

# On compactness of minimizing sequences subject to a linear differential constraint

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## Abstract

For  $\Omega \subset \mathbb{R}^N$  open (and possibly unbounded), we consider integral functionals of the form

$$F(u) := \int_{\Omega} f(x, u) dx,$$

defined on the subspace of  $L^p$  consisting of those vector fields  $u$  which satisfy  $\mathcal{A}u = 0$  on  $\Omega$  in the sense of distributions. Here,  $\mathcal{A}$  may be any linear differential operator of first order with constant coefficients satisfying Murat's condition of constant rank. The main results provide sharp conditions for the compactness of minimizing sequences with respect to the strong topology in  $L^p$ .

## 1 Introduction

We consider integral functionals of the form

$$F(u) := \int_{\Omega} f(x, u) dx, \quad u \in \mathcal{U}_{\mathcal{A}}, \quad (1.1)$$

with the class of admissible functions given by

$$\mathcal{U}_{\mathcal{A}} := \{u \in L^p(\Omega; \mathbb{R}^M) \mid \mathcal{A}u = 0 \text{ in } \Omega\}, \quad (1.2)$$

Here,  $1 < p < \infty$ ,  $\Omega \subset \mathbb{R}^N$  is open,  $\mathcal{A}$  is a linear first order differential operator as in Section 3, formally mapping  $u : \Omega \rightarrow \mathbb{R}^M$  onto  $\mathcal{A}u : \Omega \rightarrow \mathbb{R}^L$ , and the equation  $\mathcal{A}u = 0$  is understood in the sense of distributions<sup>1</sup>. Throughout, we assume that

$\mathcal{A}$  satisfies the condition of constant rank

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<sup>1</sup>i.e.,  $\int_{\Omega} u \cdot \mathcal{A}^* \varphi dx = 0$  for all  $\varphi \in C_c^\infty(\Omega, \mathbb{R}^L)$ , with  $\mathcal{A}^*$  as in Section 3

introduced by Murat [14], as specified in (3.1) in the next section. Such differential constraints arise naturally in a variety of physical models. In particular, both curl and divergence are admissible. Further examples are provided in [7, 4]. As to  $f$ , we assume that

$$f : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R} \text{ is a Carathéodory function}^2, \quad (\text{f:0})$$

$$|f(x, \mu)| \leq C(|\mu| + |h(x)|)^p, \quad (\text{f:1})$$

$$|f(x, \mu) - f(x, \eta)| \leq C(|\mu| + |\eta| + |h(x)|)^{p-1} |\mu - \eta|, \quad (\text{f:2})$$

for a.e.  $x \in \Omega$  and every  $\mu, \eta \in \mathbb{R}^M$ , where  $C > 0$  is a constant and  $h \in L^p(\Omega)$ .

The main purpose of this paper is the study of the so-called weak-strong convergence property of  $F$ , that is, we ask under which additional conditions on  $f$  and  $\Omega$  we have that

$$F(u_n) \rightarrow F(u) \text{ and } u_n \rightharpoonup u \text{ weakly in } L^p \implies u_n \rightarrow u \text{ strongly in } L^p$$

for any given sequence  $(u_n) \subset \mathcal{U}_A$ . In the case of gradients on bounded domains, where  $\mathcal{U}_A$  is replaced by  $\mathcal{U}' := \{u \in L^p(\Omega; \mathbb{R}^{N \times d}) \mid u = \nabla v \text{ for a } v \in W^{1,p}(\Omega; \mathbb{R}^d)\}$ , this question has been investigated by Visintin [18], Evans and Gariepy [5], Zhang [19] and later by Sychev [16, 17]. Results for more general<sup>3</sup>  $\mathcal{A}$ -free vector fields instead of gradients have not been obtained so far. Sychev's results provide optimal conditions for ruling out possible oscillations of  $u_n = \nabla v_n$ , but neither of the aforementioned articles attempts a comprehensive study of concentration effects. In fact, while in [5] and [19] at least sufficient conditions for ruling out concentrations are given (in the case of [5] only partially, since concentrations near the boundary are not discussed), Sychev uses a slightly different definition for the weak-strong convergence property, namely

$$F(u_n) \rightarrow F(u) \text{ and } u_n \rightharpoonup u \text{ weakly in } L^p \implies u_n \rightarrow u \text{ strongly in } L^1$$

for  $(u_n) \subset \mathcal{U}'$ . On a bounded domain, this variant allows one to ignore concentrations of  $u_n$  in  $L^p$  altogether. An alternative approach, still on bounded domains but taking concentrations into account, is possible with the methods developed in [8] and [9] for gradients, which were extended to the  $\mathcal{A}$ -free case in [6]. Our main results stated in the next section in particular provide optimal conditions for ruling out concentrations and similar effects occurring only on unbounded domains. Their proofs are collected in Section 5.

A second goal of this article and its main technical challenge is the extension of the decomposition result of [7] to unbounded domains. We employ this as an essential tool for studying the weak-strong convergence property, but it also is of independent interest. The decomposition lemma of [7] states that, up to a subsequence, any  $\mathcal{A}$ -free, bounded sequence in  $L^p$  on a bounded domain can be decomposed into the sum of two  $\mathcal{A}$ -free, bounded sequences, the first  $p$ -equiintegrable (“purely oscillating”) and the second converging to zero in measure (“purely concentrating”). On general domains, we need to split into more parts, taking into account the additional obstacles for compactness other than oscillations and concentrations which may occur if the domain has infinite measure. This is carried out in Section 4, based on some preliminary observations collected in Section 3. As in [7], we heavily rely on a projection onto  $\mathcal{A}$ -free fields defined via the Fourier transform, now on the whole space instead of in the framework of periodic functions, whose main properties are derived with the help of suitable Fourier multiplier theorems.

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<sup>2</sup>i.e.,  $f = f(x, \mu)$  is measurable in  $x \in \Omega$  for every  $\mu$  and continuous in  $\mu \in \mathbb{R}^M$  for a.e.  $x$

<sup>3</sup>note that  $\mathcal{U}' = \mathcal{U}_{\text{Curl}}$  on a bounded, simply connected domain

## 2 Main results

Just as Morrey's by now classical notion of quasiconvexity is important for functionals depending on gradients,  $\mathcal{A}$ -quasiconvexity is relevant in our setting.

**Definition 2.1.** Let  $x_0 \in \Omega$ . Following [7], we say that  $f(x_0, \cdot)$  is  $\mathcal{A}$ -quasiconvex at  $\xi \in \mathbb{R}^M$  if

$$\int_Q [f(x_0, \xi + \varphi(y)) - f(x_0, \xi)] dy \geq 0 \quad \text{for every } \varphi \in \phi_{\mathcal{A}},$$

Here,  $Q := (0, 1)^N \subset \mathbb{R}^N$  and

$$\phi_{\mathcal{A}} := \left\{ \varphi \in C_{\#}^{\infty}(\mathbb{R}^N; \mathbb{R}^M) \mid \mathcal{A}\varphi = 0 \text{ on } \mathbb{R}^N \text{ and } \int_Q \varphi = 0 \right\},$$

where  $C_{\#}^{\infty}(\mathbb{R}^N; \mathbb{R}^M)$  denotes the set of all functions  $f \in C^{\infty}(\mathbb{R}^N; \mathbb{R}^M)$  which are  $Q$ -periodic in the sense that  $f(y) = f(y + z)$  for every  $z \in \mathbb{Z}^N$  and every  $y \in \mathbb{R}^N$ . Moreover, for  $p > 1$  we say that  $f(x_0, \cdot)$  is *strictly  $p$ - $\mathcal{A}$ -quasiconvex* at  $\xi \in \mathbb{R}^M$  if

$$\int_Q [f(x_0, \xi + \varphi(y)) - f(x_0, \xi)] dy \geq g\left(\int_Q |\varphi|, \int_Q |\varphi|^p\right) \quad \text{for every } \varphi \in \phi_{\mathcal{A}},$$

with a function  $g : [0, \infty)^2 \rightarrow [0, \infty]$  which is increasing in its first variable, decreasing in the second, and satisfies  $g(t, T) > 0$  for all  $t > 0$ ,  $T \geq 0$ .

(The monotonicity of  $g$  need not be strict, and  $g$  may depend on  $x_0$  and  $\xi$ .) Finally, we say that  $f$  is *(strictly  $p$ -)  $\mathcal{A}$ -quasiconvex*, if  $f(x, \cdot)$  is (strictly  $p$ -)  $\mathcal{A}$ -quasi-convex at every  $\xi \in \mathbb{R}^M$ , for a.e.  $x \in \Omega$ .

Strict  $p$ - $\mathcal{A}$ -quasiconvexity can be characterized in the following way.

**Proposition 2.2.** *Let  $N \geq 2$ , let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open and suppose that  $f$  satisfies (f:0)–(f:2). Then for a.e.  $x \in \Omega$  and every  $\xi \in \mathbb{R}^M$ ,  $f(x, \cdot)$  is strictly  $p$ - $\mathcal{A}$ -quasiconvex at  $\xi$  if and only if*

$$\begin{aligned} & f(x, \cdot) \text{ is } \mathcal{A}\text{-quasiconvex at } \xi \text{ and for every sequence } (\varphi_n) \in \Phi_0, \\ & \int_Q f(x, \xi + \varphi_n(y)) dy \xrightarrow[n \rightarrow \infty]{} f(x, \xi) \implies \varphi_n \rightarrow 0 \text{ locally in measure}^4, \end{aligned} \quad (2.1)$$

where

$$\Phi_0 := \left\{ (\varphi_n) \subset \Phi_{\mathcal{A}} \mid \begin{array}{l} \varphi_n \rightharpoonup 0 \text{ weakly in } L^p(Q; \mathbb{R}^M) \text{ and} \\ \varphi_n \text{ is equiintegrable in } L^p(Q; \mathbb{R}^M) \end{array} \right\}.$$

Here, “equiintegrable in  $L^p$ ” is meant in the sense of Definition 2.7 below.

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<sup>4</sup>i.e.,  $|K \cap \{|\varphi_n| \geq \delta\}| \rightarrow 0$  as  $n \rightarrow \infty$ , for every  $\delta > 0$  and every compact  $K \subset \mathbb{R}^N$ .

*Remark 2.3.* Strict  $p$ - $\mathcal{A}$ -quasiconvexity can be also rephrased in terms of Young measures as follows:  $f(x, \cdot)$  is strictly  $p$ - $\mathcal{A}$ -quasiconvex at  $\xi \in \mathbb{R}^M$  if and only if

$$f(x, \cdot) \text{ is } \mathcal{A}\text{-quasiconvex at } \xi \text{ and} \\ \int_{\mathbb{R}^M} f(x, \xi + \mu) d\nu(\mu) = f(x, \xi) \implies \nu \text{ is a Dirac mass at } 0, \quad (2.2)$$

for every homogeneous Young measure  $\nu$  generated by a sequence in  $\Phi_0$ .

The equivalence of (2.1) and (2.2) essentially is a consequence of the results concerning Young measures collected in Section 5. In particular, strict  $p$ - $\mathcal{A}$ -quasiconvexity is the analogue of strict closed  $p$ -quasiconvexity as defined in [17]. Also note that if the sequence generating  $\nu$  is not required to be equiintegrable in  $L^p$ , this still gives an equivalent definition, cf. Remark 5.6.

*Remark 2.4.* If in addition to (f:0) and (f:1),  $f$  is  $\mathcal{A}$ -quasiconvex, then the  $p$ -Lipschitz condition (f:2) automatically holds for certain examples of  $\mathcal{A}$ . In particular, this is the case for the curl and the divergence (of matrix-valued fields, applied row by row) since Curl-quasiconvexity and Div-quasiconvexity both imply rank-1-convexity. For more details see [4].

As observed in [7],  $\mathcal{A}$ -quasiconvexity is vital to ensure weak lower semicontinuity of  $F$  along  $\mathcal{A}$ -free sequences and, consequently, the existence of minimizers.

**Theorem 2.5** (existence of minimizers for general domains). *Let  $N \geq 2$ , let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open, and suppose that  $f$  is  $\mathcal{A}$ -quasiconvex and satisfies (f:0)–(f:2). Moreover, suppose that  $I := \inf\{F(v) \mid v \in \mathcal{U}_{\mathcal{A}}\} > -\infty$  and that there exists a sequence  $(u_n) \subset \mathcal{U}_{\mathcal{A}}$ , bounded in  $L^p$ , such that  $F(u_n) \rightarrow I$ . Then there exists a  $u^* \in \mathcal{U}_{\mathcal{A}}$  such that  $F(u^*) = I$ .*

*Remark 2.6.* Essentially, Theorem 2.5 is a standard application of the direct methods of the calculus of variations. In particular, it suffices to show that  $F$  is lower semicontinuous along sequences in  $\mathcal{U}_{\mathcal{A}}$  which weakly converge in  $L^p$ . If  $\Omega \subset \mathbb{R}^N$  is open and bounded and  $f \geq 0$ , this is due to Theorem 3.7 in [7], and the result easily extends to unbounded domains as  $F = \sup_{k \in \mathbb{N}} F_k$  with  $F_k(u) := \int_{\Omega_k} f(x, u) dx$  defined on the bounded sets  $\Omega_k := B_k(0) \cap \Omega$ . This works even if (f:2) does not hold, and instead of  $f \geq 0$ , it actually suffices to have that  $f^-(x, u_n)$  (the negative part of  $f$ ) is weakly relatively compact in  $L^1$  for a minimizing sequence  $u_n$  which is bounded in  $L^p$ . If, on the other hand, (f:2) holds, then we can use the fact that  $F$  is bounded from below to prove weak lower semicontinuity of  $F$  without any additional assumptions on the negative part of  $f$  as shown in Section 5.

In analogy to the case of functionals depending on gradients on bounded domains [5, 16, 17], strict  $\mathcal{A}$ -quasiconvexity turns out to be the right condition to rule out possible oscillations of minimizing sequences. Of course, oscillations are not the only obstacle for compactness, and we want to investigate others as well. We employ the following terms to describe some of them, in  $L^p$  and related spaces.

**Definition 2.7.** Let  $\Omega \subset \mathbb{R}^N$  be an open set and let  $X$  be a normed space of measurable functions mapping  $\Omega$  into  $\mathbb{R}^M$  such that for every  $u \in X$  and every  $E \subset \Omega$  measurable, the

product  $\chi_E u$  also belongs to  $X$ .<sup>5</sup> Furthermore, let  $(u_n)$  be a sequence in  $X$ . We say that

$$\begin{aligned}
u_n \text{ does not concentrate in } X & \text{ if } \sup_{n \in \mathbb{N}} \sup_{E \subset \Omega, |E| \leq \delta} \|\chi_E u_n\|_X \xrightarrow{\delta \rightarrow 0^+} 0, \\
u_n \text{ is } \mathbb{R}^N\text{-tight in } X & \text{ if } \sup_{n \in \mathbb{N}} \|\chi_{\Omega \setminus B_R(0)} u_n\|_X \xrightarrow{R \rightarrow \infty} 0, \\
u_n \text{ is } \Omega\text{-tight in } X & \text{ if for every } \varepsilon > 0, \text{ there is a compact set} \\
& K \subset \Omega \text{ such that } \sup_{n \in \mathbb{N}} \|\chi_{\Omega \setminus K} u_n\|_X \leq \varepsilon, \\
u_n \text{ does not spread out in } X & \text{ if } \sup_{n \in \mathbb{N}} \|\chi_{\{|u_n| \leq \delta\}} u_n\|_X \xrightarrow{\delta \rightarrow 0^+} 0, \\
u_n \text{ is equiintegrable in } X & \text{ if } u_n \text{ does not concentrate in } X \\
& \text{and } u_n \text{ is } \mathbb{R}^N\text{-tight in } X.
\end{aligned}$$

Next, we list conditions on  $f$  to rule out possible concentrations of minimizing sequences or a lack of tightness. They all amount to requiring that

$$\begin{aligned}
& \text{for every sequence } (\varphi_n) \in \Psi, \\
& \int_{\Omega} f(x, \varphi_n(x)) dx \xrightarrow{n \rightarrow \infty} \int_{\Omega} f(x, 0) dx \implies \varphi_n \rightarrow 0 \text{ in } L^p, \tag{2.3}
\end{aligned}$$

for certain classes of sequences

$$\Psi \subset \Phi := \left\{ (\varphi_n) \subset \mathcal{U}_A \mid \begin{array}{l} (\varphi_n) \text{ is bounded in } L^p \text{ and} \\ \varphi_n \rightarrow 0 \text{ locally in measure} \end{array} \right\}$$

with suitable additional properties, each of which is stronger than the convergence to zero locally in measure required so far. In particular, we are interested in the following subsets of  $\Phi$ :

$$\begin{aligned}
\Phi_c & := \{(\varphi_n) \in \Phi \mid \varphi_n \rightarrow 0 \text{ in } L^p + L^q \text{ for every } q \in (1, p)\}, \\
\Phi_1 & := \{(\varphi_n) \in \Phi_c \mid \varphi_n \text{ is } \Omega\text{-tight in } L^p\}, \\
\Phi_2 & := \left\{ (\varphi_n) \in \Phi_c \mid \begin{array}{l} \chi_B \varphi_n \text{ is } \Omega\text{-tight in } L^p \text{ and } \chi_B \varphi_n \rightarrow 0 \text{ in } L^p \\ \text{for every bounded, measurable } B \subset \Omega \end{array} \right\}, \\
\Phi_3 & := \left\{ (\varphi_n) \in \Phi \mid \begin{array}{l} \chi_B \varphi_n \rightarrow 0 \text{ in } L^p \text{ for all bounded sets } B \subset \Omega, \\ \varphi_n \text{ does not concentrate in } L^p \text{ and} \\ \varphi_n \text{ does not spread out in } L^p, \end{array} \right\}, \\
\Phi_4 & := \{(\varphi_n) \in \Phi \mid \varphi_n \rightarrow 0 \text{ in } L^p + L^r \text{ for every } r \in (p, \infty)\}, \\
\Phi_5 & := \{(\varphi_n) \in \Phi_c \mid \chi_E \varphi_n \rightarrow 0 \text{ in } L^p \text{ for every closed } E \subset \Omega\}, \\
\Phi_{ext} & := \{(\varphi_n) \in \Phi \mid \varphi_n \rightarrow 0 \text{ in } L^p_{loc}\}.
\end{aligned}$$

Here,  $(L^p + L^q)(\Omega; \mathbb{R}^M) := \{u = v + w \in L^1_{loc}(\Omega; \mathbb{R}^M) \mid v \in L^p, w \in L^q\}$ , which is a Banach space with respect to the norm

$$\|u\|_{L^p + L^q} := \inf\{\|v\|_{L^p} + \|w\|_{L^q} \mid v \in L^p \text{ and } w \in L^q \text{ such that } u = v + w\}.$$

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<sup>5</sup>Throughout,  $\chi_E : \Omega \rightarrow \{0, 1\}$  denotes the characteristic function of  $E$ , i.e.,  $\chi_E = 1$  on  $E$  and  $\chi_E = 0$  elsewhere.

*Remark 2.8.* For a bounded sequence in  $L^p$ ,  $u_n \rightarrow 0$  in  $L^p + L^q$  for a  $q < p$  if and only if  $\chi_{\{|u_n| \leq T\}} u_n \rightarrow 0$  in  $L^p$  for every  $T > 0$  (“purely concentrating”), while  $u_n \rightarrow 0$  in  $L^p + L^r$  for a  $r > p$  if and only if  $\chi_{\{|u_n| \geq t\}} u_n \rightarrow 0$  in  $L^p$  for every  $t > 0$  (“purely spreading”).

*Remark 2.9.* The validity of (2.3) on  $\Phi_c$ ,  $\Phi_3$ ,  $\Phi_4$  and  $\Phi_{ext}$ , respectively, not only depends on  $f$  but in general also on  $\Omega$ . In particular, (2.3) automatically holds for  $\Psi = \Phi_2 \cup \Phi_3 \cup \Phi_4$  if  $\Omega$  is bounded, and  $\Phi_{ext} = \Phi_5$  in this case, consisting of sequences purely concentrating at the boundary of  $\Omega$ .

*Remark 2.10.* As opposed to the definition of strict  $p$ - $\mathcal{A}$ -quasiconvexity, (2.3) is not a pointwise property in the first variable of  $f$ . It remains an open question whether it is equivalent to a pointwise condition, at least for  $\Psi = \Phi_1$  under additional assumptions on  $f$ , in particular continuity in  $x$ .

*Remark 2.11.* It is not difficult to give sufficient conditions for (2.3) on the classes listed above. For instance, suppose that  $f$  satisfies

$$f(x, \mu) \geq V(\mu) - |h(x)|^p \quad \text{with an } h \in L^p \text{ and a } V \text{ satisfying (2.5),} \quad (2.4)$$

for every  $\mu \in \mathbb{R}^M$  and a.e.  $x \in \Omega$ , where

$$\begin{aligned} V : \mathbb{R}^M \rightarrow \mathbb{R} \text{ is continuous, } V(0) = 0, \quad |V(\mu)| \leq C |\mu|^p + C, \text{ and} \\ \int_{\Omega} V(u) dx \geq c \int_{\Omega} |u|^p dx \text{ for all } u \in \mathcal{U}_{\mathcal{A}}, \text{ with a constant } c > 0. \end{aligned} \quad (2.5)$$

Then (2.3) holds for  $\Psi = \Phi$  (and thus also for all of the subsets of  $\Phi$ ). In addition, any sequence  $(u_n) \subset \mathcal{U}_{\mathcal{A}}$  such that  $F(u_n)$  is bounded in  $\mathbb{R}$  is bounded in  $L^p$ . Note that depending on  $\mathcal{A}$ , (2.4) can be significantly weaker than a coercivity condition on  $f$  given in a purely pointwise form such as  $f(x, \mu) \geq c |\mu|^p - |h(x)|^p$ .

Our main results are the following.

**Theorem 2.12** (domains with compact boundary). *Let  $N \geq 2$ , let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open with compact boundary and let  $u \in U_{\mathcal{A}}$ . Moreover, suppose that  $f$  satisfies (f:0)–(f:2), that  $f(x, \cdot)$  is  $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x \in \Omega$  and that  $\inf\{F(v) \mid v \in \mathcal{U}_{\mathcal{A}}\} > -\infty$ . Then any bounded sequence  $(u_n) \subset U_{\mathcal{A}}$  such that  $u_n \rightharpoonup u$  weakly in  $L^p(\Omega; \mathbb{R}^M)$  and  $\limsup F(u_n) \leq F(u)$  has the following properties:*

- (i) *If  $f(x, \cdot)$  is strictly  $p$ - $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x \in \Omega$ , then  $u_n \rightarrow u$  locally in measure.*
- (ii) *If (2.3) holds for  $\Psi = \Phi_c$ , then  $u_n$  does not concentrate in  $L^p$ .*
- (iii) *If (2.3) holds for  $\Psi = \Phi_3$ , then  $\chi_{\{s^{-1} < |u_n| < s\}} u_n$  is  $\mathbb{R}^N$ -tight in  $L^p$  for every fixed  $s \geq 1$ .*
- (iv) *If (2.3) holds for  $\Psi = \Phi_4$ , then  $u_n$  does not spread out in  $L^p$ .*

*In particular, if  $f$  and  $\Omega$  are such that the assumptions of (i)–(iv) are satisfied, then  $u_n \rightarrow u$  strongly in  $L^p$ .*

Using the classes  $\Phi_1$ ,  $\Phi_2$  and  $\Phi_5$  instead of  $\Phi_c$ , possible concentrations of  $u_n$  can be studied in even greater detail, distinguishing whether they occur within  $\Omega$ , at infinity or at the boundary of  $\Omega$ .

**Corollary 2.13.** *Under the assumptions of Theorem 2.12, the following is true.*

- (ii.1) *If (2.3) holds for  $\Psi = \Phi_1$ , then  $\chi_K u_n$  does not concentrate in  $L^p$ , for every compact  $K \subset \Omega$ .*
- (ii.2) *If (2.3) holds for  $\Psi = \Phi_2$ , then  $\chi_{\Omega_{\delta_n} \setminus B_{R_n}} u_n$  does not concentrate in  $L^p$ , for every pair of sequences  $R_n \rightarrow \infty$  and  $\delta_n \rightarrow 0^+$ .*
- (ii.3) *If (2.3) holds for  $\Psi = \Phi_5$ , then  $\chi_{\Omega \setminus \Omega_{\delta_n}} u_n$  does not concentrate in  $L^p$ , for every sequence  $\delta_n \rightarrow 0^+$ .*

Here,  $B_R := \{x \in \mathbb{R}^N \mid |x| < R\}$  and  $\Omega_\delta := \{x \in \Omega \mid \text{dist}(x; \partial\Omega) > \delta\}$ . In particular,  $\Phi_c$  can be replaced by  $\Phi_1 \cup \Phi_2 \cup \Phi_5$  in Theorem 2.12 (ii).

If  $\Omega$  is an exterior domain and  $f(x, \mu)$  has a limit as  $|x| \rightarrow \infty$  which is uniform in  $\mu$  in a suitable sense, Theorem 2.12 can be partially simplified by using a more tangible characterization of (2.3) for  $\Psi = \Phi_2 \cup \Phi_3 \cup \Phi_4$  (all the cases related to the behavior of  $f$  as  $|x| \rightarrow \infty$ ):

**Proposition 2.14.** *Let  $N \geq 2$ , let  $1 < p < \infty$  and let  $\Omega \subset \mathbb{R}^N$  be the complement of a compact set. Moreover, suppose that  $f$  satisfies (f:0)–(f:2) and that there exists a function  $f_\infty : \mathbb{R}^M \rightarrow \mathbb{R}$  such that*

$$\alpha(x) := \sup_{\mu \in \mathbb{R}^M} \frac{|f_\infty(\mu) - f(x, \mu)|}{|\mu|^p + |h(x)|^p} \xrightarrow{|x| \rightarrow \infty} 0 \text{ for a suitable } h \in L^p(\mathbb{R}^N). \quad (2.6)$$

Then (2.3) holds for  $\Psi = \Phi_2 \cup \Phi_3 \cup \Phi_4$  if and only if

$$\int_{\mathbb{R}^N} f_\infty(\varphi) dx \geq g(\|\varphi\|_{L^p}) \text{ for every } \mathcal{A}\text{-free } \varphi \in L^p(\mathbb{R}^N; \mathbb{R}^M), \quad (2.7)$$

with a suitable  $g : [0, \infty) \rightarrow \mathbb{R}$  continuous such that  $g > 0$  on  $(0, \infty)$ .

If the boundary of  $\Omega$  is not compact, we can still say the following.

**Theorem 2.15** (general domains). *Let  $N \geq 2$ , let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open and let  $u \in U_{\mathcal{A}}$ . Moreover, suppose that  $f$  satisfies (f:0)–(f:2), that  $f(x, \cdot)$  is  $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x \in \Omega$  and that  $\inf\{F(v) \mid v \in U_{\mathcal{A}}\} > -\infty$ . Then any bounded sequence  $(u_n) \subset U_{\mathcal{A}}$  satisfying  $u_n \rightharpoonup u$  weakly in  $L^p(\Omega; \mathbb{R}^M)$  and  $\limsup F(u_n) \leq F(u)$  has following properties.*

- (i) *If  $f(x, \cdot)$  is strictly  $p$ - $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x \in \Omega$ , then  $u_n \rightarrow u$  locally in measure.*
- (ii) *If (2.3) holds for  $\Psi = \Phi_1$ , then  $\chi_K u_n$  does not concentrate in  $L^p$ , for every compact  $K \subset \Omega$ .*
- (iii) *If (2.3) holds for  $\Psi = \Phi_{ext}$ , then  $u_n$  is  $\Omega$ -tight in  $L^p$ .*

In particular, if  $f$  and  $\Omega$  are such that the assumptions of (i)–(iii) are satisfied, then  $u_n \rightarrow u$  strongly in  $L^p$ .

By Remark 2.11, this immediately entails the following.

**Corollary 2.16.** *Let  $N \geq 2$ , let  $1 < p < \infty$  and let  $\Omega \subset \mathbb{R}^N$  be open. Moreover, suppose that  $f$  satisfies (f:0)–(f:2) as well as (2.4) and that  $f$  is strictly  $p$ - $\mathcal{A}$ -quasiconvex. Then any minimizing sequence  $(u_n) \subset \mathcal{U}_{\mathcal{A}}$  has a subsequence which strongly converges in  $L^p$ .*

*Remark 2.17.* In fact, all results stated above as well as those of Section 4 are also true for  $N = 1$ . However, this case requires some minor technical changes in the proofs which we omit for the sake of brevity. For more details see Remark 3.3 below.

### 3 Preliminaries

Throughout this article,  $\mathcal{A}$  denotes a homogeneous linear differential operator of first order, formally mapping  $u = (u^1, \dots, u^M) : \Omega \rightarrow \mathbb{R}^M$  onto  $\mathcal{A}u = ((\mathcal{A}u)^1, \dots, (\mathcal{A}u)^L) : \Omega \rightarrow \mathbb{R}^L$  defined by

$$(\mathcal{A}u)^l := \sum_{i=1}^N \sum_{m=1}^M A_i^{lm} \partial_{x_i} u^m,$$

with  $A_i^{lm} \in \mathbb{R}$  fixed. Its formal adjoint is denoted by  $\mathcal{A}^*$ , which maps  $v = (v_1, \dots, v_L) : \Omega \rightarrow \mathbb{R}^L$  to  $\mathcal{A}^*v : \Omega \rightarrow \mathbb{R}^M$ , where

$$(\mathcal{A}^*v)^m = - \sum_{i=1}^N \sum_{l=1}^L A_i^{lm} \partial_{x_i} v_l.$$

In particular, we have  $\int_{\Omega} (\mathcal{A}u) \cdot \varphi \, dx = \int_{\Omega} u \cdot (\mathcal{A}^*\varphi) \, dx$  for all  $u \in C^1(\Omega; \mathbb{R}^M)$  and all  $\varphi \in C_c^1(\Omega; \mathbb{R}^L)$  by integration by parts. Related to  $\mathcal{A}$  via the Fourier transform is the linear matrix-valued function

$$\mathbb{A} : \mathbb{R}^N \rightarrow \mathbb{R}^{L \times M}, \quad (\mathbb{A}(\xi))^{lm} := \sum_{i=1}^N A_i^{lm} \xi_i.$$

Throughout, we assume that  $\mathcal{A}$  (and hence also  $\mathcal{A}^*$ ) satisfies the condition of constant rank, that is,

$$\text{the rank of } \mathbb{A}(\xi) \in \mathbb{R}^{L \times M} \text{ is constant as a function of } \xi \in \mathbb{R}^N \setminus \{0\}. \quad (3.1)$$

As a consequence, the orthogonal projection  $\mathbb{P}(\xi) \in \mathbb{R}^{M \times M}$  onto the kernel of  $\mathbb{A}(\xi)$  in  $\mathbb{R}^M$  is continuous as a function of  $\xi \in \mathbb{R}^N \setminus \{0\}$ . We define  $\mathbb{P}(0)$  to be the identity matrix. The Fourier multiplier  $\mathcal{P} : \mathcal{S}^M \rightarrow (\mathcal{S}^M)'$  associated to  $\mathbb{P}$  is given by

$$\mathcal{P}\varphi := \mathcal{F}^{-1}(\mathbb{P}\mathcal{F}\varphi), \quad \text{for } \varphi \in \mathcal{S}^M, \text{ with } (\mathbb{P}\mathcal{F}\varphi)(\xi) := \mathbb{P}(\xi)[\mathcal{F}\varphi(\xi)],$$

where  $\mathcal{F}$  is the Fourier transform<sup>6</sup>,  $\mathcal{S}$  denotes the Schwartz space of rapidly decaying functions of class  $C^\infty$  and  $\mathcal{S}'$  is its dual. By definition,  $\mathcal{P}$  is a projection onto the kernel of  $\mathcal{A}$ . Moreover, by the classical Hörmander-Mikhlin multiplier theorem, it extends to a continuous operator  $\mathcal{P} : L^p(\mathbb{R}^N; \mathbb{R}^M) \rightarrow L^p(\mathbb{R}^N; \mathbb{R}^M)$  projecting  $L^p(\mathbb{R}^N; \mathbb{R}^M)$  onto the kernel of  $\mathcal{A}$ . We also need this property in a broader class of weighted spaces of the form

$$L_w^p(\Omega; \mathbb{R}^M) := \left\{ u : \Omega \rightarrow \mathbb{R}^M \text{ measurable} \mid \|u\|_{L_w^p(\Omega; \mathbb{R}^M)} < \infty \right\},$$

where  $\|u\|_{L_w^p(\Omega; \mathbb{R}^M)}^p := \int_{\Omega} |u(x)|^p w(x) \, dx$

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<sup>6</sup> $(\mathcal{F}u)(\xi) := \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} u(x) \, dx$  for  $u \in L^1(\mathbb{R}^N)$  and  $\xi \in \mathbb{R}^N$ , and the definition is extended to  $u \in \mathcal{S}'$  as usual, cf. [15]. In the vector-valued case,  $\mathcal{F}$  operates component-wise.



and the weight  $w : \Omega \rightarrow (0, \infty)$  is a measurable function. Due to a result of [12],  $\mathcal{P}$  extends to a continuous projection operator on  $L_w^p(\Omega; \mathbb{R}^M)$  for various classes of weights. We only reproduce a special case which suffices for our purposes:

**Lemma 3.1.** *Let  $1 < p < \infty$ , let  $w(x) := \min\{1, |x|^\beta\}$  with a constant  $-N < \beta < N(p-1)$ , and let  $m : \mathbb{R}^N \rightarrow \mathbb{R}$  be a bounded function which is 0-homogeneous and of class  $C^N$  on  $\mathbb{R}^N \setminus \{0\}$ . Then the associated Fourier multiplier  $T$  given by  $T(u) := \mathcal{F}^{-1}(m\mathcal{F}u)$  is a bounded linear operator mapping  $L_w^p(\mathbb{R}^N)$  into itself.*

**Proof.** Since  $m(x) = m(x/|x|)$ , we have that  $|D^k m(x)| \leq C_1 |x|^{-k}$  for every  $x \in \mathbb{R}^N \setminus \{0\}$  and every  $k = 0, \dots, N$ , with a constant  $C_1 > 0$  only depending on  $m$  and  $N$ . As a consequence, for every  $s \in (1, 2]$  such that  $sk \neq N$  for  $k = 0, \dots, N$ ,

$$R^{sk-N} \int_{R < |x| < 2R} |D^k m(x)|^s dx \leq C_2 \quad \text{for every } R > 0 \text{ and every } k = 0, \dots, N,$$

with a constant  $C_2$  only depending on  $N$ ,  $s$  and  $C_1$ . This means that  $m \in M(s, N)$  in the notation of [12], and with this property established, Theorem 2 in [12] yields the assertion.  $\square$

In particular, this applies to the space  $L^p$  which corresponds to the case  $w \equiv 1$  ( $\beta = 0$ ). Lemma 3.1 can be extended to  $L^p + L^q$  as follows.

**Lemma 3.2.** *Let  $1 < q < p < \infty$  and let  $m : \mathbb{R}^N \rightarrow \mathbb{R}$  be a bounded function which is 0-homogeneous and of class  $C^N$  on  $\mathbb{R}^N \setminus \{0\}$ . Then the associated Fourier multiplier  $T$  given by  $T(u) := \mathcal{F}^{-1}[m(\mathcal{F}u)]$  is a bounded linear operator mapping  $L^q(\mathbb{R}^N; \mathbb{R}) + L^p(\mathbb{R}^N; \mathbb{R})$  into itself.*

**Proof.** For every  $\varepsilon > 0$ , there exists  $v \in L^q$  and  $w \in L^p$  with  $v + w = u$  such that  $\|v\|_{L^q} + \|w\|_{L^p} \leq \|u\|_{L^q + L^p} + \varepsilon$ . Lemma 3.1 with  $\beta = 0$  thus implies that

$$\|Tu\|_{L^q + L^p} \leq \|Tv\|_{L^q} + \|Tw\|_{L^p} \leq C(\|v\|_{L^q} + \|w\|_{L^p}) \leq C(\|u\|_{L^q + L^p} + \varepsilon)$$

for arbitrary  $\varepsilon$  with a constant  $C$  independent of  $u$  and  $\varepsilon$ .  $\square$

In the following, norms involving certain inverse derivatives will play a role, which we express by means of the operator  $(-\Delta)^{-\frac{1}{2}}$ , defined by

$$(-\Delta)^{-\frac{1}{2}} u := \mathcal{F}^{-1} |2\pi\xi|^{-1} \mathcal{F}u, \quad (3.2)$$

for any tempered distribution  $u \in \mathcal{S}'$  such that the ‘‘pointwise’’ product of  $|2\pi\xi|^{-1}$  with  $(\mathcal{F}u)(\xi)$  is well defined in  $\mathcal{S}'$ . If  $u \in L^1$ , a more explicit definition of  $(-\Delta)^{-\frac{1}{2}}$  can be given in terms of the corresponding Riesz potential, namely,

$$((-\Delta)^{-\frac{1}{2}} u)(x) = \frac{1}{\sigma} \int_{\mathbb{R}^N} |x - y|^{1-N} u(y) dy, \quad (3.3)$$

with a normalizing constant  $\sigma = \sigma(N) > 0$ , cf. [15].

*Remark 3.3.* To be precise, (3.3) only holds if  $N \geq 2$ , which is the reason for this assumption in our main results as well as in any other statement of this note directly or indirectly exploiting (3.3) in form of Lemma 3.5 below. Of course, this is just a minor technical issue. The case  $N = 1$  could easily be treated separately, for instance using the antiderivative instead of  $(-\Delta)^{-\frac{1}{2}}$ .

Extending Lemma 2.14 in [7], which in turn is largely based on ideas of [14], the properties of the projection  $\mathcal{P}$  in  $L^p + L^q$  and in  $L_w^p$  can be summarized as follows.

**Lemma 3.4.** *Let  $1 < q \leq p < \infty$  and suppose that (3.1) holds. Then we have the following.*

- (i)  $\mathcal{P} : (L^q + L^p)(\mathbb{R}^N; \mathbb{R}^M) \rightarrow (L^q + L^p)(\mathbb{R}^N; \mathbb{R}^M)$  is a linear, bounded operator.
- (ii)  $\mathcal{P}v = v$  for every  $\mathcal{A}$ -free  $v \in (L^q + L^p)(\mathbb{R}^N; \mathbb{R}^M)$ , and  $\mathcal{A} \circ \mathcal{P} = 0$ .
- (iii) Let  $u_n$  be a bounded sequence in  $(L^q + L^p)(\mathbb{R}^N; \mathbb{R}^M)$ . If  $u_n$  does not concentrate in  $L^p + L^q$ , neither does  $\mathcal{P}u_n$ . Similarly, if  $u_n$  is  $\mathbb{R}^N$ -tight in  $L^p + L^q$  then so is  $\mathcal{P}u_n$ , and if  $u_n$  does not spread out in  $L^p + L^q$ , then neither does  $\mathcal{P}u_n$ .
- (iv)  $c \|(-\Delta)^{-\frac{1}{2}} \mathcal{A}u\|_{L^p + L^q} \leq \|(I - \mathcal{P})u\|_{L^p + L^q} \leq C \|(-\Delta)^{-\frac{1}{2}} \mathcal{A}u\|_{L^p + L^q}$  for every  $u \in (L^q + L^p)(\mathbb{R}^N; \mathbb{R}^M)$ , with constants  $c, C > 0$  independent of  $u$ .

Moreover, all of the above stays true if  $L^p + L^q$  is replaced with  $L_w^p$ , where  $w$  may be any positive weight function such that Lemma 3.1 holds.

**Proof.** We essentially proceed as in [7]. As a consequence of (3.1), the projection  $\mathbb{P}(\xi)$  is a 0-homogeneous function of  $\xi$  of class  $C^\infty$  on  $\mathbb{R}^N \setminus \{0\}$ , whence Lemma 3.2 yields (i). The definition of  $\mathcal{P}$  immediately implies (ii).

For the proof of (iii) consider a bounded sequence  $u_n$  in  $L^p + L^q$ . If  $u_n$  does not concentrate in  $L^p + L^q$ , we have that  $\sup_{n \in \mathbb{N}} \|\chi_{\{|u_n| \leq h\}} u_n - u_n\|_{L^q + L^p} \rightarrow 0$  as  $h \rightarrow \infty$ , and since  $\mathcal{P}$  is continuous in  $L^q + L^p$ , we also get that

$$\sup_{n \in \mathbb{N}} \|\mathcal{P}(\chi_{\{|u_n| \leq h\}} u_n) - \mathcal{P}u_n\|_{L^q + L^p} \xrightarrow{h \rightarrow \infty} 0. \quad (3.4)$$

On the other hand, for fixed  $h$ ,  $\chi_{\{|u_n| \leq h\}} u_n$  is bounded in  $L^\infty$  and thus also in  $L^s$  for any  $s > p$ . By continuity of  $\mathcal{P}$  in  $L^s$ , this implies that  $\mathcal{P}(\chi_{\{|u_n| \leq h\}} u_n)$  is bounded in  $L^s$ . By Hölder's inequality we infer that  $\mathcal{P}(\chi_{\{|u_n| \leq h\}} u_n)$  does not concentrate in  $L^q$  since  $s > q$ , which also means that  $\mathcal{P}(\chi_{\{|u_n| \leq h\}} u_n)$  does not concentrate in  $L^p + L^q$  for fixed  $h$ , since  $p \geq q$ . Together with (3.4), this implies that  $\mathcal{P}u_n$  does not concentrate in  $L^p + L^q$ . If  $u_n$  does not spread out in  $L^p + L^q$ , an analogous argument gives that

$$\sup_{n \in \mathbb{N}} \|\mathcal{P}(\chi_{\{|u_n| \geq h\}} u_n) - \mathcal{P}u_n\|_{L^q + L^p} \xrightarrow{h \rightarrow 0^+} 0$$

and that  $\mathcal{P}(\chi_{\{|u_n| \geq h\}} u_n)$  does not spread out in  $L^p + L^q$  for fixed  $h > 0$  (since it is bounded in  $L^s$  with  $1 < s < q$ ), which implies that  $\mathcal{P}u_n$  does not spread out in  $L^p + L^q$ . Last but not least, if  $u_n$  is  $\mathbb{R}^N$ -tight in  $L^p + L^q$ , we get that

$$\sup_{n \in \mathbb{N}} \|\mathcal{P}(\chi_{B_h(0)} u_n) - \mathcal{P}u_n\|_{L^q + L^p} \xrightarrow{h \rightarrow \infty} 0$$

and that  $\mathcal{P}(\chi_{B_h(0)}u_n)$  is  $\mathbb{R}^N$ -tight in  $L^p + L^q$  for fixed  $h$  (since it is bounded in  $L^q_w$  with  $\tilde{w}(x) := \min\{1, |x|^{\tilde{\beta}}\}$ , for any  $0 < \tilde{\beta} < N(q-1)$ ), whence  $\mathcal{P}u_n$  is  $\mathbb{R}^N$ -tight in  $L^p + L^q$ .

To get (iv), first observe that

$$\begin{aligned} (I - \mathbb{P})(\xi)\mathcal{F}u(\xi) &= \mathbb{Q}(\xi)\mathbb{A}(\xi)\mathcal{F}u(\xi) = \mathbb{Q}\left(\frac{\xi}{|\xi|}\right)\frac{1}{|\xi|}\mathbb{A}(\xi)\mathcal{F}u(\xi) \\ &= 2\pi\mathbb{Q}\left(\frac{\xi}{|\xi|}\right)\mathcal{F}((-\Delta)^{-\frac{1}{2}}\mathcal{A}u)(\xi), \end{aligned} \tag{3.5}$$

where  $\mathbb{Q} : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^{L \times L}$  is defined by

$$\begin{aligned} \mathbb{Q}(\xi)\mathbb{A}(\xi)\eta &:= \eta \quad \text{for any } \eta \in (\ker \mathbb{A}(\xi))^\perp \subset \mathbb{R}^M, \\ \mathbb{Q}(\xi)\mu &:= 0 \quad \text{for any } \mu \in (\text{range } \mathbb{A}(\xi))^\perp \subset \mathbb{R}^L. \end{aligned}$$

Note that  $\mathbb{Q}$  is homogeneous of degree  $-1$  as a function of  $\xi$  since  $\mathbb{A}$  is homogeneous of degree  $1$ , which justifies the second equality in (3.5). Moreover, as a consequence of (3.1), both  $\text{range } \mathbb{A}(\xi)$  and  $(\text{range } \mathbb{A}(\xi))^\perp$  have constant dimension and vary smoothly with  $\xi \in \mathbb{R}^N \setminus \{0\}$ , and  $\mathbb{A}(\xi) : (\ker \mathbb{A}(\xi))^\perp \rightarrow \text{range } \mathbb{A}(\xi)$  is invertible with inverse smoothly depending on  $\xi$ , whence  $\mathbb{Q}$  is of class  $C^\infty$ . In particular,  $\mathbb{Q}(\frac{\xi}{|\xi|})$  gives rise to a Fourier multiplier in  $L^p + L^q$  by Lemma 3.2, whence (3.5) implies the second inequality in (iv). The first inequality follows in the same way, since  $\mathbb{A}(\frac{\xi}{|\xi|})\mathbb{Q}(\frac{\xi}{|\xi|})\mathbb{A}(\frac{\xi}{|\xi|}) = \mathbb{A}(\frac{\xi}{|\xi|})$  and  $\mathbb{A}(\frac{\xi}{|\xi|})$  also gives rise to a continuous Fourier multiplier in  $L^p + L^q$ .

Finally, note that all of the arguments above also work for  $L^p_w$  instead of  $L^p + L^q$  if we use Lemma 3.1 instead of Lemma 3.2 and suitably adapt the auxiliary spaces employed in the proof of (iii) and (iv).  $\square$

We will use Lemma 3.4 (iv) to handle domains other than the whole space, and for this purpose, the following compactness result is also crucial.

**Lemma 3.5.** *Let  $v_n$  be a bounded sequence in  $L^p(\mathbb{R}^N)$  with some  $1 < p < \infty$ . Moreover, suppose that there is a fixed compact set  $K \subset \mathbb{R}^N$  containing the support of  $v_n$  for every  $n$  and that  $\int_{\mathbb{R}^N} v_n = 0$  for every  $n$ . Then  $w_n := (-\Delta)^{-\frac{1}{2}}v_n$  is bounded in  $L^p(\mathbb{R}^N)$ , and it has a subsequence which converges strongly in  $L^p(\mathbb{R}^N)$ .*

**Proof.** Let  $B_r$  denote a ball with radius  $r$  centered at  $0$ , containing  $K$ . Observe that for fixed  $R > 0$ , (3.3) yields

$$\int_{B_R} |(-\Delta)^{-\frac{1}{2}}v_n(x)|^p dx = \|\kappa * v_n\|_{L^p(B_R)}^p \leq \|\kappa\|_{L^1(B_{R+r})}^p \|v_n\|_{L^p(B_r)}^p, \tag{3.6}$$

where  $*$  denotes the convolution and  $\kappa(z) := |z|^{1-N}$ . Moreover, for every  $R \geq 2r$ , there is a constant  $C = C(N, r) > 0$  such that

$$\sup_{y \in B_r} \left| |x-y|^{1-N} - |x|^{1-N} \right| \leq C|x|^{-N} \quad \text{for every } x \text{ with } |x| > R. \tag{3.7}$$

Since  $\int_{B_r} v_n = 0$ , (3.7) implies that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R} |(-\Delta)^{-\frac{1}{2}} v_n|^p dx &= \int_{\mathbb{R}^N \setminus B_R} \left| \int_{B_r} (|x-y|^{1-N} - |x|^{1-N}) v_n(y) dy \right|^p dx \\ &\leq \|v_n\|_{L^p}^p \int_{\mathbb{R}^N \setminus B_R} C^p |x|^{-pN} dx \end{aligned} \quad (3.8)$$

for  $R \geq 2r$ . Note that  $|x|^{-Np}$  is integrable on  $\mathbb{R}^N \setminus B_{2r}$  for every  $p > 1$ . In particular,  $(-\Delta)^{-\frac{1}{2}} v_n$  is bounded in  $L^p(\mathbb{R}^N)$  by (3.6) and (3.8) combined. In addition, (3.8) implies that

$$\int_{\mathbb{R}^N \setminus B_R} |(-\Delta)^{-\frac{1}{2}} v_n|^p dx \xrightarrow{R \rightarrow \infty} 0 \quad \text{uniformly in } n. \quad (3.9)$$

Moreover, as in (3.6) we get

$$\begin{aligned} \int_{B_R} |(-\Delta)^{-\frac{1}{2}} v_n(x) - (-\Delta)^{-\frac{1}{2}} v_n(x+h)|^p dx \\ \leq \|(\kappa(\cdot) - \kappa(\cdot+h))\|_{L^1(B_{R+r})}^p \|v_n\|_{L^p(B_r)}^p \xrightarrow{|h| \rightarrow 0} 0 \quad \text{uniformly in } n, \end{aligned} \quad (3.10)$$

for any fixed  $R > 0$ , since  $\kappa$  is integrable on bounded sets and the shift is continuous in  $L^1$ . Together, (3.9) and (3.10) imply that  $\{(-\Delta)^{-\frac{1}{2}} v_n \mid n \in \mathbb{N}\}$  is contained in a compact subset of  $L^p(\mathbb{R}^N)$ , by a standard criterion for relative compactness in  $L^p$  (e.g., [2]).  $\square$

## 4 Decomposition of $\mathcal{A}$ -free sequences

We now derive a decomposition lemma in the tradition of [1], [8], [7] and [11], here for a sequence of  $\mathcal{A}$ -free fields on the whole space. This result and suitable extensions to other unbounded domains will be our main tool for obtaining compactness of minimizing sequences.

**Lemma 4.1.** *Let  $1 < p < \infty$  and let  $\mathcal{A}$  be a linear differential operator of first order satisfying (3.1). Moreover, suppose that  $u_n$  is a bounded,  $\mathcal{A}$ -free sequence in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$  with  $u_n \rightharpoonup u$  weakly in  $L^p$ . Then there exist a subsequence  $u_{k(n)}$  of  $u_n$  and five bounded,  $\mathcal{A}$ -free sequences  $w_n^0, \dots, w_n^4$  in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$  such that*

$$u_{k(n)} = u + w_n^0 + w_n^1 + w_n^2 + w_n^3 + w_n^4 \quad \text{for every } n \in \mathbb{N}$$

and the following properties hold:

- (a)  $w_n^0 \rightharpoonup 0$  weakly in  $L^p$ , and  $w_n^0$  is equiintegrable in  $L^p$ .
- (b)  $w_n^1$  is  $\mathbb{R}^N$ -tight in  $L^p$ , and  $w_n^1 \rightarrow 0$  in  $L^p + L^q$  for every  $q \in (1, p)$ .
- (c)  $\chi_B w_n^2 \rightarrow 0$  in  $L^p$  for any bounded, measurable set  $B \subset \mathbb{R}^N$  and  $w_n^2 \rightarrow 0$  in  $L^p + L^q$  for every  $q \in (1, p)$ .
- (d)  $\chi_B w_n^3 \rightarrow 0$  in  $L^p$  for any bounded, measurable set  $B \subset \mathbb{R}^N$ .

(e)  $w_n^4 \rightarrow 0$  in  $L^r + L^p$  for every  $r \in (p, \infty)$ .

*Remark 4.2.* Using (a)–(e) to check Vitali’s criteria for compactness in  $L^p$ , it is not difficult to see that if  $u_{k(n)} = u + \tilde{w}_n^0 + \dots + \tilde{w}_n^4$  is another decomposition with the same properties, then  $w_n^j - \tilde{w}_n^j \rightarrow 0$  strongly in  $L^p$ . In this sense, the component sequences are uniquely determined.

For the proof of Lemma 4.1, we first need an auxiliary result which represents a decomposition lemma in  $L^p$ , summarizing Chacon’s biting lemma and suitable variants for unbounded domains. It is based on four different kinds of truncations of  $L^p$ -functions.

**Lemma 4.3.** *Let  $\Omega \subset \mathbb{R}^N$  be open and let  $1 \leq p < \infty$ . Then every bounded sequence  $(v_n) \subset L^p(\Omega; \mathbb{R}^M)$  has a subsequence  $(v_{k(n)})$  such that*

$$\begin{aligned} \chi_{\{|v_{k(n)}| \leq n\}} v_{k(n)} & \text{ does not concentrate in } L^p(\Omega; \mathbb{R}^M), \\ \chi_{\{|v_{k(n)}| \geq \frac{1}{n}\}} v_{k(n)} & \text{ does not spread out in } L^p(\Omega; \mathbb{R}^M), \\ \chi_{B_n(0)} v_{k(n)} & \text{ is } \mathbb{R}^N\text{-tight in } L^p(\Omega; \mathbb{R}^M) \text{ and} \\ \chi_{K_n} v_{k(n)} & \text{ is } \Omega\text{-tight in } L^p(\Omega; \mathbb{R}^M), \end{aligned} \tag{4.1}$$

where  $K_n := \{x \in \Omega \mid |x| \leq n \text{ and } \text{dist}(x; \partial\Omega) \geq \frac{1}{n}\}$ .

**Proof.** This is essentially well known. For instance, the first three lines of (4.1) immediately follow from Lemma 3.3–Lemma 3.5 in [11], and the fourth line can be obtained analogously to the third. We omit the details.  $\square$

**Proof of Lemma 4.1.** W.l.o.g. we may assume that  $u = 0$  (otherwise, since  $u$  is  $\mathcal{A}$ -free, we can decompose  $\tilde{u}_n := u_n - u$  instead). For  $j = 0, \dots, 4$ , let  $w_n^j := \mathcal{P}W_n^j \in L^p(\mathbb{R}^N; \mathbb{R}^M)$  with

$$\begin{aligned} W_n^0 & := 1 \cdot \chi_{B_n(0)} \cdot \chi_{\{|u_{k(n)}| \leq n\}} \cdot u_{k(n)}, \\ W_n^1 & := 1 \cdot \chi_{B_n(0)} \cdot (1 - \chi_{\{|u_{k(n)}| \leq n\}}) \cdot u_{k(n)}, \\ W_n^2 & := 1 \cdot (1 - \chi_{B_n(0)}) \cdot (1 - \chi_{\{|u_{k(n)}| \leq n\}}) \cdot u_{k(n)}, \\ W_n^3 & := \chi_{\{|u_{k(n)}| \geq \frac{1}{n}\}} \cdot (1 - \chi_{B_n(0)}) \cdot \chi_{\{|u_{k(n)}| \leq n\}} \cdot u_{k(n)}, \\ W_n^4 & := (1 - \chi_{\{|u_{k(n)}| \geq \frac{1}{n}\}}) \cdot (1 - \chi_{B_n(0)}) \cdot \chi_{\{|u_{k(n)}| \leq n\}} \cdot u_{k(n)}, \end{aligned}$$

where the subsequence  $u_{k(n)}$  is chosen according to Lemma 4.3 with  $v_n := u_n$ . By definition,  $u_{k(n)} = \mathcal{P}u_{k(n)} = w_n^0 + \dots + w_n^4$ , each  $w_n^j$  is  $\mathcal{A}$ -free, and the sequences  $w_n^j$  are bounded in  $L^p$  by continuity of  $\mathcal{P}$  in  $L^p$ . Moreover, due to the choice of  $u_{k(n)}$  and the definition of  $W_n^j$ , the sequences  $W_n^j$  (in place of  $w_n^j$ ) have the properties (a)–(e) listed in the assertion. The projected sequences  $w_n^j$  inherit these:  $\mathbb{R}^N$ -tightness, absence of concentration, absence of spreading and equiintegrability in  $L^p$  all survive the application of  $\mathcal{P}$  due to Lemma 3.4 (iv). Convergence in  $L^p + L^q$  or in  $L^r + L^p$  with  $1 \leq q < p$  and  $p < r < \infty$  is also preserved, as a consequence of Lemma 3.4 (i), as is weak convergence to zero in  $L^p$ . Finally, note that for a bounded sequence  $v_n$  in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$ ,  $\chi_B v_n \rightarrow 0$  in  $L^p$  for every bounded, open  $B \subset \Omega$  if and only if  $v_n \rightarrow 0$  in  $L_w^p$  with the weight  $w(x) := \min\{1, |x|^{-\frac{1}{2}}\}$  (or any other bounded weight which is locally bounded away from zero and converges to zero as  $|x| \rightarrow \infty$ ). Hence, the continuity of  $\mathcal{P}$  in  $L_w^p$  also yields that  $\chi_B w_n^{2,3} \rightarrow 0$  in  $L^p$  just as  $W_n^{2,3}$ .  $\square$

As it turns out, Lemma 4.1 can be extended to any domain but only with a somewhat coarser decomposition.

**Lemma 4.4.** *Let  $N \geq 2$ , let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open and let  $\mathcal{A}$  be a linear differential operator of first order satisfying (3.1). Moreover, suppose that  $u_n$  is a bounded,  $\mathcal{A}$ -free sequence in  $L^p(\Omega; \mathbb{R}^M)$  with  $u_n \rightharpoonup u$  weakly in  $L^p$ . Then there exist a subsequence  $u_{k(n)}$  of  $u_n$  and bounded,  $\mathcal{A}$ -free sequences  $(v_n), (w_n) \subset L^p(\mathbb{R}^N; \mathbb{R}^M)$  and  $(z_n) \subset L^p(\Omega; \mathbb{R}^M)$  such that*

$$u_{k(n)} = u + v_n + w_n + z_n \quad \text{in } \Omega \text{ for every } n \in \mathbb{N}$$

and the following properties hold:

- (a)  $v_n \rightharpoonup 0$  weakly in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$  and  $v_n$  is equiintegrable in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$ .
- (b)  $w_n \rightarrow 0$  in  $(L^p + L^q)(\mathbb{R}^N; \mathbb{R}^M)$  for every  $1 \leq q < p$ , and  $w_n$  is  $\Omega$ -tight in  $L^p(\Omega; \mathbb{R}^M)$ .
- (c)  $z_n \rightarrow 0$  in  $L^p_{loc}(\Omega; \mathbb{R}^M)$ .

**Proof.** Observe that  $\mathcal{A}u = 0$  in  $\Omega$ . We choose a sequence of cut-off functions  $(\gamma_j) \subset C^1(\mathbb{R}^N; [0, 1])$  such that

$$\begin{aligned} \{\gamma_j > 0\} &\subset \left\{ x \in \Omega \mid |x| < j \text{ and } \text{dist}(x; \partial\Omega) > \frac{1}{j} \right\} \text{ and} \\ \{0 < \gamma_j < 1\} &\subset \left\{ x \in \Omega \mid |x| > j - 1 \text{ or } \text{dist}(x; \partial\Omega) < \frac{2}{j} \right\} \end{aligned} \quad (4.2)$$

For every fixed  $j$ , we have

$$\mathcal{A}(\gamma_j(u_n - u))^l = \sum_{m=1}^M \sum_{i=1}^N A_i^{lm} (\partial_{x_i} \gamma_j) (u_n^m - u^m) \xrightarrow{n \rightarrow \infty} 0 \quad \text{weakly in } L^p(\mathbb{R}^N).$$

Since  $\text{supp } \mathcal{A}(\gamma_j(u_n - u)) \subset \text{supp } \nabla \gamma_j \subset \{0 < \gamma_j < 1\}$ , whose closure is a compact set, and since  $\int_{\mathbb{R}^N} \mathcal{A}(\gamma_j(u_n - u)) dx = 0$  due to integration by parts, Lemma 3.5 is applicable to  $\mathcal{A}(\gamma_j(u_n - u))$  and it yields that

$$\|(-\Delta)^{-\frac{1}{2}} \mathcal{A}(\gamma_j(u_n - u))\|_{L^p(\mathbb{R}^N; \mathbb{R}^L)} \xrightarrow{n \rightarrow \infty} 0 \quad (4.3)$$

for fixed  $j$ . As a consequence of (4.3), we can select a subsequence  $k(n)$  of  $n$  (fast enough) such that

$$\|(-\Delta)^{-\frac{1}{2}} \mathcal{A}(\gamma_n(u_m - u))\|_{L^p(\mathbb{R}^N; \mathbb{R}^L)} \leq \frac{1}{n} \quad \text{for every } m \geq k(n). \quad (4.4)$$

Moreover, by Lemma 4.3 we can pass to another subsequence of  $k(n)$  (not relabeled) such that

$$\gamma_n(u_{k(n)} - u) \text{ is } \Omega\text{-tight in } L^p(\Omega; \mathbb{R}^M). \quad (4.5)$$

Now define  $\tilde{u}_n := \mathcal{P}(\gamma_n(u_n - u))$ , which is a bounded sequence in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$  satisfying  $\mathcal{A}\tilde{u}_n = 0$  on  $\mathbb{R}^N$ , and decompose  $\tilde{u}_n = \tilde{w}_n^0 + \dots + \tilde{w}_n^4$  according to Lemma 4.1 (again passing to a subsequence if necessary). We claim that the decomposition  $u_n = u + v_n + w_n + z_n$  with

$$\begin{aligned} v_n &:= \tilde{w}_n^0, \\ w_n &:= \tilde{w}_n^1, \\ z_n &:= (I - \mathcal{P})[\gamma_n(u_n - u)] + (1 - \gamma_n)(u_n - u) + \tilde{w}_n^2 + \tilde{w}_n^3 + \tilde{w}_n^4, \end{aligned}$$

then has the asserted properties. First note that  $v_n$  and  $w_n$  are bounded sequences in  $L^p$  and  $\mathcal{A}$ -free on  $\Omega$  by definition, whence the same holds for  $z_n = u_n - u - v_n - w_n$ . Since  $v_n$  satisfies (a) by construction, it remains to show that (b) and (c) hold.

- (c) Since  $\gamma_n(u_{k(n)} - u)$  is supported in  $\Omega$  and  $\Omega$ -tight in  $L^p(\Omega; \mathbb{R}^M)$ , it is  $\mathbb{R}^N$ -tight in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$ . Hence,  $\tilde{u}_n$  is  $\mathbb{R}^N$ -tight in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$  by Lemma 3.4 (iii). Consequently,

$$R_n := \tilde{w}_n^2 + \tilde{w}_n^3 + \tilde{w}_n^4 = \tilde{u}_n - \tilde{w}_n^0 - \tilde{w}_n^1$$

is  $\mathbb{R}^N$ -tight in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$ , and by the properties of  $\tilde{w}_n^j$ ,  $j = 2, 3, 4$ , we also have  $R_n \rightarrow 0$  in  $L^p(B; \mathbb{R}^M)$  for every open, bounded  $B \subset \mathbb{R}^N$ . Combined, this implies that

$$R_n \rightarrow 0 \text{ in } L^p(\mathbb{R}^N; \mathbb{R}^M). \quad (4.6)$$

Moreover, by Lemma 3.4 (iv), (4.4) yields that

$$(I - \mathcal{P})[\gamma_n(u_{k(n)} - u)] \rightarrow 0 \text{ strongly in } L^p(\mathbb{R}^N, \mathbb{R}^M). \quad (4.7)$$

As a consequence (4.6), (4.7) and the second line of (4.2), we now get that  $z_n \rightarrow 0$  in  $L^p(K; \mathbb{R}^M)$  for any compact  $K \subset \Omega$ .

- (b) Combined, (4.7) and (4.5) imply that  $\tilde{u}_n = \mathcal{P}[\gamma_n(u_{k(n)} - u)]$  is  $\Omega$ -tight in  $L^p(\Omega; \mathbb{R}^M)$ . Hence,  $w_n = \tilde{w}_n^1 = \tilde{u}_n - \tilde{w}_n^0 - R_n$  is  $\Omega$ -tight in  $L^p(\Omega; \mathbb{R}^M)$  as well, where we also used that  $\tilde{w}_n^0$  and  $R_n$  are  $\Omega$ -tight in  $L^p(\Omega; \mathbb{R}^M)$ , the former since it is equintegrable in  $L^p$  and the latter because of (4.6). In addition, we clearly have  $w_n = \tilde{w}_n^1 \rightarrow 0$  in  $(L^p + L^q)(\mathbb{R}^N; \mathbb{R}^M)$  for any  $q \in (1, p)$  by definition of  $\tilde{w}_n^1$ .  $\square$

The result of Lemma 4.4 could be improved if the domain admits a continuous extension operator for  $\mathcal{A}$ -free vector fields in  $L^p$  from  $\Omega$  to  $\mathbb{R}^N$ . However, to my knowledge, extension of  $\mathcal{A}$ -free fields has not yet been investigated even on bounded domains except in a few special cases such as gradient fields (e.g. [2]) and divergence-free fields [10]. In any case, for domains with compact boundary, the ideas already used in Lemma 4.4 suffice to obtain a refined decomposition without relying on general extension results. In comparison with Lemma 4.1, the decomposition now has an additional component  $w_n^5$  which carries concentrations at the boundary.

**Lemma 4.5.** *Let  $N \geq 2$ , let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open with compact boundary and let  $\mathcal{A}$  be a linear differential operator of first order satisfying (3.1). Moreover, suppose that  $u_n$  is a bounded,  $\mathcal{A}$ -free sequence in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$  with  $u_n \rightharpoonup u$  weakly in  $L^p$ . Then there exist a subsequence  $u_{k(n)}$  of  $u_n$  and six bounded,  $\mathcal{A}$ -free sequences  $w_n^0, \dots, w_n^5$  in  $L^p(\Omega; \mathbb{R}^M)$  such that*

$$u_{k(n)} = u + w_n^0 + w_n^1 + w_n^2 + w_n^3 + w_n^4 + w_n^5 \text{ for every } n \in \mathbb{N}$$

and the following properties hold:

- (a)  $w_n^0 \rightharpoonup 0$  weakly in  $L^p$ , and  $w_n^0$  is equiintegrable in  $L^p$ .
- (b)  $w_n^1$  is  $\Omega$ -tight in  $L^p$  and  $w_n^1 \rightarrow 0$  in  $L^p + L^q$  for every  $q \in (1, p)$ .
- (c)  $\chi_B w_n^2 \rightarrow 0$  in  $L^p$  for any bounded, measurable set  $B \subset \mathbb{R}^N$  and  $w_n^2 \rightarrow 0$  in  $L^p + L^q$  for every  $q \in (1, p)$ .
- (d)  $\chi_B w_n^3 \rightarrow 0$  in  $L^p$  for any bounded, measurable set  $B \subset \mathbb{R}^N$ ,  $w_n^3$  does not spread out in  $L^p$  and  $w_n^3$  does not concentrate in  $L^p$ .
- (e)  $w_n^4 \rightarrow 0$  in  $L^r + L^p$  for every  $r \in (p, \infty)$ .
- (f)  $\chi_E w_n^5 \rightarrow 0$  in  $L^p$  for any closed set  $E \subset \Omega$ .

Moreover, the component sequences  $w_n^0, \dots, w_n^4 \in L^p(\Omega; \mathbb{R}^M)$  can be extended to bounded,  $\mathcal{A}$ -free sequences in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$  satisfying (a)–(e) even on  $\mathbb{R}^N$ .

**Proof.** Using Lemma 3.5 as in the proof of Lemma 4.4, we can find a sequence of cut-off functions  $\gamma_n \in C^1(\mathbb{R}^N; [0, 1])$  and an associated subsequence  $k(n)$  of  $n$  such that

$$\begin{aligned} \{\gamma_n > 0\} &\subset \{x \in \Omega \mid \text{dist}(x; \partial\Omega) \geq \frac{1}{n}\} \text{ and} \\ \{0 < \gamma_n < 1\} &\subset \{x \in \Omega \mid \text{dist}(x; \partial\Omega) \leq \frac{2}{n}\} \end{aligned} \quad (4.8)$$

and

$$\|(-\Delta)^{-\frac{1}{2}} \mathcal{A}(\gamma_n(u_{k(n)} - u))\|_{L^p(\mathbb{R}^N; \mathbb{R}^L)} \leq \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0. \quad (4.9)$$

Once again employing Lemma 4.3 to extract another subsequence of  $k(n)$  (if necessary; not relabeled), we may also assume that

$$\chi_K \gamma_n(u_{k(n)} - u) \text{ is } \Omega\text{-tight in } L^p(\Omega; \mathbb{R}^M), \quad (4.10)$$

where  $K$  is a fixed compact set containing  $\partial\Omega$  in its interior. (Thus (4.10) essentially means that  $\gamma_n(u_{k(n)} - u)$  does not develop concentrations “at the boundary” of  $\Omega$ .) Decomposing

$$\mathcal{P}[\gamma_n(u_{k(n)} - u)] =: \tilde{u}_n = \tilde{w}_n^0 + \dots + \tilde{w}_n^4$$

according to Lemma 4.1, we define

$$\begin{aligned} w_n^j &:= \tilde{w}_n^j \text{ for } j = 0, 1, 2, 3, 4, \\ w_n^5 &:= (I - \mathcal{P})[\gamma_n(u_{k(n)} - u)] + (1 - \gamma_n)(u_{k(n)} - u). \end{aligned}$$

By construction,  $u_{k(n)} = u + w_n^0 + \dots + w_n^5$ ,  $w_n^0, \dots, w_n^5$  are bounded,  $\mathcal{A}$ -free sequences in  $L^p(\mathbb{R}^N; \mathbb{R}^M)$ , and the properties (a), (c), (d) and (e) are satisfied. It remains to show (b) and (f).

(f) By Lemma 3.4 (iv), (4.9) yields that

$$(I - \mathcal{P})[\gamma_n(u_{k(n)} - u)] \rightarrow 0 \text{ strongly in } L^p(\mathbb{R}^N, \mathbb{R}^M). \quad (4.11)$$

Moreover, if  $E$  is a closed subset of  $\Omega$ , the compact set  $\partial\Omega$  has positive distance to  $E$ , whence  $\chi_E(1 - \gamma_n)(u_{k(n)} - u) = 0$  for every  $n$  large enough by (4.8). Together with (4.11), this implies that  $\chi_E w_n^5 \rightarrow 0$  in  $L^p$ .



(b) By construction,  $w_n^1$  is  $\mathbb{R}^N$ -tight and satisfies  $w_n^1 \rightarrow 0$  in  $L^p + L^q$  for any  $q < p$ . Hence, it suffices to show that

$$\chi_K w_n^1 = \chi_K [\tilde{u}_n - \tilde{w}_n^0 - \tilde{w}_n^2 - \tilde{w}_n^3 - \tilde{w}_n^4] \text{ is } \Omega\text{-tight in } L^p(\Omega; \mathbb{R}^M)$$

with a compact set  $K \subset \mathbb{R}^N$  containing  $\partial\Omega$  in its interior. Combined, (4.11) and (4.10) imply that  $\chi_K \tilde{u}_n = \chi_K \mathcal{P}[\gamma_n(u_{k(n)} - u)]$  is  $\Omega$ -tight in  $L^p(\Omega; \mathbb{R}^M)$ . Moreover, for  $j \in \{0, 3, 4\}$ ,  $\tilde{w}_n^j$  does not concentrate in  $L^p$  whence  $\chi_K \tilde{w}_n^j$  is  $\Omega$ -tight in  $L^p$ . Finally,  $\chi_K \tilde{w}_n^2 \rightarrow 0$  in  $L^p$  and thus also is  $\Omega$ -tight in  $L^p$ .  $\square$

## 5 Proof of the main results

The proofs are grouped into four subsections. The first subsumes various results for Young measures which are needed later. The second contains the proofs of Theorem 2.12 and Corollary 2.13 while the third is dedicated to showing Theorem 2.5 and Theorem 2.15. In the final subsection, we discuss some of the assumptions of the aforementioned theorems by proving Proposition 2.2 and Proposition 2.14.

### 5.1 Auxiliary results

Possible oscillations of minimizing sequences will be discussed with Young measures as the main tool.

**Theorem 5.1** (fundamental theorem for Young measures [3, 13]). *Let  $\Omega \subset \mathbb{R}^N$  be measurable and let  $v_n : \Omega \rightarrow \mathbb{R}^M$  be a sequence of measurable functions. Then there exists a subsequence  $(u_{k(n)})$  and a family  $\nu = (\nu_x)_{x \in \Omega}$  of nonnegative Radon measures on  $\mathbb{R}^M$ , weak\*-measurable<sup>7</sup> in  $x$ , such that the following holds:*

(i)  $\nu_x(\mathbb{R}^M) \leq 1$  for a.e.  $x \in \Omega$ .

(ii) If  $\lim_{h \rightarrow \infty} \sup_{n \in \mathbb{N}} |\{ |v_{k(n)}| \geq h \} \cap \Omega \cap B_R(0)| = 0$  for every  $R > 0$ , then  $\nu_x(\mathbb{R}^M) = 1$  for a.e.  $x \in \Omega$ .

(iii) For every Carathéodory function  $f : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}$  such that  $f(\cdot, u_{k(n)})$  is equiintegrable<sup>8</sup> in  $L^1(\Omega)$ , we have that

$$\int_{\Omega} f(x, u_{k(n)}(x)) dx \xrightarrow{k \rightarrow \infty} \int_{\Omega} \int_{\mathbb{R}^M} f(x, \mu) d\nu_x(\mu) dx.$$

As a consequence of (iii),  $\nu$  is uniquely determined by  $(u_{k(n)})$  and it is called the Young measure generated by  $u_{k(n)}$ . Moreover, if  $\nu_x = \nu_a$  for a.e.  $x \in \Omega$  with a fixed  $a \in \Omega$ , then it is called a homogeneous Young measure. Another useful consequence of (iii) is the following.

<sup>7</sup>i.e.,  $x \mapsto \int_{\mathbb{R}^M} f(\mu) d\nu_x(\mu)$  is measurable for every  $f \in C_0(\mathbb{R}^M)$

<sup>8</sup>Note that equiintegrability in  $L^1$  in the sense of Definition 2.7 is equivalent to weak relative compactness in  $L^1$  by the de la Vallée-Poussin criterion.

**Corollary 5.2.** *Let  $1 \leq p < \infty$  and let  $(u_n) \subset L^p(\Omega; \mathbb{R}^M)$  be a bounded sequence which generates a Young measure  $\nu = (\nu_x)$ . Then  $u_n \rightharpoonup u$  weakly in  $L^p$  with  $u(x) = \langle \nu_x, id \rangle := \int_{\mathbb{R}^M} \mu d\nu_x(\mu)$  for a.e.  $x \in \Omega$ , and  $u_n \rightarrow u$  locally in measure if and only if  $\nu_x = \delta_{u(x)}$  for a.e.  $x \in \Omega$ . Here,  $\delta_\mu$  denotes the Dirac mass at the point  $\mu \in \mathbb{R}^M$ .*

Young measures generated by  $\mathcal{A}$ -free sequences on bounded domains have been characterized in [7]. Here, we only employ a version of an approximation result of [7] used to “localize” the Young measure, adapted to the whole space instead of bounded domains.

**Proposition 5.3** (cf. Proposition 3.8 in [7]<sup>9</sup>). *Let  $1 \leq p < \infty$  and let  $\nu = (\nu_x)$  be a Young measure generated by a bounded sequence  $(u_n) \subset L^p(\mathbb{R}^N; \mathbb{R}^M)$  which does not concentrate in  $L^p$  and satisfies  $\|(-\Delta)^{-\frac{1}{2}} \mathcal{A}u_n\|_{L^p} \rightarrow 0$ . Then for a.e.  $a \in \mathbb{R}^N$ , there exists a sequence  $(w_n) \subset L^p_{\sharp}(\mathbb{R}^N; \mathbb{R}^M)$ , which is bounded in  $L^p(Q; \mathbb{R}^M)$ , does not concentrate in  $L^p$  and satisfies  $\mathcal{A}w_n = 0$  in  $\mathbb{R}^N$ , such that  $w_n$  generates the homogeneous Young measure  $\nu_a$  and  $\int_Q w_n dx = \langle \nu_a, id \rangle = \int_{\mathbb{R}^M} \mu d\nu_x(\mu)$  for every  $n$ .*

**Proof.** Let  $B_k$  be a sequence of concentric open balls in  $\mathbb{R}^N$  with radius  $k \in \mathbb{N}$ . For any  $k$  and  $p' := \frac{p}{p-1}$ , we have

$$\begin{aligned} & \|\mathcal{A}u_n\|_{W^{-1,p}(B_{k+1}; \mathbb{R}^M)} \\ &= \sup \left\{ \int_{\mathbb{R}^N} u_n \cdot \mathcal{A}^* \eta dx \mid \eta \in C_c^\infty(B_{k+1}; \mathbb{R}^L) \text{ with } \|\eta\|_{W^{1,p'}} \leq 1 \right\} \\ &\leq \sup \left\{ \int_{\mathbb{R}^N} u_n \cdot \mathcal{A}^* \eta dx \mid \eta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^L) \text{ with } \|\nabla \eta\|_{L^{p'}} \leq 1 \right\} \\ &\leq C \|(-\Delta)^{-\frac{1}{2}} \mathcal{A}u_n\|_{L^p(\mathbb{R}^N; \mathbb{R}^M)} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Hence, we may apply Proposition 3.8 in [7] to  $u_n$  restricted to  $B_k(0)$  (which generates  $\nu$  restricted to  $B_k(0)$ ) for every  $k \in \mathbb{N}$ , which yields the assertion.  $\square$

As an immediate consequence, we have the following.

**Proposition 5.4.** *Let  $\Omega \subset \mathbb{R}^N$  be open, let  $1 \leq p < \infty$  and let  $\nu = (\nu_x)$  be a Young measure generated a bounded sequence  $(v_n) \subset L^p(\mathbb{R}^N; \mathbb{R}^M) \cap \ker \mathcal{A}$  such that  $(v_n)$  is equiintegrable in  $L^p$  and  $v_n \rightharpoonup 0$  weakly in  $L^p$ , and suppose that  $f$  satisfies (f:0) and (f:1). Then, for a.e.  $a \in \Omega$  such that  $f(a, \cdot)$  is  $\mathcal{A}$ -quasiconvex at  $\xi \in \mathbb{R}^M$ , we have*

$$\int_{\mathbb{R}^M} f(a, \xi + \mu) d\nu_a(\mu) \geq f(a, \xi) \quad (5.1)$$

Moreover, for a.e.  $a \in \Omega$  such that  $f(a, \cdot)$  is strictly  $p$ - $\mathcal{A}$ -quasiconvex at  $\xi$ ,

$$\text{equality in (5.1) implies that } \nu_a = \delta_0, \quad (5.2)$$

<sup>9</sup>Beware that the notion of equiintegrability used in [7] is equivalent to what we term “does not concentrate” and hence coincides with our definition only on domains with finite measure.

where  $\delta_0$  denotes the Dirac mass concentrated at the point  $0 \in \mathbb{R}^M$ . In particular, given  $u \in L^p(\Omega; \mathbb{R}^M)$  such that  $f(x, \cdot)$  is  $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x \in \Omega$ , we have that

$$\liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u(x) + v_n(x)) dx \geq \int_{\Omega} f(x, u(x)) dx \quad (5.3)$$

and if  $f(x, \cdot)$  is strictly  $p$ - $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x \in \Omega$ , then

$$\text{equality in (5.3) implies that } v_n \rightarrow 0 \text{ in } L^p. \quad (5.4)$$

**Proof.** The first assertion (5.1) is a simple consequence of Proposition 5.3, Theorem 5.1 and the definition of  $\mathcal{A}$ -quasiconvexity. Here, note that we may assume that the sequence  $w_n$  of Proposition 5.3 actually belongs to  $C_{\sharp}^{\infty}(\mathbb{R}^N; \mathbb{R}^M)$ , because if not, we can replace it with a mollified sequence  $\tilde{w}_n$  (mollified as usual by convolution with a smooth kernel with small support) such that  $\tilde{w}_n - w_n \rightarrow 0$  strongly in  $L_{\sharp}^p$ , whence  $\tilde{w}_n$  inherits all properties of  $w_n$ . To show (5.2), we again employ Proposition 5.3 to choose a sequence  $w_n$  of smooth functions which is equiintegrable in  $L_{\text{per}}^p$  and generates  $\nu_a$ . If (5.1) holds with equality, Theorem 5.1 and the strict  $p$ - $\mathcal{A}$ -quasiconvexity of  $f$  imply that  $g(t_n, T) \rightarrow 0$  with  $t_n := \int_Q |w_n|$  and  $T := \sup_n \int_Q |w_n|^p$  (recall that  $g$  is decreasing in its second variable). This is possible only if  $t_n \rightarrow 0$ , whence  $w_n \rightarrow 0$  in  $L_{\sharp}^1$  and  $\nu_a = \delta_0$  due to Corollary 5.2. Finally, (5.1) and (5.2) imply (5.3) and (5.4), respectively, by Theorem 5.1. As to (5.4), we first get that  $v_n \rightarrow 0$  locally in measure, which in turn implies that  $v_n \rightarrow 0$  in  $L^p$  by Vitali's theorem, since  $v_n$  is equiintegrable in  $L^p$ .  $\square$

*Remark 5.5.* In fact, the converse of Proposition 5.4 is also true. More precisely, if  $f$  satisfies (f:0) and (f:1) then the following holds for a.e.  $a \in \Omega$  and every  $\xi \in \mathbb{R}^M$ : If (5.1) is valid for every homogeneous  $\mathcal{A}$ -free  $L_{\sharp}^p$ -Young measure<sup>10</sup>  $\nu_a$  such that  $\langle \nu_a, id \rangle = 0$ , then  $f$  is  $\mathcal{A}$ -quasiconvex at  $\xi$ , and if (5.1) and (5.2) hold for every such  $\nu_a$ , then  $f$  is strictly  $p$ - $\mathcal{A}$ -quasiconvex at  $\xi$ . Corresponding converse statements of (5.3) and (5.4) also hold, at least if  $f$  is bounded from below by a constant: If (5.3) is satisfied for every  $(v_n) \subset L^p(\Omega; \mathbb{R}^M)$  which is  $\mathcal{A}$ -free, equiintegrable in  $L^p$  and satisfies  $v_n \rightarrow 0$  weakly in  $L^p$ , then  $f(x, \cdot)$  is  $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x$ , and if (5.3) and (5.4) hold for every such  $(v_n)$ , then  $f(x, \cdot)$  is strictly  $p$ - $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x$ . The proof is omitted. It is not entirely trivial as it involves a problem of measurable selection on the level of the associated Young measures (cf. the concluding remark in [17]).

*Remark 5.6.* Given  $\Omega \subset \mathbb{R}^N$  open, any Young measure generated by a bounded,  $\mathcal{A}$ -free sequence  $(u_n) \subset L^p(\Omega; \mathbb{R}^M)$  is also generated by a bounded,  $\mathcal{A}$ -free sequence  $(\tilde{u}_n) \subset L^p(\Omega; \mathbb{R}^M)$  which is equiintegrable in  $L^p$ . For instance, one may take  $\tilde{u}_n := u + v_n$  with  $u$  and  $v_n$  defined in Lemma 4.4.

Below, (f:2) is used exclusively in form of the following simple observation.

**Proposition 5.7.** *Let  $1 \leq p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open and suppose that  $f$  satisfies (f:0) and (f:2). Then the map  $u \mapsto f(\cdot, u)$ ,  $L^p(\Omega; \mathbb{R}^M) \rightarrow L^1(\Omega)$ , is uniformly continuous on bounded subsets of  $L^p(\Omega; \mathbb{R}^M)$ .*

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<sup>10</sup>i.e., a homogeneous Young measure generated by an  $\mathcal{A}$ -free, bounded sequence  $(w_n) \in L_{\sharp}^p(\mathbb{R}^N; \mathbb{R}^M)$

**Proof.** By (f:2) and Hölder's inequality, we have

$$\int_{\Omega} |f(x, u) - f(x, v)| \, dx \leq C \left( \|u\|_{L^p}^{\frac{p-1}{p}} + \|v\|_{L^p}^{\frac{p-1}{p}} + \|h\|_{L^p}^{\frac{p-1}{p}} \right) \|u - v\|_{L^p}$$

for any  $u, v \in L^p(\Omega; \mathbb{R}^M)$ . □

## 5.2 Domains with compact boundary

As we shall see, the proof of Theorem 2.12 heavily relies on the corresponding decomposition lemma of Section 4, Lemma 4.5. In a sense made precise below, it exploits that the component sequences do not interact with each other in  $f$ , essentially due to Proposition 5.7.

**Proposition 5.8.** *Let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open with compact boundary and suppose that  $f$  satisfies (f:0)–(f:2). Moreover, let  $u_n$  be an  $\mathcal{A}$ -free, bounded sequence which weakly converges to a function  $u$  in  $L^p(\Omega; \mathbb{R}^M)$ , and let*

$$u_n = u + w_n^0 + \dots + w_n^5$$

be a decomposition as in Lemma 4.5. Then for any  $j_0 \in \{1, \dots, 5\}$ , we have

$$f(\cdot, u_n) - f(\cdot, u_n - w_n^{j_0}) - [f(\cdot, w_n^{j_0}) - f(\cdot, 0)] \xrightarrow{n \rightarrow \infty} 0 \text{ in } L^1(\Omega). \quad (5.5)$$

In particular,

$$f(\cdot, u_n) - f(\cdot, u + w_n^0) - \sum_{j=1}^5 [f(\cdot, w_n^j) - f(\cdot, 0)] \xrightarrow{n \rightarrow \infty} 0 \text{ in } L^1(\Omega). \quad (5.6)$$

This kind of result is fairly standard in the context of bounded domains, where only two component sequences appear in the decomposition lemma besides the weak limit (i.e., oscillations and concentrations); in particular, it is implicitly used in [8]. For a sequence of gradients on an unbounded domain, a corresponding result was obtained in [11]. In our present context, it would still be possible to give a proof relying on the abstract framework developed in [11], which provides a way to handle the numerous different properties of the component sequences  $w_n^j$  in a more systematic way. However, the case of functionals is somewhat simpler than that of operators mapping into a Banach space which allows a reasonably-sized self-contained proof “by hand”, although our proof of (5.5) below only discusses the case  $j_0 = 5$  in full detail, the other cases being more or less analogous.

**Proof of Proposition 5.8.** For  $\delta > 0$  let

$$\Omega_\delta := \{x \in \Omega \mid \text{dist}(x; \partial\Omega) > \delta\}.$$

We first show (5.5) for  $j_0 = 5$ . Fix  $\varepsilon > 0$  and define

$$E = E(\delta) := \Omega \setminus \Omega_\delta,$$

and choose  $\delta = \delta(\varepsilon) \in (0, 1)$  small enough such that

$$\sup_{n \in \mathbb{N}} \|\chi_{E(\delta)}(u_n - w_n^5)\|_{L^p} \leq \|\chi_{E(\delta)}u\|_{L^p} + \sum_{j=0}^4 \sup_{n \in \mathbb{N}} \|\chi_{E(\delta)}w_n^j\|_{L^p} < \varepsilon. \quad (5.7)$$

Note that such a choice of  $\delta$  is possible because the constant sequence  $u$ , as well as  $\chi_{\Omega \setminus \Omega_1} w_n^1, \dots, \chi_{\Omega \setminus \Omega_1} w_n^4$ , are  $\Omega$ -tight in  $L^p$ , the latter as a consequence of their properties (a)–(e) listed in Lemma 4.5. In addition, we have

$$\chi_{\Omega \setminus E(\delta)} w_n^5 = \chi_{\Omega_\delta} w_n^5 \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{in } L^p \quad \text{for any fixed } \delta \in (0, 1), \quad (5.8)$$

by definition of  $w_n^5$ . Together with the uniform continuity of  $v \mapsto f(\cdot, v)$ ,  $L^p \rightarrow L^1$ , on bounded subsets of  $L^p$  as derived in Proposition 5.7, (5.7) and (5.8) imply that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\Omega} |f(x, u_n) - f(x, u_n - w_n^5) - [f(x, w_n^5) - f(x, 0)]| dx \\ & \leq \limsup_{\varepsilon \rightarrow 0} \sup_{n \in \mathbb{N}} \int_E |f(x, (u_n - w_n^5) + w_n^5) - f(x, w_n^5)| dx \\ & \quad + \limsup_{\varepsilon \rightarrow 0} \sup_{n \in \mathbb{N}} \int_E |f(x, 0) - f(x, u_n - w_n^5)| dx \\ & \quad + \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\Omega \setminus E} |f(x, u_n) - f(x, u_n - w_n^5)| dx \\ & \quad + \limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\Omega \setminus E} |f(x, w_n^5) - f(x, 0)| dx \\ & = 0, \end{aligned} \quad (5.9)$$

with  $E = E(\delta(\varepsilon))$ . This concludes the proof of (5.5) for  $j_0 = 5$ . Essentially, we exploited that  $w_n^5 \rightarrow 0$  in  $L^p(\Omega \setminus E; \mathbb{R}^M)$  while at the same time the remaining components  $u$  and  $w_n^0, \dots, w_n^4$  are uniformly close to zero in  $L^p(E; \mathbb{R}^M)$  by their properties obtained in Lemma 4.5. The same kind of argument also yields (5.5) for  $j_0 = 1, \dots, 4$ , employing different choices for  $E$  which now also depend on  $n$ , adapted to the properties of the component sequence  $w_n^{j_0}$  which is separated from the rest. More precisely, we use

$$\begin{aligned} E_n &= E_n(\delta) := \Omega_\delta \cap \{|w_n^3| < \delta\} \cap \{|w_n^2| < 1\} \setminus B_{\frac{1}{5}}(0) & \text{if } j_0 = 4, \\ E_n &= E_n(\delta) := \Omega_\delta \cap \{|w_n^3| > \delta\} \cap \{|w_n^2| < 1\} \setminus B_{\frac{1}{5}}(0) & \text{if } j_0 = 3, \\ E_n &= E_n(\delta) := \Omega_\delta \cap \{|w_n^2| > \frac{1}{\delta}\} \setminus B_{\frac{1}{5}}(0) & \text{if } j_0 = 2, \\ E_n &= E_n(\delta) := \Omega_\delta \cap \{|w_n^1| > \frac{1}{\delta}\} \cap B_{\frac{1}{5}}(0) & \text{if } j_0 = 1, \end{aligned}$$

where in each case  $\delta = \delta(\varepsilon, j_0)$  is chosen small enough such that

$$\sup_{n \in \mathbb{N}} \|\chi_{E_n}(u_n - w_n^{j_0})\|_{L^p} < \varepsilon \quad \text{and} \quad \|\chi_{\Omega \setminus E_n} w_n^{j_0}\|_{L^p} \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{for fixed } \delta.$$

As before, it is not difficult to see that the choice is possible due to the properties (a)–(e) of  $w_n^j$  obtained in Lemma 4.5, and these also yield that  $\chi_{\Omega \setminus E_n} w_n^{j_0} \rightarrow 0$  in  $L^p$  in each case; we omit the (lengthy) details. Repeating (5.9) in each case then gives (5.5) for  $j_0 = 1, \dots, 4$ . Finally, note that (5.6) can be obtained by applying (5.5) successively to the sequences  $\tilde{u}_n^{(j_0)} := u + w_n^0 + \sum_{j=j_0}^5 w_n^j$ , for  $j_0 = 1, \dots, 5$ .  $\square$

*Remark 5.9.* In the preceding proof, we exploited that  $u \mapsto f(\cdot, u)$ ,  $L^p \rightarrow L^1$ , is uniformly continuous on bounded subsets of  $L^p$ , and not just on bounded subsets on  $\mathcal{U}_A$  (as a closed

subspace of  $L^p$ ). In view of the fact that we are only interested in  $F$  defined on  $\mathcal{U}_{\mathcal{A}}$ , this is a somewhat artificial assumption on the functional. It remains unclear if this is just a technical deficiency. If  $\Omega = \mathbb{R}^N$ , we can apply the projector  $\mathcal{P}$  of Section 4 to any sequences in the proof without having to face the problem of  $\mathcal{A}$ -free extension. In this case, it is possible to work under the assumption that  $F_E(u) := \int_E f(x, u) dx$  is uniformly continuous on bounded subsets of  $\mathcal{U}_{\mathcal{A}}$  for any  $E \subset \mathbb{R}^N$  measurable, with a modulus of continuity which is also uniform in  $E$ . There is still no obvious way to do the proof just using uniform continuity of  $F$  on bounded subsets of  $\mathcal{U}_{\mathcal{A}}$ , though.

If  $f$  is  $\mathcal{A}$ -quasiconvex and  $F$  is bounded from below, the assertion of Proposition 5.8 can be enhanced. As a byproduct, we get weak lower semicontinuity of  $F$  on  $\mathcal{U}_{\mathcal{A}}$ .

**Proposition 5.10.** *Let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open with compact boundary, let  $u_n$  be an  $\mathcal{A}$ -free, bounded sequence which weakly converges to a function  $u$  in  $L^p(\Omega; \mathbb{R}^M)$ , and let*

$$u_n = u + w_n^0 + \dots + w_n^5$$

*be a decomposition as in Lemma 4.5. Moreover, suppose that  $f$  satisfies (f:0)–(f:2), that  $f(x, \cdot)$  is  $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x \in \Omega$  and that  $\inf\{F(v) \mid v \in \mathcal{U}_{\mathcal{A}}\} > -\infty$ . Then*

$$\begin{aligned} \liminf_{n \rightarrow \infty} [F(u + w_n^0) - F(u)] &\geq 0, \\ \liminf_{n \rightarrow \infty} [F(w_n^j) - F(0)] &\geq 0 \quad \text{for } j = 1, \dots, 5, \end{aligned} \tag{5.10}$$

and

$$\liminf F(u_n) \geq F(u). \tag{5.11}$$

If, in addition,  $\limsup F(u_n) \leq F(u)$ , then even have that

$$F(u + w_n^0) \xrightarrow{n \rightarrow \infty} F(u) \quad \text{and} \quad F(w_n^j) \xrightarrow{n \rightarrow \infty} F(0) \quad \text{for } j = 1, \dots, 5. \tag{5.12}$$

**Proof.** The first inequality in (5.21) is an immediate consequence of Proposition 5.4, since  $w_n^0$  is equiintegrable and  $f(\cdot, u(x))$  is  $\mathcal{A}$ -quasiconvex at  $u(x)$ , for a.e.  $x \in \Omega$ . To check the remaining inequalities, fix an  $\varepsilon > 0$  and choose  $u_\varepsilon^* \in \mathcal{U}_{\mathcal{A}}$  such that  $\inf\{F(v) \mid v \in \mathcal{U}_{\mathcal{A}}\} + \varepsilon \geq F(u_\varepsilon^*)$ . In particular,

$$\liminf_{n \rightarrow \infty} [F(u_\varepsilon^* + w_n^j) - F(u_\varepsilon^*)] \geq -\varepsilon. \tag{5.13}$$

Moreover, by applying Proposition 5.8 to the sequences  $\tilde{u}_n = \tilde{u}_n(\varepsilon, j) := u_\varepsilon^* + w_n^j$  (which is also an admissible decomposition of  $\tilde{u}_n$ ) for fixed  $\varepsilon$  and  $j$ , we get that

$$[F(u_\varepsilon^* + w_n^j) - F(u_\varepsilon^*)] - [F(w_n^j) - F(0)] \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } j = 1, \dots, 5.$$

Using this to replace  $F(u_\varepsilon^* + w_n^j) \geq F(u_\varepsilon^*)$  in (5.13), we infer that

$$\liminf_{n \rightarrow \infty} [F(w_n^j) - F(0)] \geq -\varepsilon \quad \text{for } j = 1, \dots, 5.$$

Since this is true for any  $\varepsilon > 0$ , this concludes the proof of (5.21). As to the remaining assertions, first note that by (5.6) in Proposition 5.8,

$$\begin{aligned} & \liminf F(u_n) - F(u) \\ & \geq \liminf [F(u + w_n^0) - F(u)] + \sum_{j=1}^5 \liminf [F(w_n^j) - F(0)], \end{aligned} \quad (5.14)$$

whence  $\liminf F(u_n) \geq F(u)$  due to (5.21). Finally, assume that  $\limsup F(u_n) \leq F(u)$ . Proposition 5.8 then allows us to replace (5.14) by

$$0 \geq \limsup [F(u + w_n^0) - F(u)] + \sum_{j=1}^5 \liminf [F(w_n^j) - F(0)], \quad (5.15)$$

where each of the six summands is nonnegative due to (5.21). Hence

$$0 \geq \limsup [F(u + w_n^0) - F(u)] \geq \liminf [F(u + w_n^0) - F(u)] \geq 0,$$

which implies the first line of (5.23). The other lines can be obtained analogously, with suitable variants of (5.15).  $\square$

**Proof of Theorem 2.12.** From any given subsequence of  $u_n$  (not relabeled, specified later), we can extract another subsequence  $u_{k(n)}$  such that

$$u_{k(n)} = u + w_n^0 + \dots + w_n^5$$

according to Lemma 4.5. Since  $\limsup F(u_n) \leq F(u)$  by assumption, Proposition 5.10 yields that

$$F(u + w_n^0) \rightarrow F(u) \quad \text{and} \quad F(w_n^j) \rightarrow F(0) \quad \text{for } j = 1, \dots, 5. \quad (5.16)$$

With (5.16) as a starting point, we are now ready to prove (i)–(iv). Throughout, we argue by contradiction.

(i) Suppose that  $u_n$  does not converge to  $u$  locally in measure. Hence it has a subsequence (not relabeled) such that

$$\liminf_{n \rightarrow \infty} |\Omega' \cap \{|u_n - u| > \delta\}| \geq \varepsilon \quad (5.17)$$

for an  $\varepsilon > 0$ , a  $\delta > 0$  and a bounded, open set  $\Omega' \subset \Omega$ . The properties of  $w_n^j$  obtained in the decomposition lemma entail that for  $j = 1, \dots, 5$ ,  $w_n^j \rightarrow 0$  locally in measure. In particular, we can replace  $u_n$  by  $u + w_n^0$  in (5.17). But by (5.16) for  $w_n^0$  and Proposition 5.4,  $w_n^0 \rightarrow 0$  in  $L^p$  and thus also locally in measure, contradicting (5.17).

(ii) Suppose that  $u_n$  does concentrate in  $L^p$ . Then it has a subsequence (not relabeled) such that  $\liminf_{n \rightarrow \infty} \|\chi_{E_n} u_n\|_{L^p} > 0$  for suitable measurable sets  $E_n \subset \Omega$  with  $|E_n| \rightarrow 0$ . In particular,

$$\liminf_{n \rightarrow \infty} \|\chi_{E_{k(n)}} u_{k(n)}\|_{L^p} > 0 \quad (5.18)$$

Recall that by the properties of the component sequences in Lemma 4.5,  $w_n^3$  and  $w_n^4$  do not concentrate in  $L^p$ , while  $(w_n^1)$ ,  $(w_n^2)$  and  $(w_n^5)$  are elements of  $\Phi_c$ . Hence, by assumption,

(5.16) for  $j = 1, 2, 5$  implies that  $w_n^1, w_n^2$  and  $w_n^3$  converge to zero strongly in  $L^p$ . In particular,  $u_{k(n)} = u + w_n^0 + \dots + w_n^5$  does not concentrate in  $L^p$ , which contradicts (5.18).

(iii) Suppose that  $\chi_{\{s^{-1} < |u_n| < s\}} u_n$  is not  $\mathbb{R}^N$ -tight in  $L^p$  for an  $s > 1$ . Then  $u_n$  has a subsequence (not relabeled) such that  $\liminf_{n \rightarrow \infty} \|\chi_{\{s^{-1} < |u_n| < s\} \setminus B_{R_n}} u_n\|_{L^p} > 0$  for a suitable sequence of balls  $B_{R_n}$  centered at zero with radius  $R_n \rightarrow \infty$ . In particular,

$$\liminf_{n \rightarrow \infty} \|\chi_{\{s^{-1} < |u_{k(n)}| < s\} \setminus B_{R_{k(n)}}} u_{k(n)}\|_{L^p} > 0. \quad (5.19)$$

By the properties of the component sequences in Lemma 4.5, this is only possible if (5.19) also holds for  $w_n^3$  instead of  $u_{k(n)}$ . However,  $(w_n^3) \in \Phi_3$ , whence by assumption, (5.16) for  $j = 3$  implies that  $w_n^3 \rightarrow 0$  in  $L^p$ .

(iv) Suppose that  $u_n$  does spread out in  $L^p$ . Then it has a subsequence (not relabeled) such that  $\liminf_{n \rightarrow \infty} \|\chi_{|u_n| < \delta_n} u_n\|_{L^p} > 0$  for a suitable sequence  $\delta_n \rightarrow 0^+$ . In particular,

$$\liminf_{n \rightarrow \infty} \|\chi_{|u_{k(n)}| < \delta_{k(n)}} u_{k(n)}\|_{L^p} > 0. \quad (5.20)$$

As before, by Lemma 4.5,  $u$  and  $w_n^j$  for  $j \neq 4$  cannot contribute to (5.20), while  $(w_n^4) \in \Phi_4$ . Hence by assumption, (5.16) for  $j = 4$  implies that  $w_n^4 \rightarrow 0$  in  $L^p$ , contradicting (5.20).

Last but not least, observe that if the conclusions of (i)–(iv) all hold, then  $u_n$  is equiintegrable in  $L^p$  and  $u_n \rightarrow u$  locally in measure. By Vitali's theorem, this entails that  $u_n \rightarrow u$  strongly in  $L^p$ .  $\square$

**Proof of Corollary 2.13.** Essentially, (ii.1)–(ii.3) can be obtained by arguing as in (ii) in the proof of Theorem 2.15. We omit the details.  $\square$

### 5.3 General domains

In complete analogy to Proposition 5.8 and Proposition 5.10, using Lemma 4.4 instead of Lemma 4.5, we have the following.

**Proposition 5.11.** *Let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open and suppose that  $f$  satisfies (f:0)–(f:2). Moreover, let  $u_n$  be an  $\mathcal{A}$ -free, bounded sequence which weakly converges to a function  $u$  in  $L^p(\Omega; \mathbb{R}^M)$ , and let*

$$u_n = u + v_n + w_n + z_n$$

*be a decomposition as in Lemma 4.4. Then for  $q_n = w_n$  as well as for  $q_n = z_n$ , we have that*

$$f(\cdot, u_n) - f(\cdot, u_n - q_n) - [f(\cdot, q_n) - f(\cdot, 0)] \xrightarrow{n \rightarrow \infty} 0 \text{ in } L^1(\Omega).$$

*In particular,*

$$f(\cdot, u_n) - f(\cdot, u + v_n) - [f(\cdot, w_n) - f(\cdot, 0)] - [f(\cdot, z_n) - f(\cdot, 0)] \xrightarrow{n \rightarrow \infty} 0 \text{ in } L^1(\Omega).$$



**Proposition 5.12.** *Let  $1 < p < \infty$ , let  $\Omega \subset \mathbb{R}^N$  be open, let  $u_n$  be an  $\mathcal{A}$ -free, bounded sequence which weakly converges to a function  $u$  in  $L^p(\Omega; \mathbb{R}^M)$ , and let*

$$u_n = u + v_n + w_n + z_n$$

*be a decomposition as in Lemma 4.4. Moreover, suppose that  $f$  satisfies (f:0)–(f:2), that  $f(x, \cdot)$  is  $\mathcal{A}$ -quasiconvex at  $u(x)$  for a.e.  $x \in \Omega$  and that  $\inf\{F(v) \mid v \in \mathcal{U}_{\mathcal{A}}\} > -\infty$ . Then we have that*

$$\begin{aligned} \liminf_{n \rightarrow \infty} [F(u + v_n) - F(u)] &\geq 0, \\ \liminf_{n \rightarrow \infty} [F(w_n) - F(0)] &\geq 0, \quad \liminf_{n \rightarrow \infty} [F(z_n) - F(0)] \geq 0, \end{aligned} \tag{5.21}$$

and

$$\liminf F(u_n) \geq F(u). \tag{5.22}$$

If, in addition,  $\limsup F(u_n) \leq F(u)$ , then even have that

$$F(u + v_n) \xrightarrow{n \rightarrow \infty} F(u), \quad F(w_n) \xrightarrow{n \rightarrow \infty} F(0) \quad \text{and} \quad F(z_n) \xrightarrow{n \rightarrow \infty} F(0). \tag{5.23}$$

**Proof of Theorem 2.5.** As already observed in Remark 2.6, it suffices to show that  $F$  is lower semicontinuous along sequences in  $\mathcal{U}_{\mathcal{A}}$  which weakly converge in  $L^p$ , and this is due to Proposition 5.12.  $\square$

**Proof of Theorem 2.15.** The proof is analogous to the one of Theorem 2.12, substituting Lemma 4.4 for Lemma 4.5 and and Proposition 5.12 for Proposition 5.10.  $\square$

## 5.4 Proof of Proposition 2.2 and Proposition 2.14

To prove the characterization of strict  $p$ - $\mathcal{A}$ -quasiconvexity of Proposition 2.2, we need a decomposition lemma for  $\mathcal{A}$ -free sequences of periodic functions on  $\mathbb{R}^N$ .

**Lemma 5.13** (cf. Lemma 2.15 in [7]). *Let  $Q := (0, 1)^N$ , let  $1 < p < \infty$ , let  $(u_n) \subset L^p(Q; \mathbb{R}^M)$  be a bounded sequence with  $\int_Q u_n = 0$ , and suppose that  $\mathcal{A}u_n = 0$  on  $\mathbb{R}^N$ . Here, functions in  $L^p(Q; \mathbb{R}^M)$  are identified with their  $Q$ -periodic extension to  $\mathbb{R}^N$ . Then there exists a subsequence  $u_{k(n)}$  of  $u_n$  and a bounded sequence  $(v_n) \subset L^p(Q; \mathbb{R}^M)$  such that  $\mathcal{A}v_n = 0$  on  $\mathbb{R}^N$ ,  $\int_Q v_n = 0$ ,  $v_n$  is equiintegrable in  $L^p$  and  $u_{k(n)} - v_n \rightarrow 0$  locally in measure.*

**Proof.** To be precise, Lemma 2.15 in [7] is stated for functions  $u_n$  defined on a bounded domain  $\Omega \subset \mathbb{R}^N$  instead of periodic functions on  $\mathbb{R}^N$ , but the construction in the proof actually yields a sequence  $v_n \in L^p(\tilde{Q}; \mathbb{R}^M)$ , bounded and equiintegrable in  $L^p$  with  $\chi_{\Omega}(u_n - v_n) \rightarrow 0$  locally in measure, which is defined on a given open cube  $\tilde{Q} \in \mathbb{R}^N$  containing  $\Omega$ . In addition,  $v_n$  is  $\mathcal{A}$ -free on  $\mathbb{R}^N$  if extended  $\tilde{Q}$ -periodically. Since any open cube  $\tilde{Q}$  containing  $\Omega$  is admissible in [7], we may use  $\tilde{Q} := \Omega := Q$  in our context. (In fact, some of the steps in the proof could be simplified as well, as in our case there is no need to extend from  $\Omega$  to  $\tilde{Q}$ -periodic functions.)  $\square$

**Proposition 5.14.** *Let  $Q := (0, 1)^N$ , let  $1 < p < \infty$ , and suppose that  $f : \mathbb{R}^M \rightarrow \mathbb{R}$  is a continuous function which satisfies (f:1) and (f:2) where  $h(x)$  is replaced by a constant. If  $u_{k(n)}$  and  $v_n$  denote the sequences of Lemma 5.13, then we have*

$$f(u_{k(n)}) - f(v_n) - [f(u_{k(n)} - v_n) - f(0)] \xrightarrow{n \rightarrow \infty} 0 \text{ in } L^1(Q)$$

**Proof.** This is analogous to the proof of Proposition 5.8.  $\square$

**Proof of Proposition 2.2.** We want to show that  $f(x, \cdot)$  is strictly  $p$ - $\mathcal{A}$ -quasi-convex at  $\xi \in \mathbb{R}^M$  if and only if (2.1) holds. For simplicity, omit  $x$  in the following, and we assume that  $\xi = 0$  (otherwise use  $\tilde{f}(\mu) := f(x, \mu - \xi)$ ).

“only if”: Obviously, strict  $p$ - $\mathcal{A}$ -quasiconvexity at 0 implies  $\mathcal{A}$ -quasiconvexity at 0. Now suppose that

$$\int_Q [f(\varphi_n(y)) - f(0)] dy \rightarrow 0$$

for a sequence  $(\varphi_n) \in \Phi_0$ , i.e.,  $(\varphi_n) \in C_{\#}^\infty(Q; \mathbb{R}^M)$  is  $\mathcal{A}$ -free and bounded and equiintegrable in  $L^p(Q; \mathbb{R}^M)$  with weak limit zero. By the definition of strict  $p$ - $\mathcal{A}$ -quasiconvexity at 0, we infer that

$$g(\int_Q |\varphi_n|, \bar{T}) \rightarrow 0 \text{ with } \bar{T} := \sup_{n \in \mathbb{N}} \int_Q |\varphi_n|^p \quad (5.24)$$

Since  $g(t, \bar{T})$  is increasing in  $t$  and nonzero whenever  $t > 0$ , this is possible only if  $\int_Q |\varphi_n| \rightarrow 0$ , which in turn implies that  $\varphi_n \rightarrow 0$  locally in measure.

“if”: For  $t, T \geq 0$  define

$$g(t, T) := \inf \left\{ \int_Q [f(\varphi(y)) - f(0)] dy \mid \varphi \in \phi_{\mathcal{A}}, \int_Q |\varphi| \geq t, \int_Q |\varphi|^p \leq T \right\},$$

with the convention that  $g(t, T) = +\infty$  if no admissible  $\varphi$  exists. Here, recall that  $\phi_{\mathcal{A}} := \{\varphi \in C_{\#}^\infty(\mathbb{R}^N; \mathbb{R}^M) \mid \mathcal{A}\varphi = 0 \text{ on } \mathbb{R}^N \text{ and } \int_Q \varphi = 0\}$ . By construction, the inequality required in the definition of strict  $p$ - $\mathcal{A}$ -quasiconvexity at 0 is satisfied. Moreover,  $g$  is increasing in  $t$  and decreasing in  $T$ , and since  $f(x, \cdot)$  is  $\mathcal{A}$ -quasiconvex at  $\xi$ , we have  $g \geq 0$ . It remains to show that  $g(t, T) > 0$  for all  $t > 0, T \geq 0$ . Assume by contradiction that there is a  $t_0 > 0$  and a  $T_0 \geq 0$  such that  $g(t_0, T_0) = 0$ . In particular,  $T_0 > 0$  as  $g(t_0, 0) = +\infty$ , and there is a sequence  $(\tilde{\varphi}_n) \subset \phi_{\mathcal{A}}$  such that  $\int_Q |\tilde{\varphi}_n| \geq t_0, \int_Q |\tilde{\varphi}_n|^p \leq T_0$  and

$$\int_Q [f(\tilde{\varphi}_n(y)) - f(0)] dy \xrightarrow{n \rightarrow \infty} 0. \quad (5.25)$$

For  $y \in \mathbb{R}^N$  define  $\hat{\varphi}_n(y) := \tilde{\varphi}_n(ny)$ , which inherits all the properties of  $\tilde{\varphi}_n$  stated above. In particular, (5.25) turns into

$$\int_Q [f(\hat{\varphi}_n(y)) - f(0)] dy \xrightarrow{n \rightarrow \infty} 0. \quad (5.26)$$

In addition,  $\hat{\varphi}_n \rightharpoonup 0$  weakly in  $L^p(Q; \mathbb{R}^M)$ , as  $\int_Q \tilde{\varphi}_n = 0$ . By Lemma 5.13 applied to  $u_n := \hat{\varphi}_n$ , we get an  $\mathcal{A}$ -free sequence  $\varphi_n$  which is bounded and equiintegrable in  $L^p(Q; \mathbb{R}^M)$  and which still satisfies  $\int_Q \varphi_n = 0$  and  $\varphi_n \rightharpoonup 0$  weakly in  $L^p$ . Moreover,  $\lim \int_Q |\varphi_n| = \lim \int_Q |\hat{\varphi}_{k(n)}| = t_0 > 0$  since  $\hat{\varphi}_{k(n)} - \varphi_n \rightarrow 0$  locally in measure and thus in  $L^1$  as  $\hat{\varphi}_{k(n)} - \varphi_n$  is bounded in  $L^p$  and  $p > 1$ . Hence  $(\varphi_n) \in \Phi_0$  and  $\varphi_n$  does not converge to zero locally in measure. Due to Proposition 5.14, (5.26) gives

$$\int_Q [f(\varphi_n) - f(0)] + \int_Q [f(\hat{\varphi}_n - \varphi_n) - f(0)] \xrightarrow{n \rightarrow \infty} 0, \quad (5.27)$$

Since  $f$  is  $\mathcal{A}$ -quasiconvex at 0, both terms on the left hand side of (5.27) are nonnegative for every  $n$ , whence (5.27) implies that

$$\int_Q [f(\varphi_n) - f(0)] \xrightarrow{n \rightarrow \infty} 0,$$

contradicting (2.1) □

**Proof of Proposition 2.14.** We want to show that (2.7) is equivalent to (2.3) with  $\Psi = \Phi_2 \cup \Phi_3 \cup \Phi_4$ . First assume that  $\Omega = \mathbb{R}^N$ . Due to (2.6),

$$\chi_{\mathbb{R}^N \setminus B_R} |f_\infty(u) - f(\cdot, u)| \xrightarrow{R \rightarrow \infty} 0 \text{ in } L^1(\mathbb{R}^N), \text{ uniformly in } u \in U, \quad (5.28)$$

where  $U$  may be any subset of  $\mathcal{U}_A$  which is bounded in  $L^p$ . In the following, let

$$\Phi_\infty := \left\{ (\varphi_n) \in \mathcal{U}_A \left| \begin{array}{l} \varphi_n \text{ is bounded in } L^p \text{ and satisfies } \chi_B \varphi_n \rightarrow 0 \text{ in } L^p \\ \text{for every bounded, open set } B \subset \mathbb{R}^N \end{array} \right. \right\}$$

Note that  $\Phi_2 \cup \Phi_3 \cup \Phi_4 \subset \Phi_\infty$ . Since  $f_\infty$  satisfies the same growth conditions as  $f$  (i.e., (f:1), with  $h(x)$  replaced by  $0 = \liminf_{|x| \rightarrow \infty} h(x)$ ), we have that  $f_\infty(0) = 0$ , and (5.28) implies that

$$f(\cdot, \varphi_n) - f(\cdot, 0) - f_\infty(\varphi_n) \xrightarrow{n \rightarrow \infty} 0 \text{ in } L^1(\mathbb{R}^N), \text{ for every } (\varphi_n) \in \Phi_\infty. \quad (5.29)$$

As a consequence of (5.29),  $f$  can be replaced by  $f_\infty$  in (2.3) for any  $\Psi \subset \Phi_\infty$ , whence (2.7) implies (2.3) for  $\Psi = \Phi_2 \cup \Phi_3 \cup \Phi_4$ . It remains to show that the converse is also true. First suppose that there exists a  $t_0 > 0$  such that  $\{u \in \mathcal{U}_A \mid \|u\|_{L^p} = t_0\} = \emptyset$ . In this case,  $\{u \in \mathcal{U}_A \mid \|u\|_{L^p} = t\} = \emptyset$  for all  $t > 0$  since  $\mathcal{U}_A$  is invariant under multiplication with scalars. Hence  $\mathcal{U}_A = \{0\}$  and there is nothing to show. Otherwise, for  $t \in [0, \infty)$  define

$$g(t) := \inf \left\{ \int_{\mathbb{R}^N} f_\infty(\varphi) dx \left| \varphi \in L^p(\mathbb{R}^N; \mathbb{R}^M), \mathcal{A}\varphi = 0, \|u\|_{L^p} = t \right. \right\}.$$

Since  $f_\infty$  also inherits the  $p$ -Lipschitz property (f:2) (with  $h$  replaced by  $\liminf_{|x| \rightarrow \infty} h(x) = 0$ ),  $F_\infty(u) := \int_{\mathbb{R}^N} f_\infty(u) dx$  is uniformly continuous on bounded subsets of  $L^p$  by Proposition 5.7, which implies that  $g$  is continuous. It remains to show that  $g > 0$  on  $(0, \infty)$ . Suppose by contradiction that  $g(t_0) = 0$  for a  $t_0 > 0$ . Then there exists a sequence  $(\eta_n) \subset \mathcal{U}_A$  with  $\|\eta_n\|_{L^p} = t_0$  such that  $\int_{\mathbb{R}^N} f_\infty(\eta_n) dx \rightarrow 0$ . Since  $\eta_n$  is bounded in  $L^p$ , there exists a

subsequence  $k(n)$  of  $n$  and a sequence of points  $(x_n) \subset \mathbb{R}^N$  such that  $\chi_{B_n(x_n)} \eta_{k(n)} \rightarrow 0$  in  $L^p$ . For  $x \in \mathbb{R}^N$  let

$$\varphi_n(x) := \eta_{k(n)}(x - x_n)$$

By construction,  $(\varphi_n) \in \Phi_\infty$ ,  $\|\varphi_n\|_{L^p} = t_0 > 0$  and  $\int_{\mathbb{R}^N} f_\infty(\varphi_n) dx \rightarrow 0$ . By (5.29), the latter entails that

$$\int_{\mathbb{R}^N} f(x, \varphi_n) dx \rightarrow \int_{\mathbb{R}^N} f(x, 0) dx. \quad (5.30)$$

This already contradicts (2.3) for  $\Psi = \Phi_\infty$ . To get the contradiction also with the smaller set  $\Psi = \Phi_2 \cup \Phi_3 \cup \Phi_4$ , decompose  $\varphi_n = \sum_{j=0}^4 \varphi_n^j$  according to Lemma 4.1 (or a suitable subsequence, not relabeled; note that  $\varphi_n$  weakly converges to zero). We have that  $\varphi_n^0 + \varphi_n^1 \rightarrow 0$  in  $L^p$ , since  $\varphi_n^0 + \varphi_n^1 = \varphi_n - \sum_{j=2}^4 \varphi_n^j$  is  $\mathbb{R}^N$ -tight and converges to zero in  $L^p_{\text{loc}}$ . Since  $\|\varphi_n\|_{L^p} = t_0 > 0$ , this means that at least one of the three sequences  $\varphi_n^j$ ,  $j = 2, 3, 4$ , does not converge to zero strongly in  $L^p$ . Moreover, Proposition 5.8 and (5.30) imply that

$$\int_{\mathbb{R}^N} f(x, \varphi_n^j) dx \rightarrow \int_{\mathbb{R}^N} f(x, 0) dx \quad \text{for } j = 2, 3, 4. \quad (5.31)$$

As  $(\varphi_n^j) \in \Phi_j$ , this contradicts (2.3) for  $\Psi = \Phi_2 \cup \Phi_3 \cup \Phi_4$ .

The general case where  $\Omega \subset \mathbb{R}^N$  is the complement of some compact set is essentially analogous. The only additional difficulty occurs while showing that (2.7) implies (2.3) for  $\Psi = \Phi_\infty$ , because (2.7) just applies to functions defined on the whole space while the sequences in  $\Phi_2 \cup \Phi_3 \cup \Phi_4$  and  $\Phi_\infty$  now are defined only on  $\Omega$ . However, any sequence  $(\varphi_n) \in \Phi_\infty$  converges to zero strongly in  $L^p$  on any bounded set, in particular on any bounded vicinity of  $\partial\Omega$ . Using smooth cut-off functions as in the proof of Lemma 4.5 to extend before projecting back onto  $\mathcal{A}$ -free fields allows us to replace  $\varphi_n$  with an  $\mathcal{A}$ -free sequence  $\tilde{\varphi}_n$  such that  $\varphi_n - \tilde{\varphi}_n \rightarrow 0$  in  $L^p(\Omega; \mathbb{R}^M)$  and  $\tilde{\varphi}_n \rightarrow 0$  in  $L^p(\mathbb{R}^N \setminus \Omega; \mathbb{R}^M)$ .  $\square$

## 6 Concluding remarks

*Remark 6.1.* While the main results of this paper and the decomposition lemmas of Section 4 are stated for the space  $L^p$ , the method presented here can actually handle more general spaces without significant additional difficulties. In fact, the results of Section 3 are already stated in a form more general than needed if we only study  $L^p$ . In particular, it is easy to adapt the decompositions lemmas and the main results to  $L^p + L^q$  and  $L^p \cap L^q$ , respectively, with  $1 < q < p < \infty$ . This generalization is particularly useful for functionals on domains with infinite measure whose integrand does not have the same behavior near zero and near infinity, which is actually quite natural (e.g.,  $f(x, \mu) \approx |\mu|^2$  as  $|\mu| \rightarrow 0$  and  $f(x, \mu) \approx |\mu|^p$  as  $\mu \rightarrow \infty$ ). In addition, the results can be extended to weighted Lebesgue spaces, as long as the suitable results for the continuity of Fourier multipliers in these spaces are still available. Beware though that even if one is interested in one specific space only, Fourier multiplier results are still needed for a suitable family of related spaces to use the arguments employed in the proof of Lemma 3.4 (iii) and Lemma 4.1.

*Remark 6.2.* If  $\Omega = \mathbb{R}^N$ , all of the results of this paper involving a given bounded,  $\mathcal{A}$ -free sequence  $(u_n) \subset L^p(\mathbb{R}^N; \mathbb{R}^M)$  stay true if instead of  $\mathcal{A}u_n = 0$ , we only require the weaker condition  $\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}u_n\|_{L^p} \rightarrow 0$ . To see this, simply replace  $u_n$  with the  $\mathcal{A}$ -free sequence  $\tilde{u}_n := \mathcal{P}u_n$ , where  $\mathcal{P}$  is the projection on  $\mathcal{A}$ -free fields defined in Section 3. Since  $u_n - \tilde{u}_n = (I - \mathcal{P})u_n \rightarrow 0$  strongly in  $L^p$  by Lemma 3.4 (iv), the uniform continuity of  $F$  on bounded sets shown in Proposition 5.7 implies that  $F(u_n) - F(\tilde{u}_n) \rightarrow 0$ , which means that any assumption on  $F(u_n)$  used in our results will not be affected. Unfortunately, it is not clear if this also works on domains with unbounded boundary if

$$\|\mathcal{A}u_n\|_{L^{-1,p}} := \sup \left\{ \int_{\Omega} u_n \mathcal{A}^* \varphi \mid \varphi \in C_c^\infty(\Omega; \mathbb{R}^L) \text{ with } \int_{\Omega} |\nabla \varphi|^{\frac{p}{p-1}} \leq 1 \right\}$$

is used to replace  $\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}u_n\|_{L^p}$ .

*Remark 6.3.* The sufficient conditions listed in Theorem 2.12 (i)–(iv), Corollary 2.12 (ii.1)–(ii.3) and Theorem 2.15 (i)–(iii), respectively, are also necessary. For instance, if in the situation of Theorem 2.12, the assumption of (ii) does not hold, i.e., (2.3) is violated for  $\Psi = \Phi_c$ , then there exists a sequence  $\varphi_n \in \Phi_1$  with  $F(\varphi_n) \rightarrow F(0)$  and  $\varphi_n \not\rightarrow 0$  in  $L^p$  (a bounded,  $\Omega$ -tight,  $\mathcal{A}$ -free, purely concentrating sequence). In particular, for any  $u \in \mathcal{U}_{\mathcal{A}}$ ,  $u_n := u + \varphi_n$  is a bounded,  $\mathcal{A}$ -free sequence in  $L^p$  which does concentrate, and  $\lim F(u_n) - F(u) = \lim F(\varphi_n) - F(0) = 0$  as a consequence of Proposition 5.10, whence  $u_n$  is admissible for the theorem. Similar arguments also show that the conditions of Theorem 2.12 (iii),(iv), Corollary 2.12 (ii.1)–(ii.3) and and Theorem 2.15 (ii),(iii) are sharp. The necessity of strict  $\mathcal{A}$ -quasiconvexity for the local convergence in measure in part (i) of both theorems is equivalent to the converse of the second part of Proposition 5.4 discussed in Remark 5.5.

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