathrm{d}p\Big). \tag{1.5}$$

The goal of this work is to study the behavior of f^α, E_1, E_2, B as $c \to \infty$. There are several papers in the literature [1, 2, 8, 17, 19, 20, 29] that study this limit for solutions of the two and three dimensional versions of (1.1) where $f^\alpha \to 0$ as $|x| \to \infty$. The goal here is to consider solutions that do not decay as $|x| \to \infty$ and hence have infinite charge and energy. As $c \to \infty$ the limiting problem is the Vlasov-Poisson system where the lack of spatial decay is a serious issue [3, 4, 5, 6, 7, 16, 21, 22, 23, 24, 27, 28]. Thus assumptions must be made on the large $|x|$ behavior of the initial data. As in [27] assume that for each α there are $F^\alpha : \mathbb{R}^2 \to [0,\infty)$ which is C^1 and positive constants R_0 and Q_0 such that $|x| \ge R_0$ implies

$$f_0^\alpha(x,p) - F^\alpha(p) = E_{2,0}(x) = B_0(x) = 0 \tag{1.6}$$

and $|p| \geq Q_0$ implies

$$f_0^\alpha(x,p) = F^\alpha(p) = 0. \tag{1.7}$$

Further assume that

$$\sum_\alpha \left(e^\alpha \int_{\mathbb{R}^2} F^\alpha(p)\mathrm{d}p \right) = 0, \tag{1.8}$$

$$\sum_\alpha \left(e^\alpha \int_{\mathbb{R}^2} F^\alpha(p)\frac{p}{m^\alpha}\mathrm{d}p \right) = 0, \tag{1.9}$$

and

$$\sum_\alpha \left(e^\alpha \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Big(f_0^\alpha(x,p) - F^\alpha(p) \Big) \mathrm{d}p\mathrm{d}x \right) = 0. \tag{1.10}$$

Then define

$$\rho_0(x) = \sum_\alpha \left(e^\alpha \int_{\mathbb{R}^2} f_0^\alpha(x,p)\mathrm{d}p \right), \tag{1.11}$$

$$j_0(x) = \sum_\alpha \left(e^\alpha \int_{\mathbb{R}^2} f_0^\alpha(x,p)V^\alpha(p)\mathrm{d}p \right), \tag{1.12}$$

and

$$E_{1,0}(x) = 2\pi \int_{-\infty}^x \rho_0(y)\mathrm{d}y - 2\pi \int_x^\infty \rho_0(y)\mathrm{d}y. \tag{1.13}$$

Note that (1.5) follows from (1.13) and for any $|x| \geq R_0$,

$$E_{1,0}(x) = 0. \tag{1.14}$$

Then the main results of this paper are the following two theorems:

Theorem 1.1. *Let $f_0^\alpha \geq 0$, $E_{2,0}$, and B_0 be C^2 and assume that (1.8) through (1.13) hold. Then there is a global C^1 solution (f^α, E_1, E_2, B) of (1.1) and (1.4). Moreover, for every $T > 0$ and $c \geq 1$ there exists $C_0 > 0$ (depending on T and initial data, but not on c) such that*

$$|f^\alpha(t,x,p)| + |E_1(t,x)| + |E_2(t,x)| + |B(t,x)| \leq C_0 \tag{1.15}$$

for every α and every $(t,x,p) \in [0,T] \times \mathbb{R} \times \mathbb{R}^2$.

To state the second theorem we must define $f^{\alpha,\infty}$ and E_1^∞ by

$$\begin{cases} \partial_t f^{\alpha,\infty} + V_1^{\alpha,\infty}(p)\partial_x f^{\alpha,\infty} + e^\alpha E_1^\infty \partial_{p_1} f^{\alpha,\infty} = 0, \\ \rho^\infty = \sum_\alpha e^\alpha \int_{\mathbb{R}^2} f^{\alpha,\infty} dp, \\ j^\infty = \sum_\alpha e^\alpha \int_{\mathbb{R}^2} f^{\alpha,\infty} \frac{p}{m^\alpha} dp, \\ E_1^\infty = 2\pi \int_\infty^x \rho^\infty(y)dy - 2\pi \int_x^\infty \rho^\infty(y)dy, \end{cases} \tag{1.16}$$

with

$$f^{\alpha,\infty}(0,x,p) = f_0^\alpha(x,p) \tag{1.17}$$

and

$$V^{\alpha,\infty}(p) = \frac{p}{m^\alpha}. \tag{1.18}$$

From [21] it is known that (1.16) and (1.17) possesses a global C^1 solution.

Theorem 1.2. *Assume that*

$$E_{2,0} = B_0 = 0. \tag{1.19}$$

Then, with the same assumptions as in Theorem 1.1, for every $T > 0$ and $c \geq 1$ there exists $C_0 > 0$ (depending on T and initial data, but not on c) such that

$$|f^\alpha(t,x,p) - f^{\alpha,\infty}(t,x,p)| + |E_1(t,x) - E_1^\infty(t,x)| + |E_2(t,x)| + |B(t,x)| \leq C_0 c^{-1} \tag{1.20}$$

for every α and every $(t,x,p) \in [0,T] \times \mathbb{R} \times \mathbb{R}^2$.

In the following, we will use $\|\cdot\|_{L^\infty}$ to represent L^∞ norm either in $(t,x,p) \in [0,T] \times \mathbb{R} \times \mathbb{R}^2$, $(t,x) \in [0,T] \times \mathbb{R}$, or $(x,p) \in \mathbb{R} \times \mathbb{R}^2$. We will use C_0 to indicate a universal constant, which may change from line to line and may depend on T and initial data, but not on c.

The global existence of smooth solutions of (1.1) was established in [10]. This was extended to two dimensions in [11] and [12], but remains open in three dimensions. However, it was shown in [13] that solutions of the three dimensional problem can break down only if particle speeds approach the speed of light. The global existence of smooth solutions for the Vlasov-Poisson system is better understood, see [15, 18, 25, 26]. Also see [9] for a general reference on mathematical kinetic theory.

It has been suggested [14] that when studying a nonrelativistic limit it is desirable to keep the speed of light constant and analyze the limiting behavior in some other parameter. While this framework is appealing, it was not clear what other parameter to use that would not complicate both the analysis and comparison with papers such as [1, 2, 8, 17, 19, 20, 29].

This paper is organized as follows: The proof of Theorem 1.1 is in section 2. Section 3 contains the proof of Theorem 1.2. The assumptions of Theorem 1.1 are in force in section 2 and the assumptions of Theorem 1.2 are in force in section 3.

Let us also define the characteristics of the Vlasov equation $(X^\alpha(s;t,x,p), P^\alpha(s;t,x,p))$ by

$$\frac{dX^\alpha}{ds} = V_1^\alpha(P^\alpha), \tag{1.21}$$
$$\frac{dP_1^\alpha}{ds} = E_1(s,X^\alpha) + c^{-1}V_2^\alpha(P^\alpha)B(s,X^\alpha), \tag{1.22}$$
$$\frac{dP_2^\alpha}{ds} = E_2(s,X^\alpha) - c^{-1}V_1^\alpha(P^\alpha)B(s,X^\alpha), \tag{1.23}$$

with $X^\alpha(t,t,x,p) = x$, and $P^\alpha(t,t,x,p) = p$.

2. Well-Posedness of the Relativistic Vlasov-Maxwell System

The global existence stated in Theorem 1.1 follows from the global existence result of [10] by a finite speed of propagation argument. Note that it was assumed in [10] that the initial data had compact support. So to construct the solution on $(t,x) \in [0,T] \times [-L,L]$ (with $L > R_0$) with initial data f_0^α, $E_{2,0}$, B_0 as in Theorem 1.1, let $\overline{f}_0^\alpha, \overline{E}_{2,0}, \overline{B}_0$ be smooth and satisfy

$$\overline{f}_0^\alpha = f_0^\alpha, \quad \overline{E}_{2,0} = E_{2,0} \quad \overline{B}_0 = B_0 \tag{2.1}$$

if $|x| \leq L + cT$,

$$\overline{f}_0^\alpha = \overline{E}_{2,0} = \overline{B}_0 = 0 \tag{2.2}$$

if $|x| \geq L + cT + 1$ and

$$\sum_\alpha e^\alpha \int_{\mathbb{R}^2} \overline{f}_0^\alpha dp = 0 \tag{2.3}$$

if $|x| \geq L + cT$. By [10] (1.1) possesses a global C^1 solution $\overline{f}^\alpha, \overline{E}_1, \overline{E}_2, \overline{B}$ with initial data $\overline{f}_0^\alpha, \overline{E}_{2,0}, \overline{B}_0$. Since increasing L and T will not change $\overline{f}^\alpha, \overline{E}_1, \overline{E}_2, \overline{B}$ on the set $\{(t,x) : 0 \leq t \text{ and } |x| \leq L + c(T-t)\}$, it follows that

$$(f^\alpha, E_1, E_2, B) = \lim_{L,T\to\infty} (\overline{f}^\alpha, \overline{E}_1, \overline{E}_2, \overline{B}) \tag{2.4}$$

is a global smooth solution of (1.1).

Note that by use of the characteristics of the Vlasov equation it follows that

$$0 \le f^\alpha \le \max f_0^\alpha \le C_0. \tag{2.5}$$

2.1. Estimate of E_2 and B.

Lemma 2.1. *We have*

$$\|E_2\|_{L^\infty} + \|B\|_{L^\infty} \le C_0 \tag{2.6}$$

Proof. We divide the proof into several steps:

Step 1: Relativistic energy estimate.
We multiply $\sqrt{(m^\alpha)^2 + c^{-2}p^2}$ on both sides of the Vlasov equation, sum up over α, and integrate over $p \in \mathbb{R}^2$ to obtain

$$\begin{aligned}\sum_\alpha \int_{\mathbb{R}^2} \partial_t f^\alpha \sqrt{(m^\alpha)^2 + c^{-2}p^2}\mathrm{d}p + \sum_\alpha \int_{\mathbb{R}^2} p_1 \partial_x f^\alpha \mathrm{d}p \\ + \sum_\alpha \int_{\mathbb{R}^2} e^\alpha \Big(E_1 \sqrt{(m^\alpha)^2 + c^{-2}p^2} + c^{-1}p_2 B\Big) \partial_{p_1} f^\alpha \mathrm{d}p \\ + \sum_\alpha \int_{\mathbb{R}^2} e^\alpha \Big(E_2 \sqrt{(m^\alpha)^2 + c^{-2}p^2} - c^{-1}p_1 B\Big) \partial_{p_2} f^\alpha \mathrm{d}p = 0.\end{aligned} \tag{2.7}$$

Integrating by parts in (2.7), we get

$$\begin{aligned}\partial_t\Big(\sum_\alpha \int_{\mathbb{R}^2} f^\alpha \sqrt{(m^\alpha)^2 + c^{-2}p^2}\mathrm{d}p\Big) + \partial_x\Big(\sum_\alpha \int_{\mathbb{R}^2} p_1 f^\alpha \mathrm{d}p\Big) \\ - \sum_\alpha \int_{\mathbb{R}^2} e^\alpha E_1 \frac{c^{-2}p_1}{\sqrt{(m^\alpha)^2 + c^{-2}p^2}} f^\alpha \mathrm{d}p - \sum_\alpha \int_{\mathbb{R}^2} e^\alpha E_2 \frac{c^{-2}p_2}{\sqrt{(m^\alpha)^2 + c^{-2}p^2}} f^\alpha \mathrm{d}p = 0,\end{aligned} \tag{2.8}$$

which further implies

$$\begin{aligned}\partial_t\Big(\sum_\alpha \int_{\mathbb{R}^2} f^\alpha \sqrt{(m^\alpha)^2 + c^{-2}p^2}\mathrm{d}p\Big) + \partial_x\Big(\sum_\alpha \int_{\mathbb{R}^2} p_1 f^\alpha \mathrm{d}p\Big) \\ - E_1 \sum_\alpha \int_{\mathbb{R}^2} e^\alpha c^{-2} f^\alpha V_1^\alpha(p)\mathrm{d}p - E_2 \sum_\alpha \int_{\mathbb{R}^2} e^\alpha c^{-2} f^\alpha V_2^\alpha(p)\mathrm{d}p = 0.\end{aligned} \tag{2.9}$$

Based on the definition of $j(t,x)$, from (2.9), we deduce that

$$\partial_t\Big(c^2 \sum_\alpha \int_{\mathbb{R}^2} f^\alpha \sqrt{(m^\alpha)^2 + c^{-2}p^2}\mathrm{d}p\Big) + \partial_x\Big(c^2 \sum_\alpha \int_{\mathbb{R}^2} f^\alpha p_1 \mathrm{d}p\Big) - (E_1 j_1 + E_2 j_2) = 0. \tag{2.10}$$

Multiplying E_1, E_2 and B on the corresponding Maxwell equations, we obtain

$$E_1 \partial_t E_1 = -4\pi E_1 j_1, \tag{2.11}$$

$$E_2 \partial_t E_2 + cE_2 \partial_x B = -4\pi E_2 j_2, \tag{2.12}$$

$$B \partial_t B + cB \partial_x E_2 = 0. \tag{2.13}$$

Summing them up yields

$$\frac{1}{2}\partial_t(E_1^2 + E_2^2 + B^2) + c\partial_x(E_2 B) = -4\pi\Big(E_1 j_1 + E_2 j_2\Big). \tag{2.14}$$

Substituting (2.14) into (2.10), we have

$$\begin{aligned}\partial_t\Big(c^2 \sum_\alpha \int_{\mathbb{R}^2} f^\alpha \sqrt{(m^\alpha)^2 + c^{-2}p^2}\mathrm{d}p\Big) + \partial_x\Big(c^2 \sum_\alpha \int_{\mathbb{R}^2} f^\alpha p_1 \mathrm{d}p\Big) \\ + \frac{1}{8\pi}\partial_t\Big(E_1^2 + E_2^2 + B^2\Big) + \frac{c}{4\pi}\partial_x(E_2 B) = 0.\end{aligned} \tag{2.15}$$

Based on (2.15), we define

$$\mathcal{E} = c^2\Big(\sum_\alpha \int_{\mathbb{R}^2} f^\alpha \sqrt{(m^\alpha)^2 + c^{-2}p^2}\mathrm{d}p\Big) + \frac{1}{8\pi}\Big(E_1^2 + E_2^2 + B^2\Big), \tag{2.16}$$

$$\mathcal{M} = c^2\Big(\sum_\alpha \int_{\mathbb{R}^2} f^\alpha p_1 \mathrm{d}p\Big) + \frac{c}{4\pi}(E_2 B), \tag{2.17}$$

which satisfies

$$\partial_t \mathcal{E} + \partial_x \mathcal{M} = 0. \tag{2.18}$$

In [10] (2.18) was the crucial ingredient. Since we need bounds independent of c here, we further define

$$\mathfrak{E} = \mathcal{E} - c^2\Big(\sum_\alpha \int_{\mathbb{R}^2} m^\alpha f^\alpha \mathrm{d}p\Big), \tag{2.19}$$

$$\mathfrak{M} = \mathcal{M} - c^2\Big(\sum_\alpha \int_{\mathbb{R}^2} m^\alpha f^\alpha V_1^\alpha(p)\mathrm{d}p\Big), \tag{2.20}$$

and use the Vlasov equation and integration by parts to verify that

$$\begin{aligned}
\partial_t \mathfrak{E} + \partial_x \mathfrak{M} &= (\partial_t \mathcal{E} + \partial_x \mathcal{M}) - c^2 \sum_\alpha \int_{\mathbb{R}^2} m^\alpha \Big(\partial_t f^\alpha + V_1^\alpha(p)\partial_x f^\alpha\Big)\mathrm{d}p \\
&= c^2 \sum_\alpha \int_{\mathbb{R}^2} m^\alpha \Big(e^\alpha \left(E_1 + c^{-1}V_2^\alpha(p)B\right)\partial_{p_1} f^\alpha + e^\alpha \left(E_2 - c^{-1}V_1^\alpha(p)B\right)\partial_{p_2} f^\alpha\Big)\mathrm{d}p \\
&= \ -c^2 \sum_\alpha \int_{\mathbb{R}^2} m^\alpha \Big(-e^\alpha \frac{c^{-3}Bp_1p_2}{(\sqrt{(m^\alpha)^2 + c^{-2}p^2})^3} f^\alpha + e^\alpha \frac{c^{-3}Bp_1p_2}{(\sqrt{(m^\alpha)^2 + c^{-2}p^2})^3} f^\alpha\Big)\mathrm{d}p \\
&= 0.
\end{aligned} \tag{2.21}$$

Step 2: Characteristic triangle.
We consider the point $(t,x) \in [0,T] \times \mathbb{R}$ in the time-space plane and the triangle bounded by $\tau = 0$, $y = x - c(t-\tau)$ and $y = x + c(t-\tau)$ for $\tau \in [0,t]$. Integrating (2.21) over this triangular region and applying the divergence theorem we find that

$$0 = \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} (\partial_\tau \mathfrak{E} + \partial_y \mathfrak{M})\mathrm{d}y\mathrm{d}\tau = L + M + N, \tag{2.22}$$

where

$$L = \int_{x-ct}^{x+ct} \Big(\langle \mathfrak{E}, \mathfrak{M}\rangle\Big|_{(0,y)} \cdot \langle -1, 0\rangle\Big)\mathrm{d}y, \tag{2.23}$$

$$M = \int_0^t \Big(\langle \mathfrak{E}, \mathfrak{M}\rangle\Big|_{(\tau, x+c(t-\tau))} \cdot \frac{\langle c, 1\rangle}{\sqrt{1+c^2}}\Big)\sqrt{1+c^2}\mathrm{d}\tau, \tag{2.24}$$

$$N = \int_0^t \Big(\langle \mathfrak{E}, \mathfrak{M}\rangle\Big|_{(\tau, x-c(t-\tau))} \cdot \frac{\langle c, -1\rangle}{\sqrt{1+c^2}}\Big)\sqrt{1+c^2}\mathrm{d}\tau. \tag{2.25}$$

Simplifying (2.22), we obtain

$$\int_{x-ct}^{x+ct} \mathfrak{E}(0,y)\mathrm{d}y = \int_0^t \Big(c\mathfrak{E} + \mathfrak{M}\Big)(\tau, x+c(t-\tau))\mathrm{d}\tau + \int_0^t \Big(c\mathfrak{E} - \mathfrak{M}\Big)(\tau, x-c(t-\tau))\mathrm{d}\tau, \tag{2.26}$$

which further implies

$$c^{-1}\int_{x-ct}^{x+ct} \mathfrak{E}(0,y)\mathrm{d}y = \int_0^t \Big(\mathfrak{E} + c^{-1}\mathfrak{M}\Big)(\tau, x+c(t-\tau))\mathrm{d}\tau + \int_0^t \Big(\mathfrak{E} - c^{-1}\mathfrak{M}\Big)(\tau, x-c(t-\tau))\mathrm{d}\tau. \tag{2.27}$$

We can simply denote (2.27) as $I = II + III$. Then we need to estimate I, II and III.

Step 3: Estimate of I.
Note that

$$\begin{aligned}\mathfrak{E} &= c^2\bigg(\sum_\alpha\int_{\mathbb{R}^2} f^\alpha\sqrt{(m^\alpha)^2+c^{-2}p^2}\mathrm{d}p\bigg)+\frac{1}{8\pi}\Big(E_1^2+E_2^2+B^2\Big)-c^2\bigg(\sum_\alpha\int_{\mathbb{R}^2} m^\alpha f^\alpha\mathrm{d}p\bigg) \\ &= c^2\bigg(\sum_\alpha\int_{\mathbb{R}^2} f^\alpha\left(\sqrt{(m^\alpha)^2+c^{-2}p^2}-m^\alpha\right)\mathrm{d}p\bigg)+\frac{1}{8\pi}\Big(E_1^2+E_2^2+B^2\Big) \\ &= c^2\bigg(\sum_\alpha\int_{\mathbb{R}^2} f^\alpha\left(\frac{c^{-2}p^2}{\sqrt{(m^\alpha)^2+c^{-2}p^2}+m^\alpha}\right)\mathrm{d}p\bigg)+\frac{1}{8\pi}\Big(E_1^2+E_2^2+B^2\Big) \\ &\leq \bigg(\sum_\alpha\frac{1}{2m^\alpha}\int_{\mathbb{R}^2}\left|f^\alpha\right|p^2\mathrm{d}p\bigg)+\frac{1}{8\pi}\Big(E_1^2+E_2^2+B^2\Big).\end{aligned} \tag{2.28}$$

Therefore, we have

$$\begin{aligned}I &= c^{-1}\int_{x-ct}^{x+ct}\mathfrak{E}(0,y) \\ &\leq c^{-1}\int_{x-ct}^{x+ct}\bigg(\sum_\alpha\frac{1}{2m^\alpha}\int_{\mathbb{R}^2}\left|f_0^\alpha\right|p^2\mathrm{d}p\bigg)\mathrm{d}y+c^{-1}\int_{x-ct}^{x+ct}\frac{1}{8\pi}\Big(E_{1,0}^2+E_{2,0}^2+B_0^2\Big)\mathrm{d}y \\ &\leq 2t\max_x\bigg\{\sum_\alpha\frac{1}{2m^\alpha}\int_{\mathbb{R}^2}\left|f_0^\alpha\right|p^2\mathrm{d}p\bigg\}+\frac{t}{4\pi}\max_x\Big\{E_{1,0}^2+E_{2,0}^2+B_0^2\Big\} \\ &\leq C_0,\end{aligned} \tag{2.29}$$

which is uniform in c.

Step 4: Estimate of II and III.
Note that

$$\begin{aligned}\mathfrak{E}\pm c^{-1}\mathfrak{M} &= \Bigg(c^2\bigg(\sum_\alpha\int_{\mathbb{R}^2} f^\alpha\sqrt{(m^\alpha)^2+c^{-2}p^2}\mathrm{d}p\bigg)+\frac{1}{8\pi}\Big(E_1^2+E_2^2+B^2\Big)-c^2\bigg(\sum_\alpha\int_{\mathbb{R}^2} m^\alpha f^\alpha\mathrm{d}p\bigg)\Bigg) \\ &\quad\pm c^{-1}\Bigg(c^2\bigg(\sum_\alpha\int_{\mathbb{R}^2} f^\alpha p_1\mathrm{d}p\bigg)+\frac{c}{4\pi}(E_2B)-c^2\bigg(\sum_\alpha\int_{\mathbb{R}^2} m^\alpha f^\alpha V_1^\alpha(p)\mathrm{d}p\bigg)\Bigg) \\ &= \bigg(c^2\sum_\alpha\int_{\mathbb{R}^2} f^\alpha\left(\sqrt{(m^\alpha)^2+c^{-2}p^2}-m^\alpha\pm c^{-1}(p_1-m^\alpha V_1^\alpha)\right)\mathrm{d}p\bigg) \\ &\quad+\frac{1}{8\pi}\Big(E_1^2+E_2^2+B^2\pm 2E_2B\Big) \\ &= \bigg(c^2\sum_\alpha\int_{\mathbb{R}^2} f^\alpha\left(\sqrt{(m^\alpha)^2+c^{-2}p^2}-m^\alpha\right)\left(1\pm c^{-1}V_1^\alpha\right)\mathrm{d}p\bigg)+\frac{1}{8\pi}\Big(E_1^2+(E_2\pm B)^2\Big).\end{aligned} \tag{2.30}$$

Since

$$E_1^2+(E_2\pm B)^2\geq 0, \tag{2.31}$$

$$\left|c^{-1}V_1^\alpha\right| = \left|\frac{c^{-1}p_1}{\sqrt{(m^\alpha)^2+c^{-2}p^2}}\right| \leq \left|\frac{c^{-1}p}{\sqrt{(m^\alpha)^2+c^{-2}p^2}}\right| < 1, \tag{2.32}$$

$$\sqrt{(m^\alpha)^2+c^{-2}p^2}-m^\alpha\geq 0, \tag{2.33}$$

and $f^\alpha\geq 0$, we deduce

$$\mathfrak{E}\pm c^{-1}\mathfrak{M}\geq c^2\sum_\alpha\int_{\mathbb{R}^2} f^\alpha\left(\sqrt{(m^\alpha)^2+c^{-2}p^2}-m^\alpha\right)\left(1\pm c^{-1}V_1^\alpha\right)\mathrm{d}p\geq 0. \tag{2.34}$$

Define

$$\Gamma^\alpha = \sqrt{(m^\alpha)^2 + c^{-2}p^2}. \tag{2.35}$$

Then based on (2.30), we have

$$\begin{aligned} II &= \int_0^t \left(\mathfrak{E} + c^{-1}\mathfrak{M}\right)(\tau, x + c(t-\tau))\mathrm{d}\tau \\ &\geq \int_0^t \left(c^2 \sum_\alpha \int_{\mathbb{R}^2} f^\alpha \left(\Gamma^\alpha - m^\alpha\right)\left(1 + c^{-1}V_1^\alpha\right) \mathrm{d}p\right)(\tau, x + c(t-\tau))\mathrm{d}\tau \geq 0, \end{aligned} \tag{2.36}$$

$$\begin{aligned} III &= \int_0^t \left(\mathfrak{E} - c^{-1}\mathfrak{M}\right)(\tau, x - c(t-\tau))\mathrm{d}\tau \\ &\geq \int_0^t \left(c^2 \sum_\alpha \int_{\mathbb{R}^2} f^\alpha \left(\Gamma^\alpha - m^\alpha\right)\left(1 - c^{-1}V_1^\alpha\right) \mathrm{d}p\right)(\tau, x - c(t-\tau))\mathrm{d}\tau \geq 0. \end{aligned} \tag{2.37}$$

Step 5: Synthesis.
Collecting the results in (2.29), (2.36) and (2.37) in (2.27), we obtain

$$0 \leq \int_0^t k_+(\tau, x + c(t-\tau))\mathrm{d}\tau \leq C_0, \tag{2.38}$$

$$0 \leq \int_0^t k_-(\tau, x - c(t-\tau))\mathrm{d}\tau \leq C_0, \tag{2.39}$$

where

$$k_\pm = c^2 \sum_\alpha \int_{\mathbb{R}^2} f^\alpha \left(\Gamma^\alpha - m^\alpha\right)\left(1 \pm c^{-1}V_1^\alpha\right) \mathrm{d}p. \tag{2.40}$$

This is the starting point for the following estimates. For convenience, we denote

$$\sigma_\pm^\alpha = c^2 \left(\Gamma^\alpha - m^\alpha\right)\left(1 \pm c^{-1}V_1^\alpha\right), \tag{2.41}$$

so that

$$k_\pm = \sum_\alpha \int_{\mathbb{R}^2} f^\alpha \sigma_\pm^\alpha \mathrm{d}p. \tag{2.42}$$

Step 6: Estimate of j_2.
Since

$$\sigma_\pm^\alpha = c^2 \left(\frac{c^{-2}p^2}{\Gamma^\alpha + m^\alpha}\right)\left(1 \pm c^{-1}V_1^\alpha\right) = \left(\frac{p^2}{\Gamma^\alpha + m^\alpha}\right)\left(1 \pm c^{-1}V_1^\alpha\right), \tag{2.43}$$

we directly estimate

$$\begin{aligned} \sigma_\pm^\alpha &= \left(\frac{p^2}{\sqrt{(m^\alpha)^2 + c^{-2}p^2} + m^\alpha}\right)\left(\frac{\sqrt{(m^\alpha)^2 + c^{-2}p^2} \pm c^{-1}p_1}{\sqrt{(m^\alpha)^2 + c^{-2}p^2}}\right) \\ &\geq \left(\frac{p^2}{\sqrt{(m^\alpha)^2 + c^{-2}p^2} + m^\alpha}\right)\left(\frac{\sqrt{(m^\alpha)^2 + c^{-2}p^2} - c^{-1}|p_1|}{\sqrt{(m^\alpha)^2 + c^{-2}p^2}}\right) \\ &= \left(\frac{p^2}{\sqrt{(m^\alpha)^2 + c^{-2}p^2} + m^\alpha}\right)\left(\frac{(m^\alpha)^2 + c^{-2}p_2^2}{\left(\sqrt{(m^\alpha)^2 + c^{-2}p^2}\right)\left(\sqrt{(m^\alpha)^2 + c^{-2}p^2} + c^{-1}|p_1|\right)}\right) \\ &\geq \frac{p^2\left((m^\alpha)^2 + c^{-2}p_2^2\right)}{4\left(\sqrt{(m^\alpha)^2 + c^{-2}p^2}\right)^3} \\ &= \frac{p^2\left((m^\alpha)^2 + c^{-2}p_2^2\right)}{4(\Gamma^\alpha)^3}. \end{aligned} \tag{2.44}$$

$\sigma_{\pm}^{\alpha}$ builds the bridge between j_2 and $k_{\pm}$. In detail, we can decompose

$$\begin{aligned} j_2 =& \sum_{\alpha}\left(e^{\alpha}\int_{\mathbb{R}^2}V_2^{\alpha}f^{\alpha}(t,x,p)\mathrm{d}p\right) \\ =& \sum_{\alpha}\left(e^{\alpha}\int_{|p|\geq c}V_2^{\alpha}f^{\alpha}(t,x,p)\mathrm{d}p\right)+\sum_{\alpha}\left(e^{\alpha}\int_{1\leq|p|\leq c}V_2^{\alpha}f^{\alpha}(t,x,p)\mathrm{d}p\right) \\ &+\sum_{\alpha}\left(e^{\alpha}\int_{|p|\leq 1}V_2^{\alpha}f^{\alpha}(t,x,p)\mathrm{d}p\right). \end{aligned} \tag{2.45}$$

Here we estimate these three terms separately:

- For $|p|\geq c$, we have

$$\begin{aligned} \sigma_{\pm}^{\alpha} &\geq \frac{p^2\left((m^{\alpha})^2+c^{-2}p_2^2\right)}{4(\Gamma^{\alpha})^3}\geq\frac{p^2 2m^{\alpha}c^{-1}|p_2|}{4(\Gamma^{\alpha})^3} \\ &= \frac{1}{2}m^{\alpha}c|V_2^{\alpha}|\frac{c^{-2}p^2}{(m^{\alpha})^2+c^{-2}p^2}\geq\frac{1}{2}m^{\alpha}c|V_2^{\alpha}|\frac{1}{(m^{\alpha})^2+1}\geq C_0|V_2^{\alpha}|. \end{aligned} \tag{2.46}$$

  Hence, we know

$$\left|\sum_{\alpha}\left(e^{\alpha}\int_{|p|\geq c}V_2^{\alpha}f^{\alpha}(t,x,p)\mathrm{d}p\right)\right|\leq\sum_{\alpha}\left(|e^{\alpha}|\int_{|p|\geq c}\frac{\sigma_{\pm}^{\alpha}}{C_0}f^{\alpha}(t,x,p)\mathrm{d}p\right)\leq C_0k_{\pm}. \tag{2.47}$$

- For $1\leq|p|\leq c$, we have

$$\sigma_{\pm}^{\alpha}\geq\frac{p^2(m^{\alpha})^2}{4(\Gamma^{\alpha})^3}\geq\frac{|p_2|}{4\Gamma^{\alpha}}\frac{|p|(m^{\alpha})^2}{(\Gamma^{\alpha})^2}\geq\frac{1}{4}|V_2^{\alpha}|\frac{(m^{\alpha})^2}{(m^{\alpha})^2+1}. \tag{2.48}$$

  Hence, we know

$$\begin{aligned} \left|\sum_{\alpha}\left(e^{\alpha}\int_{1\leq|p|\leq c}V_2^{\alpha}f^{\alpha}(t,x,p)\mathrm{d}p\right)\right| &\leq 4\sum_{\alpha}\left(|e^{\alpha}|\int_{1\leq|p|\leq c}\frac{(m^{\alpha})^2+1}{(m^{\alpha})^2}\sigma_{\pm}^{\alpha}f^{\alpha}(t,x,p)\mathrm{d}p\right) \\ &\leq 4\max_{\alpha}|e^{\alpha}|\frac{(m^{\alpha})^2+1}{(m^{\alpha})^2}k_{\pm}\leq C_0k_{\pm}. \end{aligned} \tag{2.49}$$

- For $|p|\leq 1$, we know

$$\begin{aligned} \left|\sum_{\alpha}\left(e^{\alpha}\int_{|p|\leq 1}V_2^{\alpha}f^{\alpha}(t,x,p)\mathrm{d}p\right)\right| &\leq \max_{\alpha}\frac{1}{m^{\alpha}}\sum_{\alpha}\left(|e^{\alpha}|\int_{|p|\leq 1}f^{\alpha}(t,x,p)\mathrm{d}p\right) \\ &\leq C_0\sum_{\alpha}\left(|e^{\alpha}|\int_{|p|\leq 1}\mathrm{d}p\right)\leq C_0. \end{aligned} \tag{2.50}$$

Collecting the results in (2.45), (2.47), (2.49) and (2.50), we have

$$|j_2|\leq C_0(1+k_{\pm}). \tag{2.51}$$

Step 7: Estimate of E_2 and B.
From Maxwell equations, we know

$$\partial_t E_2+c\partial_x B=-4\pi j_2, \tag{2.52}$$

$$\partial_t B+c\partial_x E_2=0. \tag{2.53}$$

Therefore, we have

$$\partial_t(E_2\pm B)\pm c\partial_x(E_2\pm B)=-4\pi j_2. \tag{2.54}$$

Hence, we have

$$(E_2\pm B)(t,x)=(E_2\pm B)(0,x\mp ct)-4\pi\int_0^t j_2(\tau,x\mp c(t-\tau))\mathrm{d}\tau, \tag{2.55}$$

which further implies

$$|(E_2 \pm B)(t,x)| \le |(E_2 \pm B)(0, x \mp ct)| + 4\pi \int_0^t |j_2|(\tau, x \mp c(t-\tau))\mathrm{d}\tau \tag{2.56}$$

$$\le C_0 + C_0 \int_0^t (1 + k_\pm)(\tau, x \mp c(t-\tau))\mathrm{d}\tau. \tag{2.57}$$

Based on (2.38) and (2.39), we conclude

$$|(E_2 \pm B)(t,x)| \le C_0, \tag{2.58}$$

which further implies

$$\|E_2\|_{L^\infty} \le C_0, \tag{2.59}$$

$$\|B\|_{L^\infty} \le C_0. \tag{2.60}$$

□

2.2. **Estimate of E_1 and f^α.**

Lemma 2.2. *We have*

$$\|E_1\|_{L^\infty} + \|f^\alpha\|_{L^\infty} \le C_0. \tag{2.61}$$

Also, for each $t > 0$, there exists $Q(t)$ (independent of c) such that $f^\alpha(s,x,p) = 0$ for any $|p| \ge Q(t)$ and $0 \le s \le t$.

Proof. We divide the proof into several steps:

Step 1: Truncated system.
Define a C^∞ cut-off function $\psi : \mathbb{R} \to [0,1]$ satisfying

$$\psi(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 0 & \text{if } x \ge 1. \end{cases} \tag{2.62}$$

For $L > R_0$, define

$$f_0^{\alpha,L}(x,p) = f_0^\alpha(x,p)\psi(|x| - L), \tag{2.63}$$

$$E_{2,0}^L(x) = E_{2,0}(x)\psi(|x| - L), \tag{2.64}$$

$$B_0^L(x) = B_0(x)\psi(|x| - L), \tag{2.65}$$

$$\rho_0^L(x) = \sum_\alpha e^\alpha \int_{\mathbb{R}^2} f_0^{\alpha,L}(x,p)\mathrm{d}p = \rho_0(x)\psi(|x| - L), \tag{2.66}$$

$$j_0^L(x) = \sum_\alpha e^\alpha \int_{\mathbb{R}^2} f_0^{\alpha,L}(x,p)V^\alpha(p)\mathrm{d}p = j_0(x)\psi(|x| - L), \tag{2.67}$$

$$E_{1,0}^L(x) = 2\pi \int_{-\infty}^x \rho_0^L(y)\mathrm{d}(y) - 2\pi \int_x^\infty \rho_0^L(y)\mathrm{d}(y). \tag{2.68}$$

We can directly verify the truncated data satisfy the compatibility condition

$$\partial_x E_{1,0}^L(x) = 4\pi\rho_0^L(x). \tag{2.69}$$

The truncated initial data $f_0^{\alpha,L}(x,p)$, $E_{1,0}^L(x)$, $E_{2,0}^L(x)$, $B_0^L(x)$, have compact support both in space and momenta, so with fixed light speed c, we can apply the main theorem in [10] to obtain a global smooth solution $f^{\alpha,L}$, E_1^L, E_2^L, B^L.

Step 2: Characteristics.
Define the maximum velocity support of $f^{\alpha,L}$ as

$$Q^L(t) = \sup\Big\{ |p| : f^{\alpha,L}(t,x,p) \ne 0 \text{ for some } s \in [0,t], x \in \mathbb{R} \Big\}, \tag{2.70}$$

and characteristics $X^{\alpha,L}(s;t,x,p)$, $P^{\alpha,L}(s;t,x,p)$ of the truncated system by

$$\left\{\begin{array}{rcl}\dfrac{\mathrm{d}X^{\alpha,L}}{\mathrm{d}s} &=& V_1^{\alpha}(P^{\alpha,L}),\\ \dfrac{\mathrm{d}P_1^{\alpha,L}}{\mathrm{d}s} &=& e^{\alpha}\Big(E_1^L(s,X^{\alpha,L})+c^{-1}V_2^{\alpha}(P^{\alpha,L})B^L(s,X^{\alpha,L})\Big),\\ \dfrac{\mathrm{d}P_2^{\alpha,L}}{\mathrm{d}s} &=& e^{\alpha}\Big(E_2^L(s,X^{\alpha,L})-c^{-1}V_1^{\alpha}(P^{\alpha,L})B^L(s,X^{\alpha,L})\Big),\\ X^{\alpha,L}(t;t,x,p) &=& x,\\ P_1^{\alpha,L}(t;t,x,p) &=& p_1,\\ P_2^{\alpha,L}(t;t,x,p) &=& p_2,\end{array}\right. \tag{2.71}$$

Since

$$\left|V^{\alpha}(P^{\alpha,L})\right|\leq c, \tag{2.72}$$

then for any $|x|\geq L+ct$, we have

$$f^{\alpha,L}(t,x,p)=E_1^L(t,x)=E_2^L(t,x)=B^L(t,x)=\rho^L(t,x)=j^L(t,x)=0, \tag{2.73}$$

which means they are still compactly supported in space for any t.

Step 3: Estimate of E_1^L.
Integrating the Vlasov equation over $p\in\mathbb{R}^2$ and summing up over α, we obtain

$$\partial_t\rho^L+\partial_x j^L=0. \tag{2.74}$$

Since ρ^L and j^L are of compact support, we can further integrate over $x\in\mathbb{R}$ to obtain (since $L>R_0$)

$$\int_{\mathbb{R}}\rho^L(t,x)\mathrm{d}x=\int_{\mathbb{R}}\rho_0^L(x)\mathrm{d}x=\int_{\mathbb{R}}\rho_0(x)\mathrm{d}x=0. \tag{2.75}$$

Hence, from the equation $\partial_x E_{1,0}^L(x)=4\pi\rho_0^L(x)$, we obtain

$$\begin{aligned}E_1^L(t,x)&=2\pi\int_{-\infty}^{x}\rho^L(t,y)\mathrm{d}y-2\pi\int_{x}^{\infty}\rho^L(t,y)\mathrm{d}y=4\pi\int_{-\infty}^{x}\rho^L(t,y)\mathrm{d}y\\&=4\pi\int_{\mathbb{R}}\rho^L(t,y)\mathbf{1}_{\{y\leq x\}}\mathrm{d}y,\end{aligned} \tag{2.76}$$

Therefore, we have

$$\begin{aligned}E_1^L(t,x)-E_{1,0}^L(x)&=4\pi\int_{\mathbb{R}}\Big(\rho^L(t,y)-\rho_0^L(y)\Big)\mathbf{1}_{\{y\leq x\}}\mathrm{d}y\\&=4\pi\sum_{\alpha}e^{\alpha}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\Big(f^{\alpha,L}(t,y,p)-f_0^{\alpha,L}(y,p)\Big)\mathbf{1}_{\{y\leq x\}}\mathrm{d}p\mathrm{d}y.\end{aligned} \tag{2.77}$$

Define the substitution $(y,p)\to(\tilde{y},\tilde{p})$ as

$$\left\{\begin{array}{rcl}\tilde{y}&=&X^{\alpha,L}(0;t,y,p),\\ \tilde{p}&=&P^{\alpha,L}(0;t,y,p).\end{array}\right. \tag{2.78}$$

It is a classical result that the Jacobian of this substitution is 1. Then we have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^2}f^{\alpha,L}(t,y,p)\mathbf{1}_{\{y\leq x\}}\mathrm{d}p\mathrm{d}y&=\int_{\mathbb{R}}\int_{\mathbb{R}^2}f_0^{\alpha,L}(\tilde{y},\tilde{p})\mathbf{1}_{\{y\leq x\}}\mathrm{d}\tilde{p}\mathrm{d}\tilde{y}\\&=\int_{\mathbb{R}}\int_{\mathbb{R}^2}f_0^{\alpha,L}(\tilde{y},\tilde{p})\mathbf{1}_{\{X^{\alpha,L}(t;0,\tilde{y},\tilde{p})\leq x\}}\mathrm{d}\tilde{p}\mathrm{d}\tilde{y}.\end{aligned} \tag{2.79}$$

Substituting (2.79) into (2.77) and rewriting the dummy variables in (2.79) as (y,p) we get

$$E_1^L(t,x)-E_{1,0}^L(x)=4\pi\sum_{\alpha}e^{\alpha}\int_{\mathbb{R}}\int_{\mathbb{R}^2}f_0^{\alpha,L}(y,p)\Big(\mathbf{1}_{\{X^{\alpha,L}(t;0,y,p)\leq x\}}-\mathbf{1}_{\{y\leq x\}}\Big)\mathrm{d}p\mathrm{d}y. \tag{2.80}$$

For (y,p) such that $f_0^{\alpha,L}(y,p)\neq 0$, we have

$$\left|P^{\alpha,L}(s;0,y,p)\right| \le Q^L(s), \tag{2.81}$$

which further implies

$$\left|V^\alpha\Big(P^{\alpha,L}(s;0,y,p)\Big)\right| \le \left|\frac{P^{\alpha,L}(s;0,y,p)}{m^\alpha}\right| \le \frac{Q^L(s)}{m^\alpha}. \tag{2.82}$$

Therefore, we get the bound for maximal distance between initial position and position at time t,

$$\left|X^{\alpha,L}(t;0,y,p)-y\right| = \left|\int_0^t V_1^\alpha\Big(P^{\alpha,L}(s;0,y,p)\Big)\mathrm{d}s\right| \le \frac{1}{m^\alpha}\int_0^t Q^L(s)\mathrm{d}s. \tag{2.83}$$

We decompose the integral over $y\in\mathbb{R}$ in (2.80) to get

$$E_1^L(t,x)-E_{1,0}^L(x) = I+II+III, \tag{2.84}$$

where I is the integral over $y \in \Big(x+\frac{1}{m^\alpha}\int_0^t Q^L(s)\mathrm{d}s, +\infty\Big)$, II is the integral over $y\in\Big(-\infty, x-\frac{1}{m^\alpha}\int_0^t Q^L(s)\mathrm{d}s\Big)$, and III is the integral over $y\in\Big[x-\frac{1}{m^\alpha}\int_0^t Q^L(s)\mathrm{d}s, x+\frac{1}{m^\alpha}\int_0^t Q^L(s)\mathrm{d}s\Big]$.

- In the integral I, we have $\mathbf{1}_{\{X^{\alpha,L}(t;0,y,p)\le x\}}-\mathbf{1}_{\{y\le x\}} = 1-1 = 0$, which implies $I=0$;
- In the integral II, we have $\mathbf{1}_{\{X^{\alpha,L}(t;0,y,p)\le x\}}-\mathbf{1}_{\{y\le x\}} = 0-0 = 0$, which implies $II=0$;

Therefore, we have

$$|E_1^L(t,x)-E_{1,0}^L(x)| = |III| \le 4\pi\sum_\alpha |e^\alpha|\max_{\alpha,x,p} f_0^{\alpha,L}\bigg(\frac{2}{m^\alpha}\int_0^t Q^L(s)\mathrm{d}s\bigg)(2Q_0)^2 \le C_0\int_0^t Q^L(s)\mathrm{d}s, \tag{2.85}$$

which means we may take the supremum to obtain

$$\sup_{[0,t]\times\mathbb{R}}|E_1^L(s,x)| \le C_0 + C_0\int_0^t Q^L(s)\mathrm{d}s. \tag{2.86}$$

On the other hand, based on the characteristic equation (2.71) and Lemma 2.1, we know for $t\in[0,T]$,

$$Q^L(t)-Q_0 \le C_0\int_0^t \sup_x\Big(\left|E_1^L(s,x)\right|+\left|E_2^L(s,x)\right|+\left|B^L(s,x)\right|\Big)\mathrm{d}s \le C_0 + C_0\sup_{[0,t]\times\mathbb{R}}|E_1^L(s,x)|. \tag{2.87}$$

Combining (2.86) and (2.87), we obtain

$$Q^L(t) \le C_0 + C_0\int_0^t Q^L(s)\mathrm{d}s. \tag{2.88}$$

By Gronwall's inequality, we have for $t\in[0,T]$,

$$0 < Q^L(t) \le C_0, \tag{2.89}$$

which further implies

$$\left\|E_1^L\right\|_{L^\infty} \le C_0, \tag{2.90}$$

where C_0 only depends on T and the initial data and is independent of L and c.

Step 4: Synthesis:
For any finite c, the domain of dependence of the point (t,x,p) is bounded in $[0,T]\times\mathbb{R}\times\mathbb{R}^2$, so we can always take L large enough that the solution $f^\alpha(t,x,p) = f^{\alpha,L}(t,x,p)$. Hence, we have shown

$$\|E_1\|_{L^\infty} \le C_0. \tag{2.91}$$

The existence of $Q(t)$ is guaranteed by the analysis of $Q^L(t)$. □

The proof of Theorem 1.1 is now complete.

3. The Limit as c Tends to Infinity

Regarding $f^{\alpha,\infty}, \rho^{\infty}, j^{\infty}, E_1^{\infty}$ as defined in (1.16), we have the following:

Theorem 3.1. *We have*

$$\|E_1^\infty\|_{L^\infty} + \|f^{\alpha,\infty}\|_{L^\infty} + \|\partial_x f^{\alpha,\infty}\|_{L^\infty} + \|\partial_{p_1} f^{\alpha,\infty}\|_{L^\infty} + \|\partial_{p_2} f^{\alpha,\infty}\|_{L^\infty} \le C_0. \tag{3.1}$$

Also, for each $t>0$, there exists $Q^\infty(t)$ such that $f^{\alpha,\infty}(s,x,p)=0$ for any $|p|\ge Q^\infty(t)$ and $0\le s\le t\le T$.

Proof. The bound on $\|E_1^\infty\|_{L^\infty} + \|f^{\alpha,\infty}\|_{L^\infty}$ and the construction of $Q^\infty(t)$ may be obtained by the methods of the previous section. To bound the derivatives define $R_x^\alpha = \partial_x f^{\alpha,\infty}$, $R_{p_1}^\alpha = \partial_{p_1} f^{\alpha,\infty}$ and $R_{p_2}^\alpha = \partial_{p_2} f^{\alpha,\infty}$. Then they satisfy

$$\partial_t R_x^\alpha + V_1^{\alpha,\infty}(p)\partial_x R_x^\alpha + e^\alpha E_1^\infty \partial_{p_1} R_x^\alpha = -e^\alpha \partial_x E_1^\infty R_{p_1}^\alpha, \tag{3.2}$$

$$\partial_t R_{p_1}^\alpha + V_1^{\alpha,\infty}(p)\partial_x R_{p_1}^\alpha + e^\alpha E_1^\infty \partial_{p_1} R_{p_1}^\alpha = -\partial_{p_1} V_1^{\alpha,\infty}(p) R_x^\alpha, \tag{3.3}$$

$$\partial_t R_{p_2}^\alpha + V_1^{\alpha,\infty}(p)\partial_x R_{p_2}^\alpha + e^\alpha E_1^\infty \partial_{p_1} R_{p_2}^\alpha = 0, \tag{3.4}$$

with initial data

$$R_x^\alpha(0,x,p) = \partial_x f_0^\alpha, \tag{3.5}$$

$$R_{p_1}^\alpha(0,x,p) = \partial_{p_1} f_0^\alpha, \tag{3.6}$$

$$R_{p_2}^\alpha(0,x,p) = \partial_{p_2} f_0^\alpha. \tag{3.7}$$

By the bound on $f^{\alpha,\infty}$ and using $Q^\infty(t)$ we have

$$|\partial_x E_1^\infty| = |4\pi\rho^\infty| \le C_0. \tag{3.8}$$

So integration along the characteristics of (1.16) yields

$$|R_x^\alpha(t,x,p)| \le C_0 + C_0\int_0^t \sup_{[0,s]\times\mathbb{R}\times\mathbb{R}^2} \left|R_{p_1}^\alpha\right| \mathrm{d}s, \tag{3.9}$$

$$\left|R_{p_1}^\alpha(t,x,p)\right| \le C_0 + C_0\int_0^t \sup_{[0,s]\times\mathbb{R}\times\mathbb{R}^2} |R_x^\alpha| \,\mathrm{d}s, \tag{3.10}$$

$$\left|R_{p_2}^\alpha(t,x,p)\right| \le C_0. \tag{3.11}$$

Combining (3.9) and (3.10) and yields

$$\begin{aligned}\sup_{[0,t]\times\mathbb{R}\times\mathbb{R}^2} |R_x^\alpha| &\le C_0 + C_0 t + C_0 t\int_0^t \sup_{[0,s]\times\mathbb{R}\times\mathbb{R}^2} |R_x^\alpha| \,\mathrm{d}s \\ &\le C_0 + C_0\int_0^t \sup_{[0,s]\times\mathbb{R}\times\mathbb{R}^2} |R_x^\alpha| \,\mathrm{d}s.\end{aligned} \tag{3.12}$$

By Gronwall's inequality, we have

$$\sup_{[0,t]\times\mathbb{R}\times\mathbb{R}^2} |R_x^\alpha| \le C_0. \tag{3.13}$$

Similarly, we can prove

$$\sup_{[0,t]\times\mathbb{R}\times\mathbb{R}^2} \left|R_{p_1}^\alpha\right| \le C_0. \tag{3.14}$$

□

3.1. **Estimate of E_2 and B.**

Lemma 3.2. *We have*

$$\|E_2\|_{L^\infty} + \|B\|_{L^\infty} \le C_0 c^{-1}. \tag{3.15}$$

Proof. We divide the proof into several steps:

Step 1: Setup.
Define

$$D_0 = R_0 + (1+T)\sup_{[0,T]}(Q(t)+Q^\infty(t)) \tag{3.16}$$

and note that $|x| > D_0$ and $f^\alpha(t,x,p)\neq 0$ implies

$$f^\alpha(t,x,p) = F^\alpha(X^\alpha(0;t,x,p)). \tag{3.17}$$

Denote $m_0 = \min_\alpha m^\alpha$ and note that $f^\alpha(t,x,p)\neq 0$ implies $|V^\alpha| \le \dfrac{D_0}{m_0}$. Let

$$\Lambda = \left\{(t,x) : t\in[0,T], |x| \ge D_0\left(1+\frac{t}{m_0}\right)\right\}. \tag{3.18}$$

Thus, if $(t,x)\in\Lambda$, with $s\in[0,t]$ and $f^\alpha(t,x,p)\neq 0$, then

$$\begin{aligned} |X^\alpha(s;t,x,p)| \ &\ge\ |x| - \int_s^t \left|V_1^\alpha\Big(P^\alpha(\tau;t,x,p)\Big)\right| \mathrm{d}\tau \ge |x| - \frac{D_0}{m_0}(t-s) \\ &\ge D_0\left(1+\frac{t}{m_0}\right) - \frac{D_0}{m_0}(t-s) = D_0\left(1+\frac{s}{m_0}\right), \end{aligned} \tag{3.19}$$

which implies $(s, X^\alpha(s;t,x,p))\in\Lambda$. Denote

$$\|\sigma(t)\|_\Lambda = \sup\left\{|\sigma(s,x)| : s\in[0,t], (s,x)\in\Lambda\right\}. \tag{3.20}$$

Step 2: Estimate of j in Λ.
Consider (t,x,p) with $(t,x)\in\Lambda$ and $f^\alpha(t,x,p)\neq 0$. Then

$$\frac{\mathrm{d}P_1^\alpha}{\mathrm{d}s} = e^\alpha\Big(E_1(s,X^\alpha) + c^{-1}V_2^\alpha(P^\alpha)B(s,X^\alpha)\Big), \tag{3.21}$$

$$\frac{\mathrm{d}P_2^\alpha}{\mathrm{d}s} = e^\alpha\Big(E_2(s,X^\alpha) - c^{-1}V_1^\alpha(P^\alpha)B(s,X^\alpha)\Big), \tag{3.22}$$

which implies

$$\left|\frac{\mathrm{d}P_1^\alpha}{\mathrm{d}s}\right| \le C_0\Big(\ \|E_1\|_\Lambda + \|B\|_\Lambda\Big), \tag{3.23}$$

$$\left|\frac{\mathrm{d}P_2^\alpha}{\mathrm{d}s}\right| \le C_0\Big(\ \|E_2\|_\Lambda + \|B\|_\Lambda\Big). \tag{3.24}$$

Therefore,

$$|P^\alpha(0;t,x,p) - p| \le C_0\Big(\ \|E_1\|_\Lambda + \|E_2\|_\Lambda + \|B\|_\Lambda\Big), \tag{3.25}$$

and

$$|f^\alpha(t,x,p) - F^\alpha(p)| \ =\ |F^\alpha(P^\alpha(0;t,x,p)) - F^\alpha(p)| \le C_0\Big(\ \|E_1\|_\Lambda + \|E_2\|_\Lambda + \|B\|_\Lambda\Big). \tag{3.26}$$

It follows from assumption (1.9) that

$$\begin{aligned} \left|\sum_\alpha\left(e^\alpha\int_{\mathbb{R}^2} F^\alpha V^\alpha \mathrm{d}p\right)\right| &= \left|\sum_\alpha\left(e^\alpha\int_{\mathbb{R}^2} F^\alpha (V^\alpha - p/m^\alpha)\mathrm{d}p\right)\right| \\ &\le \sum_\alpha |e^\alpha| \int_{\mathbb{R}^2} F^\alpha c^{-2}\frac{|p|^3}{(m^\alpha)^3}dp \le C_0 c^{-2}, \end{aligned} \tag{3.27}$$

so

$$|j(t,x)| = \left|\sum_\alpha\left(e^\alpha\int_{\mathbb{R}^2}f^\alpha V^\alpha \mathrm{d}p\right)\right| \leq C_0c^{-2}+\left|\sum_\alpha\left(e^\alpha\int_{\mathbb{R}^2}(f^\alpha-F^\alpha)V^\alpha \mathrm{d}p\right)\right| \tag{3.28}$$
$$\leq C_0c^{-2}+C_0\bigg(\left\|E_1\right\|_\Lambda+\left\|E_2\right\|_\Lambda+\left\|B\right\|_\Lambda\bigg).$$

Hence, we know

$$\left\|j(t)\right\|_\Lambda \leq C_0c^{-2}+C_0\bigg(\left\|E_1(t)\right\|_\Lambda+\left\|E_2(t)\right\|_\Lambda+\left\|B(t)\right\|_\Lambda\bigg). \tag{3.29}$$

Step 3: Estimate of E_1 in Λ.
Since E_1 satisfies $\partial_t E_1=-4\pi j_1$, we have

$$E_1(t,x)=E_1(0,x)-4\pi\int_0^t j_1(s,x)\mathrm{d}s. \tag{3.30}$$

For $(t,x)\in\Lambda$, we know $E_1(0,x)=0$. Thus,

$$|E_1(t,x)|\leq 4\pi\int_0^t |j_1(s,x)|\,\mathrm{d}s. \tag{3.31}$$

Therefore, we have

$$\left\|E_1(t)\right\|_\Lambda\leq C_0c^{-2}+C_0\int_0^t\bigg(\left\|E_1(s)\right\|_\Lambda+\left\|E_2(s)\right\|_\Lambda+\left\|B(s)\right\|_\Lambda\bigg)\mathrm{d}s. \tag{3.32}$$

Step 4: Estimate of E_2 and B in Λ.
Next, we consider E_2 and B for any $(t,x)\in[0,T]\times\mathbb{R}$. Based on (2.54), we can directly obtain

$$E_2 = -2\pi\int_0^t\bigg(j_2(\tau,x-c(t-\tau))+j_2(\tau,x+c(t-\tau))\bigg)\mathrm{d}\tau, \tag{3.33}$$

$$B = -2\pi\int_0^t\bigg(j_2(\tau,x-c(t-\tau))-j_2(\tau,x+c(t-\tau))\bigg)\mathrm{d}\tau. \tag{3.34}$$

Hence, in order to estimate E_2 and B, we need to bound $\left|\int_0^t j_2(\tau,x-c(t-\tau))\mathrm{d}\tau\right|$ and $\left|\int_0^t j_2(\tau,x+c(t-\tau))\mathrm{d}\tau\right|$.
By substitution, we need to bound

$$\int_0^t j_2(\tau,x-c(t-\tau))\mathrm{d}\tau=c^{-1}\int_{x-ct}^x j_2\left(t-\frac{x-y}{c},y\right)\mathrm{d}y, \tag{3.35}$$

and

$$\int_0^t j_2(\tau,x+c(t-\tau))\mathrm{d}\tau=c^{-1}\int_x^{x+ct} j_2\left(t-\frac{y-x}{c},y\right)\mathrm{d}y. \tag{3.36}$$

Note that $(s,y)\notin\Lambda\Rightarrow|y|<D_0(1+s/m_0)\leq D_0(1+T/m_0)=C_0$ so by (3.28) and Theorem 1.1

$$c^{-1}\left|\int_{x-ct}^x j_2\left(t-\frac{x-y}{c},y\right)\mathrm{d}y\right| \leq c^{-1}\int_{-C_0}^{C_0}\left\|j_2\right\|_{L^\infty}dy \tag{3.37}$$
$$+c^{-1}\int_{x-ct}^x C_0\bigg(c^{-2}+\left\|E_1(t-\frac{x-y}{c})\right\|_\Lambda+\left\|E_2(t-\frac{x-y}{c})\right\|_\Lambda+\left\|B(t-\frac{x-y}{c})\right\|_\Lambda\bigg)\mathrm{d}y$$
$$\leq C_0c^{-1}+C_0\int_0^t\bigg(\left\|E_1(s)\right\|_\Lambda+\left\|E_2(s)\right\|_\Lambda+\left\|B(s)\right\|_\Lambda\bigg)\mathrm{d}s,$$

and similarly

$$\left|c^{-1}\int_x^{x+ct} j_2\left(t-\frac{y-x}{c},y\right)\mathrm{d}y\right|\leq C_0c^{-1}+C_0\int_0^t\bigg(\left\|E_1(s)\right\|_\Lambda+\left\|E_2(s)\right\|_\Lambda+\left\|B(s)\right\|_\Lambda\bigg)\mathrm{d}s. \tag{3.38}$$

This implies

$$|E_2(t,x)| + |B(t,x)| \le C_0 c^{-1} + C_0 \int_0^t \Big(\|E_1(s)\|_\Lambda + \|E_2(s)\|_\Lambda + \|B(s)\|_\Lambda \Big) \mathrm{d}s. \tag{3.39}$$

We may restrict (3.39) to Λ to obtain

$$\|E_2(t)\|_\Lambda + \|B(t)\|_\Lambda \le C_0 c^{-1} + C_0 \int_0^t \Big(\|E_1(s)\|_\Lambda + \|E_2(s)\|_\Lambda + \|B(s)\|_\Lambda \Big) \mathrm{d}s. \tag{3.40}$$

Step 5: Synthesis.
In summary, we know from (3.32) and (3.40) that

$$\|E_1(t)\|_\Lambda + \|E_2(t)\|_\Lambda + \|B(t)\|_\Lambda \le C_0 c^{-1} + C_0 \int_0^t \Big(\|E_1(s)\|_\Lambda + \|E_2(s)\|_\Lambda + \|B(s)\|_\Lambda \Big) \mathrm{d}s. \tag{3.41}$$

By Gronwall's inequality, we have

$$\|E_1(t)\|_\Lambda + \|E_2(t)\|_\Lambda + \|B(t)\|_\Lambda \le C_0 c^{-1} \mathrm{e}^{C_0 t} \le C_0 c^{-1}. \tag{3.42}$$

Therefore, by (3.39) we have

$$\|E_2(t,x)\|_{L^\infty} + \|B(t,x)\|_{L^\infty} \le C_0 c^{-1}. \tag{3.43}$$

□

3.2. **Estimate of $E_1 - E_1^\infty$ and $f^\alpha - f^\alpha_\infty$.**

Lemma 3.3. *We have*

$$\|E_1 - E_1^\infty\|_{L^\infty} + \|f^\alpha - f^\alpha_\infty\|_{L^\infty} \le C_0 c^{-1}. \tag{3.44}$$

Proof. Since $\partial_t E_1 = -4\pi j_1$ and $\partial_t E_1^\infty = -4\pi j_1^\infty$, we have

$$\begin{aligned} |E_1 - E_1^\infty| &\le 4\pi \int_0^t |j_1(s,x) - j_1^\infty(s,x)|\, \mathrm{d}s \\ &\le C_0 \sum_\alpha \int_0^t \int_{\mathbb{R}^2} |h^\alpha|\, \mathrm{d}p\mathrm{d}s \end{aligned} \tag{3.45}$$

where $h^\alpha(t,x,p) = f^\alpha(t,x,p) - f^{\alpha,\infty}(t,x,p)$. Note that h^α satisfies the equation

$$\begin{aligned} &\partial_t h^\alpha + V_1^\alpha(p)\partial_x h^\alpha + e^\alpha \left(E_1 + c^{-1}V_2^\alpha(p)B\right)\partial_{p_1} h^\alpha + e^\alpha \left(E_2 - c^{-1}V_1^\alpha(p)B\right)\partial_{p_2} h^\alpha \\ &= (V_1^{\alpha,\infty}(p) - V_1^\alpha(p))\partial_x f^{\alpha,\infty} + e^\alpha \left(E_1^\infty - E_1 - c^{-1}V_2^\alpha(p)B\right)\partial_{p_1} f^{\alpha,\infty} - e^\alpha \left(E_2 - c^{-1}V_1^\alpha(p)B\right)\partial_{p_2} f^{\alpha,\infty} \end{aligned} \tag{3.46}$$

and

$$h^\alpha(0,x,p) = 0. \tag{3.47}$$

By Theorem 3.1 and by Lemma 3.2 we have

$$\left| \Big(V_1^{\alpha,\infty}(p) - V_1^\alpha(p) \Big) \partial_x f^{\alpha,\infty} \right| \le C_0 c^{-2}, \tag{3.48}$$

$$\left| e^\alpha \left(E_1^\infty - E_1 - c^{-1}V_2^\alpha(p)B\right)\partial_{p_1} f^{\alpha,\infty} \right| \le C_0 c^{-1} + \sup_{[0,t]\times\mathbb{R}} |E_1 - E_1^\infty|, \tag{3.49}$$

$$\left| e^\alpha \left(E_2 - c^{-1}V_1^\alpha(p)B\right)\partial_{p_2} f^{\alpha,\infty} \right| \le C_0 c^{-1}. \tag{3.50}$$

Then integrating along the characteristics of the Vlasov equation, we have

$$|h^\alpha(t,x,p)| \le C_0 c^{-1} + C_0 \int_0^t \sup_{[0,s]\times\mathbb{R}} |E_1 - E_1^\infty|\, \mathrm{d}s. \tag{3.51}$$

Combining (3.45) and (3.51), we have

$$\begin{aligned} |E_1 - E_1^\infty| &\le C_0 \sum_\alpha \int_0^t \int_{|p|<Q(s)+Q^\infty(s)} \sup_{[0,s]\times\mathbb{R}\times\mathbb{R}^2} |h^\alpha| dpds \\ &\le C_0 t (c^{-1} + \int_0^t \sup_{[0,s]\times\mathbb{R}} |E_1 - E_1^\infty|\, \mathrm{d}s). \end{aligned} \tag{3.52}$$

Taking the supremum yields

$$\sup_{[0,t]\times\mathbb{R}} |E_1 - E_1^\infty| \le C_0 T(c^{-1} + \int_0^t \sup_{[0,s]\times\mathbb{R}} |E_1 - E_1^\infty| \, \mathrm{d}s). \tag{3.53}$$

Using Gronwall's inequality, for $t \in [0,T]$ we obtain

$$\sup_{[0,t]\times\mathbb{R}} |E_1 - E_1^\infty| \le C_0 c^{-1}, \tag{3.54}$$

which is

$$\|E_1 - E_1^\infty\|_{L^\infty} \le C_0 c^{-1}. \tag{3.55}$$

By (3.51) this implies

$$\|f^\alpha - f_\infty^\alpha\|_{L^\infty} \le C_0 c^{-1}. \tag{3.56}$$

□

The proof of Theorem 1.2 is now complete.

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