WEAK SOLUTIONS OF NONLINEAR WAVE EQUATIONS

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Abstract. We show local in time existence of a weak solution to the nonlinear wave equation with power-like damping and source terms, when the source has a subcritical exponent or its primitive is positive. The damping is present through an increasing function bounded by a power.

In this work we study weak solutions to the equation:

$$\text{(NLW)} \qquad \begin{cases} u_{tt} - \Delta u + f(x,t,u) + g(x,t,u_t) = 0 \text{ a.e. in } \mathbb{R}^n \times [0,\infty); \\ u\big|_{t=0} = u_0; \\ u_t\big|_{t=0} = u_1. \end{cases}$$

By a weak solution of (NLW) we will understand a solution in the sense of distributions, i.e. in the sense given by the following definition.

Definition 0.1. *Let $\Omega_T := (0,T) \times \mathbb{R}^n$, the functions f, g under the assumptions (A2), (A4) from below, and the initial data $u_0 \in H_0^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n) \cap L^{m+1}(\mathbb{R}^n)$. A weak solution of (NLW) on $\mathbb{R}^n$, up to time T, is any function u with*

$$u \in H_0^1(\Omega_T) \cap L^{p+1}(\Omega_T), \quad u_t \in L^2(\Omega_T) \cap L^{m+1}(\Omega_T),$$

such that for every $\phi \in C_c^\infty(\Omega_T)$:

$$\int_{\Omega_T} u_t(x,s)\phi_t(x,s) + \nabla u(x,s) \cdot \nabla \phi(x,s) + f(x,t,u)\phi(x,s) + g(x,t,u_t)\phi(x,s)\, dxds = 0$$

and $u(x,0) = u_0(x)$, $u_t(x.0) = u_1(x)$ a.e.

1. Introduction

Some of the early results for the equation (NLW) were obtained in the simplified cases where one of the functions f or g vanishes, while the other one is a power function. These are classical results due to J.L. Lions [10] and K. Jörgens [6]. A landmark paper where the interaction between the source term and the damping term is treated is authored by J.L. Lions and W.A. Strauss [12]. The literature of this subject is vast, with results covering existence results, as well as blow-up theorems in the noncoercive cases. We mention the contributions made by R. T. Glassey [3], [4], J. Schaeffer [15], and F. John [7], regarding blow-up in finite time of solutions, and the works regarding existence of solutions by P. Brenner and W. von Wahl [2], L. Kapitanskii [8], S. Klainerman [9], and more recently by G. Todorova, V. Georgiev, J. Serrin and E. Vitillaro in a series of papers [5], [16], [21], [22].

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The work presented here brings an improvement to the local existence result by J. Serrin, G. Todorova and E. Vitillaro in [16], by extending the range for the exponent m, while it provides an alternative proof for the common range of the exponents p and m. More precisely, we can cover the entire range $0 \leq m$ for $p < 2^*$ ($2^* = \dfrac{2n}{n-2}$) while the local result in [16] deals only with $m > 2^*/(2^*-p)$ in the case $1+2^*/2 < p < 2^*$ (region III in Theorem 1 of [16]). Under the assumption that $F(u) = \int_0^u f(x,t,v)dv \geq 0$, we obtain existence without imposing any growth conditions on F. The shortcoming in our work is that we impose more conditions on the initial data, mainly that $(u_0, u_1) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, whereas the natural condition is $(u_0, u_1) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. (Note: the stronger assumptions are imposed by the Theorem 3.1, but we suspect that they can be removed). Our assumption (A7) is more restrictive than (Q4) (pg. 8 in [16]), but our work has the advantage of allowing some dependence on t for f, and of having no assumptions on g_v ((Q5)-(Q6) pg.8 in [16]).

The main tools used in this paper are based on the method of potentials which goes back to D. H. Sattinger in [14], and an idea due to by M. Crandall and L. Tartar used in [20], which will be explained later.

The paper is organized as follows: we will continue with a section that explains the notation used and lists the assumptions under which we work; some preliminary results follow in a subsequent section, together with their proofs (some of these results are known in the literature, while others are new results that will be needed later). The main result is presented in a separate section. Some directions for further improvement are contained in the last section.

2. Notation and assumptions

Throughout we will use the following notation:
$x \in \mathbb{R}^n$ denotes a vector with components x_i and length $|x|$;
$|\cdot|_{q,\Omega}$ is the norm in $L^q(\Omega)$ and $|\cdot|_q$ is the norm in $L^q(\mathbb{R}^n)$;
$||\cdot||_{H_0^1(\Omega)}$ will also be denoted by $||\cdot||_\Omega$;
$|\Omega| =$ the Lebesgue measure of $\Omega \subset \mathbb{R}^n$
$\omega_n =$ the measure of the unit ball in $\mathbb{R}^n$
$B(x,R) =$ the open ball centered at x, of radius R;
$B(R) =$ the open ball centered at the origin, of radius R;
The space dimension is $n \geq 3$ with the Sobolev exponent $\dfrac{2n}{n-2}$ denoted by 2^*.

The nonlinear terms f and g will be taken to satisfy the following assumptions:
(A0) f is measurable in x and differentiable in t and continuously differentiable in u;
(A1) g is measurable in t and it is differentiable in x and u_t;
(A2) (a)$|F(x,t,u)| \leq m_1|u|^p + m_2|u|^q$ such that $2 < q < p < 2^*$, $m_1, m_2 > 0$, where $F(x,t,u) = \displaystyle\int_0^u f(x,t,v)dv$, so we also have that $f(x,t,0) = 0$;
Or, (b)$F(x,t,u) \geq 0$ and $f(x,t,0) = 0$;
(A3) Monotonicity: $g_v(x,t,v) \geq 0$, $t, v \in \mathbb{R}$;
(A4) $vg(x,t,v) \geq C|v|^m$ and $|g(x,t,v)| \leq C|v|^{m-1}$ for some $m \geq 1$; we note that as a consequence $g(x,t,0) = 0$;
(A5) $|f_t(x,t,u)| \leq C$ for some $C > 0$;
(A6) $|\nabla_x g(x,t,v)| \leq C|v|$;
(A7) $|g_t(x,t,v)| \leq C|v|$.

3. Preliminary Results

This section contains all the results that we will use in order to prove our main theorem. At first, we record for completeness the following result, whose ideas can be found in the classical works of V. Barbu [1], and J. L. Lions [12]:

Theorem 3.1. *(Existence and uniqueness of solutions for dissipative wave equations with Lipschitz source terms)*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $f(x,t,u)$ and $g(x,t,v)$ under the assumptions (A0)-(A1),(A3)-(A7) and, additionally:

$$|f(x,t,u) - f(x,t,v)| \leq L|u-v|,$$

for every $x \in \mathbb{R}^n$ and every $t,u,v \in \mathbb{R}$. Let $u_0, u_1 \in H_0^1(\Omega)$ with $u_0 \in H^2(\Omega)$, $G(x,0,u_1) \in L^1(\Omega)$, where G is the antiderivative with respect to v of g, i.e. $G(x,t,v) = \int_0^v g(x,t,y)dy$.

Then the Cauchy problem:

$$\text{(LD)} \qquad \begin{cases} u_{tt} - \Delta u + f(x,t,u) + g(x,t,u_t) = 0 \text{ in } \Omega \times (0,T); \\ (u,u_t)|_{t=0} = (u_0,u_1); \\ u = 0 \text{ on } \partial\Omega \times (0,T). \end{cases}$$

admits a unique solution

$$u \in C^1([0,T];L^2(\Omega)) \cap C([0,T];H_0^1(\Omega)) \text{ with } u_{tt},\ \Delta u \in L^2(0,T;L^2(\Omega)).$$

Proof. **Existence:** Consider the approximate equation:

$$(\text{LD}_\lambda) \qquad \begin{cases} u_{tt}^\lambda - \Delta u^\lambda + f(x,t,u^\lambda) + g^\lambda(x,t,u_t^\lambda) = 0 \text{ in } \Omega \times (0,T); \\ (u^\lambda, u_t^\lambda)|_{t=0} = (u_0,u_1); \\ u^\lambda = 0 \text{ on } \partial\Omega \times (0,T), \end{cases}$$

where

$$g^\lambda(v) = g \circ (I + \lambda g)^{-1}(v), \tag{3.1}$$

i.e. $g^\lambda(x,t,v + \lambda g(x,t,v)) = g(v))$, is the Yosida approximation for g. We denote by:

$$G^\lambda(x,t,v) = \int_0^v g^\lambda(x,t,y)dy.$$

In the subsequent lemma we collect a series of results about g^λ that will be needed in our proof.

Lemma 3.2. *(Properties of the Yosida approximation)*

For g an increasing, differentiable function in v we define g^λ as in (??). Then, the following hold:

1. *$(A3)_\lambda$: $\quad g_v^\lambda \geq 0$;*
2. *For each $\lambda > 0$, g^λ is a Lipschitz function in v such that there exists $L(\lambda) > 0$ with:*
$$|g^\lambda(x,t,v_1) - g^\lambda(x,t,v_2)| \leq L(\lambda)|v_1 - v_2|;$$
3. *$\lambda g^\lambda(v) = v - (I + \lambda g)^{-1}(v)$;*
4. *G^λ is a positive function;*
5. *$(A6)$ implies $(A6)_\lambda$: $\quad |\nabla_x g^\lambda(x,t,v)| \leq C|v|$;*
6. *$(A7)$ implies $(A7)_\lambda$: $\quad |G_t^\lambda(x,t,v)| \leq C|v|^2$;*
7. *$G^\lambda(x,t,v) \leq G(x,t,v)$, for every x,t,v; hence $||G^\lambda(t,v)||_{L^1(\Omega)} \leq ||G(t,v)||_{L^1(\Omega)}$.*

Proof. (1) We differentiate with respect to v the equality $g^\lambda(v+\lambda g(v)) = g(v)$ (we drop the x, t dependence, since it doesn't affect our computations) and obtain:

$$g_v^\lambda(v + \lambda g(v)) = \frac{g_v(v)}{1 + \lambda g_v(v)}. \tag{3.2}$$

Since $g_v(v) \geq 0$ for every v, we get $g_v^\lambda \geq 0$.

(2) By (1) and (3.1) $|g_v^\lambda| \leq \frac{1}{\lambda}$. We have

$$|g^\lambda(x,t,v_1) - g^\lambda(x,t,v_2)| \leq \int_{v_1}^{v_2} g_v^\lambda(x,t,y)dy \leq \frac{1}{\lambda}|v_1 - v_2|.$$

Hence g^λ is Lipschitz in v and $L(\lambda) = \frac{1}{\lambda}$.

(3) It is enough to show that $(I + \lambda g)^{-1} = I - \lambda g^\lambda$. This equality is true, since $(I - \lambda g^\lambda)(I + \lambda g) = I$ is the same as $g = g^\lambda(I + \lambda g)$, relation equivalent to (**??**).

(4) We have $g^\lambda(x,t,0) = 0$ (by the definition of g^λ and by $g(x,t,0) = 0$). Therefore, by (1) $g^\lambda(v) \geq 0$, if $v \geq 0$ and $g^\lambda(v) < 0$, if $v < 0$. By analyzing both cases $v \geq 0$ and $v < 0$ and by using the definition, we obtain that $G^\lambda \geq 0$.

(5) For simplicity, denote by $v_\lambda(x,t,v) := (I + \lambda g(x,t))^{-1}(v)$, so that

$$v_\lambda + \lambda g(v_\lambda) = v. \tag{3.3}$$

Then, by the definition of g^λ, $g^\lambda(x,t,v) = g(x,t,v_\lambda)$. We differentiate (3.2) with respect to x and obtain

$$\nabla_x v_\lambda(1 + \lambda g_v(v_\lambda) = -\lambda \nabla_x g(v_\lambda).$$

Hence,

$$|\nabla_x g^\lambda(v)| = \frac{1}{\lambda}|\nabla_x v_\lambda| = \frac{|\nabla_x g(v_\lambda)|}{1 + \lambda g_v(v_\lambda)} \leq |\nabla_x g(v_\lambda)|,$$

since $\lambda, g_v \geq 0$. The facts that g is increasing and $g(x,t,0) = 0$ imply that $v_\lambda g(v_\lambda) \geq 0$, so by squaring (3.2), we obtain $|v_\lambda| \leq |v|$, which together with the previous inequality and the hypothesis conclude the proof.

(6) As in the previous case, we prove that $|g_t^\lambda(x,t,v)| \leq C|v|$. By integrating with respect to v, we obtain $(A7)_\lambda$.

(7) We use the notation of (5) and the fact that $|v_\lambda| \leq |v|$. In the case $0 \leq v_\lambda \leq v$, by the monotonicity of g, we have that $0 \leq g(v_\lambda) = g^\lambda(v) \leq g(v)$. By integration with respect to v we obtain $G^\lambda(x,t,v) \leq G(x,t,v)$, for $v \geq 0$. In analogous way we treat the case $v \leq 0 \leq v_\lambda$. By integration with respect to the x variable, we obtain the desired inequality of the L^1 norms.

□

For the proof of our theorem, we will need the following estimates:

$$|u_t^\lambda(t)|^2_{L^2(\Omega)} + ||u^\lambda(t)||^2_{H_0^1(\Omega)} \leq C; \tag{3.4}$$

$$||u_t^\lambda(t)||^2_{H_0^1(\Omega)} + |\Delta u^\lambda(t)|^2_{L^2(\Omega)} ds \leq C; \tag{3.5}$$

$$\int_0^T |u_{tt}^\lambda(t)|^2_{L^2(\Omega)} \leq C; \tag{3.6}$$

$$\int_0^T \int_\Omega |g^\lambda(x,t,u_t^\lambda)|^2 dx\, ds \leq C, \tag{3.7}$$

C representing throughout a generic constant, independent of λ.

These estimates are obtained by multiplying the equation (LD_λ) by appropriate quantities. In order to obtain (3.3) we use the multiplier u_t^λ, integrate over the space Ω and obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t^\lambda|^2 + |\nabla u^\lambda|^2 dx \leq -\int_\Omega f(x,t,u^\lambda)u_t^\lambda dx,$$

by the monotonicity of g^λ. Integration in t and the Lipschitz assumptions on f yield:

$$\begin{aligned}\int_\Omega |u_t^\lambda(t)|^2 + |\nabla u^\lambda(t)|^2 dx &\leq \int_\Omega u_1^2 + |\nabla u_0|^2 + 2L\int_0^t\int_\Omega |u^\lambda(s)||u_t^\lambda(s)|dxds \\ &\leq \int_\Omega u_1^2 + |\nabla u_0|^2 + L\int_0^t\int_\Omega |u^\lambda(s)|^2 + |u_t^\lambda(s)|^2 dx\,ds,\end{aligned}$$

which by Poincaré's inequality is

$$\leq \int_\Omega u_1^2 + |\nabla u_0|^2 + L\int_0^t\int_\Omega C|\nabla u^\lambda(s)|^2 + |u_t^\lambda(s)|^2 dx\,ds.$$

These inequalities hold for any $t \in (0,T)$, so by Gronwall we get:

$$|u_t^\lambda(t)|^2_{L^2(\Omega)} + ||u^\lambda(t)||^2_{H_0^1(\Omega)} \leq \left(\int_\Omega u_1^2 + |\nabla u_0|^2\right) e^{CT} < C.$$

The second estimate is obtained by multiplying the equation by $-\Delta u_t^\lambda$ and integrating in x. Hence,

$$\int_\Omega [-u_{tt}^\lambda \Delta u_t^\lambda + \Delta u^\lambda \Delta u_t^\lambda - f(x,t,u^\lambda)\Delta u_t^\lambda - g^\lambda(x,t,u_t^\lambda)\Delta u_t^\lambda]dx = 0.$$

Green's formula, the fact that $g_v^\lambda(x,t,v) \geq 0$ and the consequence of (A6) for g^λ will give us:

$$\begin{aligned}\int_\Omega g^\lambda(x,t,u_t^\lambda)\Delta u_t^\lambda dx &= -\int_\Omega g_v^\lambda(x,t,u_t^\lambda)|\nabla u_t^\lambda|^2 dx - \int_\Omega \nabla_x g^\lambda(x,t,u_t^\lambda)\cdot\nabla u_t^\lambda dx \\ &\leq -\int_\Omega \nabla_x g^\lambda(x,t,u_t^\lambda)\cdot \nabla u_t^\lambda dx \leq C\int_\Omega |u_t^\lambda(x,t)|^2 + |\nabla u_t^\lambda(x,t)|^2 dx,\end{aligned}$$

hence:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_t^\lambda|^2 + |\Delta u^\lambda|^2 dx \leq \int_\Omega f(x,t,u^\lambda)\Delta u_t^\lambda dx + C\int_\Omega |u_t^\lambda|^2 + |\nabla u_t^\lambda|^2 dx.$$

We integrate in time so

$$\begin{aligned}\int_\Omega |\nabla u_t^\lambda(t)|^2 + |\Delta u^\lambda(t)|^2 dx \leq \int_\Omega |\nabla u_1|^2 + |\Delta u_0|^2 dx + \\ \int_0^t\int_\Omega (2f(x,t,u^\lambda)\Delta u_t^\lambda + C|\nabla u_t^\lambda|^2 + C|u_t^\lambda|^2)dxds.\end{aligned}$$

Use (3.3) to bound $\int_0^t \int_\Omega |u_t^\lambda(x,s)|^2)dxds$ and in the term that contains f, integrate by parts (in the weak sense) with respect to t to obtain :

$$\int_\Omega |\nabla u_t^\lambda(t)|^2 + |\Delta u^\lambda(t)|^2 dx \le C + \int_\Omega |\nabla u_1|^2 + |\Delta u_0|^2 dx + C\int_0^t \int_\Omega |\nabla u_t^\lambda(s)|^2 dxds$$
$$+ 2\int_\Omega f(x,t,u^\lambda)\Delta u^\lambda dx|_{s=0}^{s=t} - 2\int_0^t \int_\Omega [f_t(x,t,u^\lambda) + f_u(x,t,u^\lambda)u_t^\lambda]\Delta u^\lambda dxds,$$

and since $|f_u| < L$, $|f_t| < C$, with the help of Young's inequality we obtain:

$$\le C + \int_\Omega |\nabla u_1|^2 + |\Delta u_0|^2 dx + \frac{L}{\alpha}\left(\int_0^t \int_\Omega |u^\lambda(x,s)|^2 dxds + \int_\Omega |u_0(x)|^2 dx\right)$$
$$+ L\alpha\left(\int_0^t \int_\Omega |\Delta u^\lambda(x,s)|^2 dxds + \int_\Omega |\Delta u_0|^2 dx\right) + C\int_0^t \int_\Omega |\nabla u_t^\lambda(s)|^2 dxds$$
$$+ (L+C)\int_0^t \int_\Omega (|u_t^\lambda(s)|^2 + |\Delta u^\lambda(s)|^2)dxds,$$

which with the right choice for α, the aid of Poincaré's inequality, and by using the estimate (3.3) will yield a Gronwall type inequality, which will imply (3.4).

The third estimate in our list is obtained with the aid of the multiplier u_{tt}^λ, and by integrating in space and time. Hence:

$$\int_0^T \int_\Omega |u_{tt}^\lambda|^2 dxds + \int_\Omega G^\lambda(x,T,u_t^\lambda(T)))dx = \int_\Omega G^\lambda(x,0,u_1)dx + \int_0^T \int_\Omega G_t^\lambda(x,s,u_t)dxds$$
$$+ \int_0^T \int_\Omega \Delta u^\lambda u_{tt}^\lambda dxds - \int_0^T \int_\Omega f(x,t,u^\lambda)u_{tt}^\lambda dxds \overset{\text{by }(A7)_\lambda}{\le} \int_\Omega G^\lambda(x,0,u_1)dx + C\int_0^T \int_\Omega |u_t|^2 dxds$$
$$+ \frac{1}{2\varepsilon}\int_0^T |\Delta u^\lambda|^2_{L^2(\Omega)}ds + 2\varepsilon\int_0^T \int_\Omega |u_{tt}^\lambda|^2 dxds + \frac{L}{2\eta}\int_0^T \int_\Omega |u^\lambda|^2 dxds + \frac{L\eta}{2}\int_0^T \int_\Omega |u_{tt}^\lambda|^2 dxds,$$

where we made use of Young's inequality with coefficients $\varepsilon, \frac{1}{\varepsilon}, \eta, \frac{1}{\eta}$. Poincaré's inequality combined with the bounds from (3.3) and (3.4) will yield, after choosing ε and η small enough:

$$C\int_0^T \int_\Omega |u_{tt}^\lambda|^2 dxds + \int_\Omega G^\lambda(x,T,u_t^\lambda(T))dx \le \int_\Omega G^\lambda(x,0,u_1)dx + C.$$

G^λ is a positive function by Lemma 3.2_3 and $||G^\lambda(0,u_1)||_{L^1(\Omega)} \le C$ by the hypothesis and Lemma 3.2_4. These facts will imply (3.5).

We follow the same kind of argument for the last estimate, multiplying by $g^\lambda(u_t^\lambda)$ and integrating over $(0,T)\times\Omega$.

$$\int_0^T \int_\Omega |g^\lambda(x,t,u_t^\lambda)|^2 dxds = \int_0^T \int_\Omega \left(\Delta u^\lambda g^\lambda(x,t,u_t^\lambda) - u_{tt}^\lambda g^\lambda(x,t,u_t^\lambda)\right) dxds$$
$$-f(x,t,u^\lambda)g^\lambda(x,t,u_t^\lambda)dxds \le \int_0^T \int_\Omega \frac{1}{2\varepsilon}|\Delta u^\lambda|^2 + \frac{\varepsilon}{2}|g^\lambda(x,t,u_t^\lambda)|^2 + \frac{1}{2\eta}|u_{tt}^\lambda|^2 + \frac{\eta}{2}|g^\lambda(x,t,u_t^\lambda)|^2$$
$$+ \frac{L}{2\zeta}|u^\lambda|^2 + \frac{\zeta}{2}|g^\lambda(x,t,u_t^\lambda)|^2 dxds.$$

Again, choose ε, η, ζ small enough in Young's inequality and use (3.3),(3.4) and (3.5) to obtain (3.6).

Next, we will show that $(u^\lambda)_{\lambda\geq 0}$ is a Cauchy sequence in the $H_0^1(\Omega)$ and $(u_t^\lambda)_{\lambda\geq 0}$ is Cauchy in $L^2(\Omega)$. We subtract the equation (LD$_\mu$) from (LD$_\lambda$), multiply the result by the difference $u_t^\lambda - u_t^\mu$, and integrate over Ω to obtain:

$$\begin{aligned}(3.8)\quad \frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_t^\lambda - u_t^\mu)^2 + |\nabla u^\lambda - \nabla u^\mu|^2 dx\right) + \int_\Omega (f(x,t,u^\lambda) - f(x,t,u^\mu))\\ \cdot (u_t^\lambda - u_t^\mu)dx + \int_\Omega (g^\lambda(x,t,u_t^\lambda) - g^\mu(x,t,u_t^\mu))(u_t^\lambda - u_t^\mu)dx = 0\end{aligned}$$

By Lemma 3.2_2 we have the identity:

$$u_t^\lambda = \lambda g^\lambda(u_t^\lambda) + (1+\lambda g)^{-1}(u_t^\lambda),$$

and a similar relation for u^μ. We employ again the monotonicity of g, the Lipschitz assumption on f, and integrate (3.7) with respect to time to arrive at the following inequalities:

$$\begin{aligned}|u_t^\lambda(t) - u_t^\mu(t)|^2_{L^2(\Omega)} + ||u^\lambda(t) - u^\mu(t)||^2_{H_0^1(\Omega)} &\leq 2L\int_0^t\int_\Omega |u^\lambda - u^\mu||u_t^\lambda - u_t^\mu|dxds\\ &- 2\int_0^t\int_\Omega (g^\lambda(x,t,u_t^\lambda) - g^\mu(x,t,u_t^\mu))(\lambda g^\lambda(x,t,u_t^\lambda) - \mu g^\mu(x,t,u_t^\mu))dxds\\ &\leq L\int_0^t\int_\Omega \left(|u^\lambda - u^\mu|^2 + |u_t^\lambda - u_t^\mu|^2\right)dxds + C|\lambda - \mu|\\ &\overset{\text{by Poincaré}}{\leq} C\int_0^t \left(|u_t^\lambda(s) - u_t^\mu(s)|^2_{L^2(\Omega)} + ||u^\lambda(t) - u^\mu(t)||^2_{H_0^1(\Omega)}\right)ds + C|\lambda - \mu|,\end{aligned}$$

which with the help of Gronwall's inequality will show that our sequence is Cauchy. Further explanation is due in the above argument where we used that

$$-\int_0^t\int_\Omega (g^\lambda(x,t,u_t^\lambda) - g^\mu(x,t,u_t^\mu))(\lambda g^\lambda(x,t,u_t^\lambda) - \mu g^\mu(x,t,u_t^\mu))dxds \leq C(\lambda - \mu).$$

For simplicity, let us denote by a and b the following quantities:

$$a = g^\lambda(x,t,u_t^\lambda), \quad b = g^\mu(x,t,u_t^\mu)$$

Then, it will be enough to show that:

$$(3.9)\qquad -(\lambda a - \mu b)(a-b) \leq C(\lambda - \mu)$$

for some C, which can be positive or negative. There are two cases: either $\lambda = \mu$, which is trivial, or $\lambda \neq \mu$. In this second case, we can choose $C = \dfrac{b^2 - a^2}{2}$, so the following inequalities

$$a^2 - ab + C \geq 0, \quad b^2 - ab - C \geq 0$$

will hold and imply

$$-\lambda(a^2 - ab + C) - \mu(b^2 - ab - C) \leq 0,$$

which is (3.8) rearranged. The argument is finished as we observe that $\int_0^T \int_\Omega a^2 dxds$, respectively $\int_0^T \int_\Omega b^2 dxds$, are finite due to (3.6). Thus, we obtained:

$$
\begin{aligned}
&u^\lambda(t) \to u(t) \text{ uniformly in rapport with } t \text{ in } H_0^1(\Omega)\\
&u_t^\lambda(t) \to u_t(t) \text{ uniformly in rapport with } t \text{ in } L^2(\Omega).
\end{aligned}
\tag{3.10}
$$

In order to conclude the proof of existence (and regularity) of the solution, we remark that the following convergences take place:

$u_{tt}^\lambda \rightharpoonup u_{tt}$ weakly in $L^2(0,T;L^2(\Omega))$ due to (3.5);

$\Delta u^\lambda \rightharpoonup \Delta u$ weakly in $L^2(0,T;L^2(\Omega))$ due to (3.4);

$f(x,t,u^\lambda) \to f(x,t,u)$ in $L^2(0,T;L^2(\Omega))$ due to the Lipschitz assumptions;

$g^\lambda(x,t,u_t^\lambda) \rightharpoonup \xi$ weakly in $L^2(0,T;L^2(\Omega))$ for some $\xi \in L^2(0,T;L^2(\Omega))$, due to (3.6).

Finally, $(3.9)_2$ implies $\xi = g(u_t)$ in the sense of distributions.

Uniqueness: Suppose that u and v are two solutions of (LD), then the difference $u-v$ satisfies the equation:

$$(u-v)_{tt} - \Delta(u-v) + f(x,t,u) - f(x,t,v) + g(x,t,u_t) - g(x,t,v_t) = 0,$$

with initial and boundary data identically zero. As usual, we multiply the equation by $(u-v)_t$, which is allowed due to the regularity obtained above, and integrate in space and time. Therefore:

$$
\begin{aligned}
\int_\Omega [(u-v)_t]^2(t,x) + |\nabla(u-v)|^2(t,x)dx \le -\int_0^t \int_\Omega (f(x,t,u) - f(x,t,v))(u(s,x) - v(s,x))_t \\
+ (g(x,s,u_t) - g(x,s,v_t)(u_t(s,x) - v_t(s,x))dsdx.
\end{aligned}
$$

The same ingredients that we used before, the Lipschitz assumptions on f, the monotonicity of g, the Cauchy and Gronwall inequalities give us $u - v = 0$. Thus, the solutions of (LD) are unique.

□

A classical technique that we will use is to approximate the initial data with smooth functions and then pass to the limit in the sequence of approximate solutions. The following theorem will justify this argument.

Theorem 3.3. *(Convergence of a sequence of smooth solutions)*

Under the assumptions of Theorem 3.1, if $(u_{0_\eta}, u_{1_\eta})_{\eta\ge 1}$ is a sequence of smooth functions such that $(u_{0_\eta}, u_{1_\eta}) \to (u_0, u_1)$ in $H_0^1(\Omega) \times L^2(\Omega)$, then $u_\eta(t) \to u(t)$ in $H_0^1(\Omega)$ for every $t > 0$.

Proof. We use the same techniques that helped us prove Theorem 3.1, i.e. multiply the equation by $u_{\eta_t} - u_t$, integrate with respect to the space and then the time variables and use the Lipschitz assumptions for f and the monotonicity for g. Doing so yields:

$$
\begin{aligned}
&\int_\Omega |(u_\eta(x,t) - u(x,t))_t|^2 + |\nabla u_\eta(x,t) - \nabla u(x,t)|^2 dx \le \int_\Omega (u_{1_\eta}(x) - u_1(x))^2\\
&+ |\nabla u_{0_\eta}(x) - \nabla u_0(x)|^2 dx + \int_0^t \int_\Omega L(|(u_\eta(x,s) - u(x,s))_t|^2 + |\nabla u_\eta(x,s) - \nabla u(x,s)|^2) dxds
\end{aligned}
$$

Poincaré's followed by the Gronwall's inequalities will finish the proof. □

For the proof of our main result we will need a finite propagation speed result and the energy identity, which are stated and proved below.

Proposition 3.4. *(Finite propagation of speed) Under the hypothesis of Theorem 3.1,*

(1) *if the initial data* u_0, u_1 *is compactly supported inside the ball* $B(x_0, R)$ *then* $u(x,t) = 0$ *outside* $B(x_0,\ R+t)$*;*
(2) *if* $(u_0,u_1), (v_0,v_1)$ *are two pairs of initial data with compact support for (LD), with the corresponding solutions* $u(x,t)$*, respectively* $v(x,t)$*, then* $u(x,t) = v(x,t)$ *inside* $B(x_0,\ R-t)$ *for any* $t < R$*.*

Proof. **Part (1)** The proof presented here extends an argument used for the linear wave equation by L. Tartar [18].

Assume for now that $f(x,t,u) = 0$ for $|x - x_0| \geq R + t$. Since the equation is invariant by translations, without loss of generality we can take $x_0 = 0$. First we approximate the initial data uniformly by smooth functions (u_{0_η}, u_{1_η}) with compact support inside $B(R_\eta)$, with $R_\eta \to R$ as $\eta \to 0$. For any $T > 0$, the solution for :

$$\begin{cases} u_{\eta tt} - \Delta u_\eta + f(x,t,u_\eta) + g(x,t,u_{\eta t}) = 0; \\ (u_\eta, u_{\eta t})|_{t=0} = (u_{\eta_0}, u_{\eta_1}). \end{cases} \tag{3.11}$$

exists up to T.

Consider a function ϕ_η with $\phi_\eta(r) = 0$ on $(-\infty, R_\eta]$, $\phi_\eta(r) > 0$ on (R_η, ∞), such that $\phi'(r) \geq 0$ on $\mathbb{R}$. Since $u_{\eta_t} \in L^\infty(0,T; H_0^1(B(R_\eta)))$, we are allowed to multiply (NLW) by $u_{\eta t}(t,x)\phi_\eta(|x| - t)$, $0 < t < T$. The quantity:

$$I_\eta(t) = \int_{\mathbb{R}^n} (|u_{\eta t}(x,t)|^2 + |\nabla u_\eta(x,t)|^2)\phi_\eta(|x| - t)\,dx$$

is well defined and assume for now that $\dfrac{dI_\eta}{dt} \leq 0$. It can be easily seen that $I_\eta(0) = 0$, for we have:

$$I_\eta(0) = \int_{|x|<R_\eta} (|u_{1_\eta}(x)|^2 + |\nabla u_{0_\eta}(x)|^2)\phi_\eta(|x|)\,dx \\ + \int_{|x|>R_\eta} (|u_{1_\eta}(x)|^2 + |\nabla u_{0_\eta}(x)|^2)\phi_\eta(|x|)\,dx.$$

The first integral is 0 since $\phi_\eta(|x|) = 0$ for $|x| < R_\eta$. The initial data has support inside the domain $|x| < R_\eta$, so the second integral is zero.

The assumption that $t \to I_\eta(t)$ is decreasing leads us to $I_\eta(t) \leq I_\eta(0) = 0$, which means that $u_\eta(x,t) = 0$ if $|x| - t > R_\eta$. We pass to the limit in η (see Theorem 3.3) to obtain $u(x,t) = 0$ for $|x| - t > R$, and this concludes the proof.

It suffices then to prove that $\dfrac{dI_\eta}{dt} \leq 0$. For the regularized initial data we have $u_\eta \in L^\infty(0,T; H_0^1(B(R_\eta)))$, $u_{\eta_t} \in L^\infty(0,T; H_0^1(B(R_\eta)))$, $u_{\eta_{tt}} \in L^1(0,T; L^2(B(R_\eta)))$, which enables us to compute (we drop the subscript η in the remainder of the proof):

$$\frac{dI}{dt}(t) = \int_{\mathbb{R}^n} 2\phi(|x|-t)(u_t u_{tt} + \sum_{i=1}^n u_{x_i} u_{tx_i})(x,t)\,dx - \int_{\mathbb{R}^n} \phi'(|x|-t)(u_t^2 + |\nabla u|^2)(x,t)\,dx$$
$$= \int_{\mathbb{R}^n} 2\phi(|x|-t)(u_t u_{tt})(x,t)\,dx - \int_{\mathbb{R}^n} \sum_{i=1}^n ((2\phi(|x|-t)u_{x_i})_{x_i} u_t)(x,t)\,dx$$
$$- \int_{\mathbb{R}^n} \phi'(|x|-t)(u_t^2 + |\nabla u|^2)(x,t)\,dx = \int_{\mathbb{R}^n} 2\phi(|x|-t)((u_{tt} - \Delta u)u_t)(x,t)\,dx$$
$$- \int_{\mathbb{R}^n} \sum_{i=1}^n 2\phi'(|x|-t)\frac{x_i}{r}(u_{x_i} u_t)(x,t)\,dx - \int_{\mathbb{R}^n} \phi'(|x|-t)(u_t^2 + |\nabla u|^2)(x,t)\,dx$$

By (NLW):

$$u_{tt} - \Delta u = -f(x,t,u) - g(x,t,u_t),$$

hence the assumptions on the support of ϕ and f, together with the fact that g is nondecreasing, make the first term of the last equality negative. We factor out $\phi'(|x|-t)$ in the other two terms, and since $\phi'(r) \geq 0$ for every r, it is enough to show that:

$$u_t^2 + |\nabla u|^2 + 2\sum_{i=1}^n \frac{x_i}{|x|} u_{x_i} u_t \geq 0.$$

The Cauchy-Schwarz inequality:

$$(\frac{x_i}{|x|} u_t + u_{x_i})^2 \geq 0 \tag{3.12}$$

is summed over all $i = 1..n$ to obtain the conclusion.

At this point, the function f might not vanish for $|x-x_0| \geq R+t$. A fixed point argument will overcome this shortcoming. Consider the iterative equation:

$$\begin{cases} u_{tt}^{k+1} - \Delta u^{k+1} + f(x,t,u^k) + g(x,t,u_t^{k+1}) = 0 \\ (u^{k+1}, u_t^{k+1})|_{t=0} = (u_0, u_1), \end{cases}$$

for every $k \in \mathbb{N}$, with $(u^0, u_t^0) = (u_0, u_1)$. An induction argument, together with the first part of the proof, will show that $u^k(x,t) = 0$ for $|x-x_0| > R+t$, for every $k \in \mathbb{N}$. It would be enough then to show that $u^k(x,t) \to u(x,t)$ a.e. as $k \to \infty$. Since f is Lipschitz we obtain that $f(x,t,u^k(x,t))$, which is zero for $|x-x_0| \geq R+t$, converges a.e. to $f(x,t,u(x,t))$, hence f vanishes outside the cone $|x-x_0| < R+t$. The sequence of difference functions $v^k(x,t) := u^k(x,t) - u(x,t)$ satisfies:

$$\begin{cases} v_{tt}^{k+1} - \Delta v^{k+1} + f(x,t,v^k+u) - f(x,t,u) + g(x,t,(v^{k+1}+u)_t) - g(x,t,u_t) = 0 \\ (v^{k+1}, v_t^{k+1})|_{t=0} = (0,0). \end{cases}$$

Upon multiplication by v_t^{k+1}, integration over $(0,t) \times \mathbb{R}^n$ and by using the monotonicity of g we derive the following inequality:

$$\int_{\mathbb{R}^n} (v_t^{k+1}(x,t))^2 + |\nabla v^{k+1}(x,t)|^2\,dx \leq \int_0^t \int_{\mathbb{R}^n} 2|f(x,t,v^k+u) - f(x,t,u)||v_t^{k+1}(x,s)|dxds,$$

which by the Lipschitz assumptions is

$$\leq \int_0^t 2L|v^k(s)|_2 |v_t^{k+1}(s)|_2\,ds \leq L\int_0^t |v^k(s)|_2^2 + |v_t^{k+1}(s)|_2^2\,ds.$$

We need now a bound for $\int_0^t |v^k(s)|_2^2 ds$, which we obtain by writing:

$$|v^k(t)|_2^2 = 2\int_0^t \int_{\mathbb{R}^n} v^k(x,s)v_t^k(x,s)dxds \le \int_0^t \int_{\mathbb{R}^n} (v^k(x,s))^2 + (v_t^k(x,s))^2 dxds.$$

Gronwall's inequality for the function $|v^k(s)|_2^2$ will give us for any $t < T$ the bound:

$$|v^k(t)|_2^2 \le e^T \int_0^t |v_t^k(s)|_2^2 ds.$$

Therefore, we have for the function $\phi^k(t) = \int_0^t |v_t^k(s)|_2^2 + |\nabla v^k(t)|_2^2 ds$ the inequality:

$$\phi_t^{k+1}(t) \le L\phi^{k+1}(t) + C\phi^k(t),$$

which after integration becomes:

$$\phi^{k+1}(t) \le C\int_0^t e^{L(t-s)}\phi^k(s)ds \le Ce^{LT}\int_0^t \phi^k(s)ds$$

A simple induction argument will show that:

$$\phi^{k+1}(t) \le K\frac{C^{k+1}e^{LT(k+1)}t^{k+1}}{(k+1)!}$$

where K is a bound on $|\phi^1(t)|$ for all t in $[0,T]$. Thus we proved the convergence for $u^k(x,t)$ a.e. (x,t), so $u(x,t) = 0$ outside the domain of dependence, i.e. for $|x - x_0| \ge R + t$.

Part (2) We follow here a similar argument as in part (1). Initially, we work under the assumption that

$$f(x,t,u(x,t)) = f(x,t,v(x,t)) \tag{3.13}$$

for $|x| < R - t$ (again, take $x_0 = 0$). The difference $u - v$ satisfies:

$$\begin{cases} (u-v)_{tt} - \Delta(u-v) + f(x,t,u) - f(x,t,v) + g(x,t,u_t) - g(x,t,v_t) = 0; \\ ((u-v),(u-v)_t)\,|_{t=0} = (0,0). \end{cases} \tag{3.14}$$

We take the function ψ strictly positive on $(-\infty, R)$, such that $\psi(r) = 0$ on $[R,\infty)$ and $\psi'(r) \le 0$ everywhere. Define the quantity $J(t)$ by

$$J(t) := \int_{\mathbb{R}^n} \left((u_t(x,t) - v_t(x,t))^2 + |\nabla(u(x,t) - v(x,t)|^2\right)\psi(|x| - t)dx.$$

We claim that $\frac{dJ}{dt} \le 0$. As before, this will show that $u(x,t) = v(x,t)$ on the support of $\psi(|x| + t)$, i.e. if $|x| < R - t$. The proof of the claim is similar to that in (1):

$$\frac{dJ}{dt} = \int_{\mathbb{R}^n} 2\psi(|x|+t)[((u-v)_{tt} - \Delta(u-v))(u-v)_t](x,t)\,dx$$

$$+\int_{\mathbb{R}^n}\sum_{i=1}^n 2\psi'(|x|+t)\frac{x_i}{r}[(u-v)_{x_i}(u-v)_t](x,t)\,dx + \int_{\mathbb{R}^n}\psi'(|x|+t)[(u-v)_t^2 + |\nabla(u-v)|^2](x,t)\,dx$$

We use the equality $(u-v)_{tt}-\Delta(u-v)=-f(x,t,u)+f(x,t,v)-g(x,t,u_t)+g(x,t,v_t)$, (3.11), (3.12) and the assumptions on the support of ψ to obtain that J is decreasing. In order to prove (3.12), we use the following iterative schemes with $u^1=u_0$ and $v^1=v_0$:

$$\begin{cases} u_{tt}^{k+1}-\Delta u^{k+1}+f(x,t,u^k)+g(x,t,u_t^{k+1})=0 \\ (u^{k+1},u_t^{k+1})|_{t=0}=(u_0,u_1) \end{cases}$$

and

$$\begin{cases} v_{tt}^{k+1}-\Delta v^{k+1}+f(x,t,v^k)+g(x,t,v_t^{k+1})=0 \\ (v^{k+1},v_t^{k+1})|_{t=0}=(v_0,v_1). \end{cases}$$

By the first part of the proof, since $(u_0,u_1)=(v_0,v_1)$ we have that $u^k=v^k$, which implies that $u^{k+1}=v^{k+1}$ on the desired domain. Since we also have that $u^1=v^1$, by the induction principle, the equality $u^k=v^k$ holds true for all k's. The argument will be complete after showing $u^k-v^k\to 0$ a.e. as $k\to\infty$. This is accomplished by using the procedure as in (1). We subtract the above iterative schemes, so that:

$$\begin{cases} (u^{k+1}-v^{k+1})_{tt}-\Delta(u^{k+1}-v^{k+1})+f(x,t,u^k)-f(x,t,v^k) \\ \qquad\qquad +g(x,t,u_t^{k+1})-g(x,t,v_t^{k+1})=0 \\ (u^{k+1}-v^{k+1},(u^{k+1}-v^{k+1})_t)|_{t=0}=(0,0). \end{cases}$$

An argument identical to the one used in (1) concludes the proof. □

The proof of the following proposition is contained in the proof of Theorem 3.1, but we state the result explicitly below:

Proposition 3.5. *(The energy identity)*
Under the assumptions of Theorem 3.1 we have the following equality:

(3.15)
$$E(t):=\frac{1}{2}|u_t(t)|_{2,\Omega}^2+\frac{1}{2}|\nabla u(t)|_{2,\Omega}^2+\int_\Omega F(x,t,u(x,t))\,dx-\int_0^t\int_\Omega f_t(x,s,u(x,s))\,dx\,ds$$
$$+\int_0^t\int_\Omega g(x,s,u_t(x,s))u_t(x,s)\,dx\,ds=E(0)=const.$$

4. The Main Result

We state the main result of this paper in the following theorem. We remark that the bounded set Ω is replaced by $\mathbb{R}^n$.

Theorem 4.1. *(Local existence in time of weak solutions):*
For $(u_0,u_1)\in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$, and $G(x,0,u_1)\in L^1(\mathbb{R}^n)$, where $G(x,t,v)=\int_0^v g(x,t,u)du$, under the assumptions (A0)-(A7), there exists a time $T>0$ up to which the (NLW) equation admits a weak solution.

Proof. STEP 1: We will discuss first the case where f satisfies (A2)(a). We assume that f is a Lipschitz function of constant L, i.e.

$$|f(x,t,u)-f(x,t,v)|\le L|u-v|, \tag{4.1}$$

that $f(x,t,u)=g(x,t,u_t)=0$ if $|x|\ge R$ for some positive R, and that $g_v(x,t,v)\ge 0$. We restrict u_0,u_1 such that they are compactly supported inside the ball of radius R, so with such f, g, u_0, u_1 the existence and regularity results of Section 2 are available. At this point we additionally impose that the following conditions are satisfied:

$$|\nabla u_0|_\Omega < \alpha, \quad \frac{1}{2}|u_1|_\Omega^2 + \frac{1}{2}|\nabla u_0|_\Omega^2 + \int_\Omega F(x, 0, u_0(x))\, dx < \Phi(\alpha), \tag{4.2}$$

where α, Φ and Ω will be chosen later. We will prove that $|\nabla u(t)|_\Omega < \alpha$ for any $t < 1$.

In the next step we will eliminate the restrictions (4.2), and then approximate the nonlinearity f that satisfies the assumptions (A0),(A2),(A5) by Lipschitz functions f_ε. We apply the results obtained in the first step (where the bounds will not depend on ε) and pass to the limit. We will discuss first the case where f satisfies (A2)(a).

Due to the finite propagation speed we have that $u(x,t)$ is zero outside $B(R+t)$, hence, for $t < 1$, $u(x,t)$ is supported inside the set $\Omega := B(R+1)$.

For $p < 2^*$, $v \in H_0^1(\Omega)$ we will need the following inequality:

$$|v|_{p,\Omega} \le C(R+1)^{n\frac{2^*-p}{2^*p}} |\nabla v|_\Omega. \tag{4.3}$$

This is a consequence of the Hölder inequality:

$$|v|_{p,\Omega} \le \left(\int_\Omega |v|^{\frac{2^*}{p}p} dx\right)^{\frac{1}{2^*}} \left(\int_\Omega dx\right)^{\frac{2^*-p}{2^*p}} \le |v|_{2^*}(\omega_n(R+1)^n)^{\frac{2^*-p}{2^*p}}$$

and of the Sobolev Imbedding theorem:

$$|v|_{2^*} \le C^*|\nabla v|_\Omega,$$

where C^* depends only on n, and $C = C^*\omega_n^{\frac{2^*-p}{2^*p}}$.

Since $u(t) \in H_0^1(B(R+t))$ implies $u(t) \in H_0^1(\Omega)$ for $t < 1$, (4.3) will hold for any $u(t)$.

By the assumptions (A3) and (A5) (combined with the fact that $t < 1$), and the conservation of energy formula, the following relation holds:

$$\frac{1}{2}|u_t(t)|_\Omega^2 + \frac{1}{2}|\nabla u(t)|_\Omega^2 + \int_\Omega F(x,t,u(x,t))\, dx \le C + \frac{1}{2}|u_1|_\Omega^2 + \frac{1}{2}|\nabla u_0|_\Omega^2 + \int_\Omega F(x,0,u_0(x))\, dx = E(0) =: E_0. \tag{4.4}$$

The assumptions on F, followed by an application of the inequality (4.3) will yield:

$$\frac{1}{2}|\nabla u(t)|_\Omega^2 + \int_\Omega F(x,t,u(x,t))\, dx \ge \frac{1}{2}|\nabla u(t)|_\Omega^2 - m_1|u(t)|_{p,\Omega}^p - m_2|u(t)|_{q,\Omega}^q \ge \frac{1}{2}|\nabla u(t)|_\Omega^2 - m_1C^p|\nabla u(t)|_\Omega^p(R+1)^{n\frac{2^*-p}{2^*}} - m_2C^q|\nabla u(t)|_\Omega^q(R+1)^{n\frac{2^*-q}{2^*}}. \tag{4.5}$$

The right hand side of the above inequality will be analyzed with the aid of the function:

$$\Phi(x) = \frac{x^2}{2} - Ax^p - Bx^q, \quad x \ge 0. \tag{4.6}$$

where $A, B > 0$.

For p, $q > 2$, Φ has exactly 2 critical points: $x = 0$ and $x = \alpha$, where α is the only positive root of the equation $1 = pA\alpha^{p-2} + qB\alpha^{q-2}$. At $x = 0$ there is a local minimum and at $x = \alpha$ a global maximum. We point out that α depends on R, which measures the size of the support of the initial data.

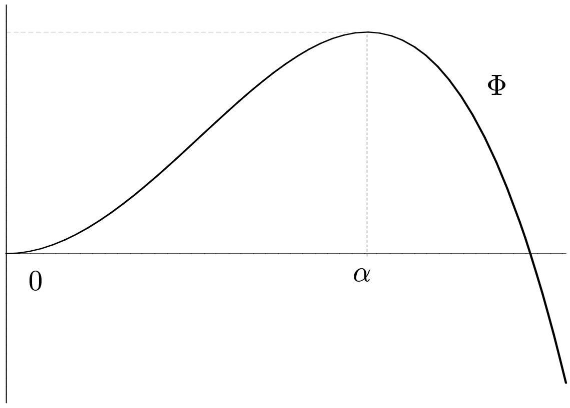

FIGURE 1. Graph of Φ

In (4.6) we take:

$$A = m_1 C^p (R+1)^{n\frac{2^*-p}{2^*}}$$

$$B = m_2 C^q (R+1)^{n\frac{2^*-q}{2^*}}.$$

($R+1$ is related to the restriction $t < 1$).

Assume that $|\nabla u(s)|_\Omega < \alpha$ for all $s \in [0, s_0)$, for some s_0 ($s_0 > 0$, as $|\nabla u_0|_\Omega < \alpha$ and $t \to |\nabla u(t)|_\Omega$ is continuous). With the new notation, (4.5) becomes:

$$\frac{1}{2}|\nabla u(t)|_\Omega^2 + \int_\Omega F(x,t,u(x,t))\,dx \geq \frac{1}{2}|\nabla u(t)|_\Omega^2 - A|\nabla u(t)|_\Omega^p - B|\nabla u(t)|_\Omega^q,$$

which combined with (4.4) gives us:

$$\begin{aligned}(4.7)\quad & \frac{1}{2}|u_t(t)|_\Omega^2 + \frac{1}{2}|\nabla u(t)|_\Omega^2 - A|\nabla u(t)|_\Omega^p - B|\nabla u(t)|_\Omega^q \leq \\ & \frac{1}{2}|u_t(t)|_\Omega^2 + \frac{1}{2}|\nabla u(t)|_\Omega^2 + \int_\Omega F(x,t,u(x,t))\,dx \leq \\ & \frac{1}{2}|u_1|_\Omega^2 + \frac{1}{2}|\nabla u_0|_\Omega^2 + \int_\Omega F(x,0,u_0(x))\,dx < \Phi(\alpha),\end{aligned}$$

the last inequality being part of the hypothesis for Step 1. Therefore

$$\frac{1}{2}|\nabla u(t)|_\Omega^2 - A|\nabla u(t)|_\Omega^p - B|\nabla u(t)|_\Omega^q \leq \Phi(\alpha)$$

so, by the continuity in time of $|\nabla u(t)|_\Omega$ we get:

$$|\nabla u(t)|_\Omega < \alpha \tag{4.8}$$

for any $t < 1$, otherwise $\Phi(|\nabla u(t)|) \geq \Phi(\alpha)$ for the t which did not satisfy (4.8), so we would obtain a contradiction. We make the remark that the bound $t < 1$ is imposed by the choice of the domain $\Omega = B(R+1)$.

If f satisfies (A2)(b), we easily obtain the bound $|\nabla u(t)|_\Omega < C(u_0, u_1)$ from the energy identity, since we have that $F \geq 0$.

STEP 2: Under the Lipschitz assumption on f we eliminate the restrictions (4.2) and the fact that f and g have compact support in the variable x. The idea follows an argument of M. Crandall and L. Tartar who applied it in order to show global existence of a solution for the Broadwell model in [20].

Consider a pair of initial data $(u_0, u_1) \in H_0^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, $u_0 \in H^2(\mathbb{R}^3)$ and $G(x, 0, u_1) \in L^1(\mathbb{R}^n)$. Fix $t < 1$ and $x_0 \in \mathbb{R}^n$. We will find a domain around x_0, small enough, such that (4.2) are satisfied inside. Then we construct a new pair of initial data (u_0^*, u_1^*) for which we apply the results of Step 1 to obtain bounds for the new solution u^* for $t < 1$.

From now on let α be the critical point of the function Φ, from Step 1, with the coefficients A, B corresponding to $R = 1$. Denote by $(\overline{u}_0, \overline{u}_1)$ the initial data which coincides with (u_0, u_1) inside $B(x_0, 1)$ and is 0 outside.

We find $\rho < 1$ small enough such that:

$$\begin{aligned} &|\nabla \overline{u}_0|_{B(x_0,\rho)} < \frac{\alpha}{2} \quad \text{and} \quad |\nabla \overline{u}_0|^2_{B(x_0,\rho)} \le \frac{\Phi(\alpha)}{6}, \\ &2(C^*\omega_n)^{\frac{1}{n}} \left(|\nabla \overline{u}_0|_{B(x_0,\rho)} + |\overline{u}_0|_{B(x_0,\rho)} \right) \le \frac{\alpha}{2}, \quad \text{and} \quad 4{C^*}^2 \omega_n^{\frac{2^*-2}{2^*}} |\overline{u}_0|^2_{B(x_0,\rho)} \le \frac{\Phi(\alpha)}{6}, \\ &\frac{1}{2} |\overline{u}_1|^2_{B(x_0,\rho)} \le \frac{\Phi(\alpha)}{3}, \\ &m_1 (C^*)^p \left(|\nabla \overline{u}_0|_{B(x_0,\rho)} + |\overline{u}_0|_{B(x_0,\rho)} (\frac{2}{\rho} + 1) \right)^p \le \frac{\Phi(\alpha)}{6}, \\ &m_2 (C^*)^q \left(|\nabla \overline{u}_0|_{B(x_0,\rho)} + |\overline{u}_0|_{B(x_0,\rho)} (\frac{2}{\rho} + 1) \right)^q \le \frac{\Phi(\alpha)}{6}, \end{aligned} \tag{4.9}$$

where C^* is the constant of the Sobolev inequality. To have these conditions satisfied, it is enough to take ρ such that:

$$|\nabla \overline{u}_0|_{B(x_0,\rho)} < \min \left\{ \frac{\alpha}{2}, \sqrt{\frac{\Phi(\alpha)}{6}}, \frac{\alpha}{8(C^*\omega_n)^{\frac{1}{n}}}, \frac{1}{2C^*} \left(\frac{\Phi(\alpha)}{6m_1} \right)^{1/p}, \frac{1}{2C^*} \left(\frac{\Phi(\alpha)}{6m_2} \right)^{1/q} \right\}, \tag{4.10}$$

$$|\overline{u}_0|_{B(x_0,\rho)} < \min \left\{ \frac{\alpha}{8(C^*\omega_n)^{\frac{1}{n}}}, \frac{1}{\omega_n C^*} \sqrt{\frac{\Phi(\alpha)}{24 \omega_n^{\frac{2^*-2}{2^*}}}}, \frac{1}{6C^*} \left(\frac{\Phi(\alpha)}{6m_1} \right)^{1/p}, \frac{1}{6C^*} \left(\frac{\Phi(\alpha)}{6m_2} \right)^{1/q} \right\},$$

$$|\overline{u}_1|^2_{B(x_0,\rho)} \le \sqrt{\frac{\Phi(\alpha)}{6}}.$$

The existence of such a ρ, independent of $x_0 \in \mathbb{R}^n$, is motivated by the equiintegrability of the functions $\overline{u}_0, \nabla \overline{u}_0, \overline{u}_1$ (see [11]).

It is possible that $\overline{u}_0 \notin H_0^1(B(x_0, \rho))$ since it does not necessarily have zero trace on the boundary, so in order to apply the results of Step 1, we multiply $\overline{u}_0$ by θ, a twice differentiable cutoff function, obtained by smoothing the Lipschitz graph:

$$\theta_0(x) = \begin{cases} 1, \ |x - x_0| \le \frac{\rho}{2} \\ 1 - \frac{2|x - x_0|}{\rho}, \ \frac{\rho}{2} \le |x - x_0| \le \rho \\ 0, \ |x - x_0| \ge \rho. \end{cases}$$

With an appropriate choice of the smoothing operator for θ_0 we will have:

$$|\theta|_{\infty, B(x_0,\rho)} \le 1, \quad |\nabla \theta|_{\infty, B(x_0,\rho)} \le \frac{2}{\rho}. \tag{4.11}$$

Now, the product $\theta\overline{u}_0 =: u_0^*$ belongs to $H_0^1(B(x_0,\rho)) \cap H^2(B(x_0,\rho))$, and we have:

$$\nabla u_0^* = \theta \nabla \overline{u}_0 + \overline{u}_0 \nabla \theta.$$

Hence:

$$|\nabla u_0^*|_{B(x_0,\rho)} \leq |\theta|_{\infty,B(x_0,\rho)} |\nabla \overline{u}_0|_{B(x_0,\rho)} + |\nabla \theta|_{\infty,B(x_0,\rho)} |\overline{u}_0|_{B(x_0,\rho)}.$$

By (4.11) and by applying Hölder's inequality, the above quantity is:

$$\leq \frac{\alpha}{2} + |\overline{u}_0|_{2*,B(x_0,\rho)} |B(x_0,\rho)|^{\frac{1}{n}} \frac{2}{\rho},$$

which, with the aid of the Sobolev's inequality, becomes:

$$\leq \frac{\alpha}{2} + 2(C^*\omega_n)^{\frac{1}{n}} \left(|\nabla \overline{u}_0|_{B(x_0,\rho)} + |\overline{u}_0|_{B(x_0,\rho)} \right) \overset{(4.10)}{\leq} \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

We claim that $(u_0^*, u_1^*) := (\theta \overline{u}_0, \overline{u}_1)$ satisfies the second inequality of (4.2), i.e.

$$\frac{1}{2}|u_1^*|^2_{B(x_0,\rho)} + \frac{1}{2}|\nabla u_0^*|^2_{B(x_0,\rho)} + \int_{B(x_0,\rho)} F(x,0,u_0^*(x))\,dx \leq \Phi(\alpha),$$

and we can prove this fact in a similar manner as before. For the first term we have:

$$\frac{1}{2}|u_1^*|^2_{B(x_0,\rho)} = \frac{1}{2}|\overline{u}_1|^2_{B(x_0,\rho)} \leq \frac{\Phi(\alpha)}{3}.$$

Also,

$$\begin{aligned}
(4.12)\quad \frac{1}{2}|\nabla u_0^*|^2_{B(x_0,\rho)} &\leq |\theta|^2_{\infty,B(x_0,\rho)} |\nabla \overline{u}_0|^2_{B(x_0,\rho)} + |\nabla\theta|^2_{\infty,B(x_0,\rho)} |\overline{u}_0|^2_{B(x_0,\rho)} \\
&\leq \frac{\Phi(\alpha)}{6} + \frac{4}{\rho^2} C^{*2} |B(x_0,\rho)|^{\frac{2^*-2}{2^*}} |\overline{u}_0|^2_{B(x_0,\rho)} = \frac{\Phi(\alpha)}{6} + 4C^{*2}\omega_n^{\frac{2^*-2}{2^*}} |\overline{u}_0|^2_{B(x_0,\rho)} \\
&\leq \frac{\Phi(\alpha)}{6} + \frac{\Phi(\alpha)}{6} = \frac{\Phi(\alpha)}{3}.
\end{aligned}$$

For the last term, the assumption (A2)(a), the Sobolev embedding theorem and the restrictions for ρ in (4.10) give us:

$$\begin{aligned}
(4.13)\quad \int_{B(x_0,\rho)} F(x,0,u_0^*(x))\,dx &\leq \int_{B(x_0,\rho)} (m_1|u_0^*(x)|^p + m_2|u_0^*(x)|^q)\,dx \\
&\leq m_1(C^*)^p \left(|\nabla u_0^*|_{B(x_0,\rho)} + |u_0^*|_{B(x_0,\rho)}\right)^p + m_2(C^*)^q \left(|\nabla u_0^*|_{B(x_0,\rho)} + |u_0^*|_{B(x_0,\rho)}\right)^q \\
&\leq m_1(C^*)^p \left(|\nabla \overline{u}_0|_{B(x_0,\rho)} + |\overline{u}_0|_{B(x_0,\rho)}(\frac{2}{\rho}+1) \right)^p + m_2(C^*)^q \left(|\nabla \overline{u}_0|_{B(x_0,\rho)} \right. \\
&\left. \quad + |\overline{u}_0|_{B(x_0,\rho)}(\frac{2}{\rho}+1) \right)^q \leq \frac{\Phi(\alpha)}{6} + \frac{\Phi(\alpha)}{6} = \frac{\Phi(\alpha)}{3}.
\end{aligned}$$

Next we intend to use the finite speed property to prove that in a neighborhood of x_0, for any time $t < \frac{\rho}{2}$ the solution $u^*(\cdot,t)$, given by (u_0^*, u_1^*), coincides with $u(\cdot,t)$.

Having the finite propagation speed, we obtain the uniqueness $u^* = u$ up to time $\rho/2$ in a neighborhood of x_0. By moving x_0 in $\mathbb{R}^n$ and "patching" all the local solutions u^*, we can eliminate the constraint of compact support for the initial data.

STEP 3: Approximate f by the Lipschitz functions f_ε given by:

$$f_\varepsilon(x,t,u) = \begin{cases} f(x,t,u), \text{ if } |f_u(x,t,u)| \le 1/\varepsilon, \\ f(x,t,u_+) + \frac{u-u_+}{\varepsilon}, \text{ if } f_u(x,t,u) \ge f_u(x,t,u_+) = 1/\varepsilon, \\ f(x,t,u_-) - \frac{u-u_-}{\varepsilon}, \text{ if } f_u(x,t,u) \le f_u(x,t,u_-) = -1/\varepsilon. \end{cases}$$

As f is continuously differentiable in u, the functions f_ε approximate f maintaining a continuous derivative at u_-, respectively, at u_+. We notice that for this Lipschitz nonlinearity f_ε, in the inequality (A2)(a) we can choose the same constants m_1, m_2 as for f, so they will remain independent of ε. Hence, the coefficients A, B from (4.6), corresponding to f_ε, as well as the root α, and the radius ρ, chosen in Step 2, will not vary with ε.

We solve the problem with the nonlinearities $f_\varepsilon, g_\varepsilon$ and initial data (u_0^*, u_1^*) as in the previous steps and obtain the estimates:

$$|\nabla u_\varepsilon^*(t)|_{B(x_0,\rho)} < \alpha. \tag{4.14}$$

From the energy identity (3.14), (A5) and the fact that g is increasing the following inequality results:

$$|u_{\varepsilon t}^*(t)|^2_{2,B(x_0,\rho)} + |\nabla u_\varepsilon^*(t)|^2_{2,B(x_0,\rho)} + \int_{B(x_0,\rho)} F_\varepsilon(x,t,u_\varepsilon^*(x,t))dx \le C, \tag{4.15}$$

so, with the growth condition on F^ε, Sobolev's inequality and (4.14) we obtain the bound:

$$\begin{aligned} |u_{\varepsilon t}^*(t)|^2_{2,B(x_0,\rho)} + |\nabla u_\varepsilon^*(t)|^2_{2,B(x_0,\rho)} &\le C + \int_{B(x_0,\rho)} m_1|u_\varepsilon^*(x,s)|^p + m_2|u_\varepsilon^*(x,s)|^q dx \\ &\le C + C(\rho,m_1)|\nabla u_\varepsilon^*(t)|^p_{2,B(x_0,\rho)} + C(\rho,m_2)|\nabla u_\varepsilon^*(t)|^q_{2,B(x_0,\rho)} < C. \end{aligned} \tag{4.16}$$

By integrating (4.16) in time up to $\rho/2$, we deduce from the Dunford-Pettis Theorem (see [23]) the existence of a subsequence, denoted also by u_ε^*, for which we have the convergences:

$$\begin{aligned} &u_\varepsilon^* \overset{*}{\rightharpoonup} u^* \text{ in } L^\infty(0,\rho/2; H_0^1(B(x_0,\rho)) \\ &u_{\varepsilon t}^* \overset{*}{\rightharpoonup} u_t^* \text{ in } L^\infty(0,\rho/2; L^2(B(x_0,\rho)). \end{aligned} \tag{4.17}$$

The Sobolev compactness embedding theorem yields the convergence $u_\varepsilon^* \to u^*$ strongly in $L^2((0,\rho/2) \times B(x_0,\rho))$ so $u_\varepsilon^*(x,t) \to u^*(x,t)$ a.e. $(x,t) \in (0,\rho/2) \times B(x_0,\rho)$. Hence $f_\varepsilon(x,t,u_\varepsilon^*(x,t)) \to f(x,t,u^*(x,t))$ a.e. The growth bound on f_ε and (4.14) will suffice to transform the a.e. convergence in weak convergence for $f_\epsilon(x,t,u_\varepsilon^*(x,t))$, by Lebesgue dominated convergence theorem (see [10] pg.12 for details).

Since $(u_0,u_1) = (u_0^*, u_1^*)$ for $|y - x_0| < \dfrac{\rho}{2}$, due to the uniqueness result stated in Theorem 3.1, we have $u_\varepsilon^*(x,t) = u_\varepsilon(x,t)$ for $|x - x_0| < \frac{\rho}{2} + t - s$; hence $u^*(x,t) = u(x,t)$ on the same backward cone.

A monotonicity argument will be applied in order to pass to the limit in the nonlinear damping term $g(x,t,u_{\varepsilon_t})$. From (A4), the energy identity and the bounds on F_ε obtained above, we have:

$$\begin{aligned} &|u_{\varepsilon_t}(t)|^2_{2,B(x_0,\rho)} + |\nabla u_\varepsilon(t)|^2_{2,B(x_0,\rho)} + \int_0^t \int_{B(x_0,\rho)} |u_{\varepsilon_t}(x,s)|^m dxds \\ &\le |u_{\varepsilon_t}(t)|^2_{2,B(x_0,\rho)} + |\nabla u_\varepsilon(t)|^2_{2,B(x_0,\rho)} + \int_0^t \int_{B(x_0,\rho)} g(x,s,u_{\varepsilon_t}(x,s))u_{\varepsilon_t}(x,s)dxds \le C. \end{aligned} \tag{4.18}$$

Therefore, again we can extract a subsequence u^*_ε such that:

$$\begin{aligned} &u_{\varepsilon t} \rightharpoonup u_t \text{ in } L^m((0,\rho/2)\times B(x_0,\rho)) \\ &g(x,t,u_{\varepsilon t}) \overset{*}{\rightharpoonup} \xi \text{ in } L^{m'}((0,\rho/2)\times B(x_0,\rho)). \end{aligned} \tag{4.19}$$

Passing to the limit in ε we obtain

$$u_{tt} - \Delta u + f(x,t,u) + \xi = 0 \text{ in the sense of distributions.} \tag{4.20}$$

We need to verify that $\xi = g(x,t,u_t)$. By (4.17) and (4.18),

$$|u_t(t)|^2_{2,B(x_0,\rho)} + |\nabla u(t)|^2_{2,B(x_0,\rho)} + \liminf_{\varepsilon\to 0}\int_0^t\int_{B(x_0,\rho)} g(x,s,u_{\varepsilon t}(x,s))u_{\varepsilon t}(x,s)dxds \le C. \tag{4.21}$$

By the monotonicity of g we also have that:

$$\int_0^t\int_{B(x_0,\rho)} (g(x,s,u_{\varepsilon t}(x,s)) - g(x,s,\phi(x,s)))(u_{\varepsilon t}(x,s) - \phi(x,s))dxds \ge 0. \tag{4.22}$$

for every $\phi \in L^m((0,t)\times B(x_0,\rho)$. We have that:

$$\begin{aligned} \liminf_{\varepsilon\to 0}\int_0^t\int_{B(x_0,\rho)} &(g(x,s,u_{\varepsilon t}(x,s)) - g(x,s,\phi(x,s)))(u_{\varepsilon t} - \phi(x,s))dxds \\ &= \int_0^t\int_{B(x_0,\rho)} (\xi(x,s) - g(x,s,\phi(x,s)))(u_t(x,s) - \phi(x,s))dxds, \end{aligned} \tag{4.23}$$

as it can be checked by using (4.20) that

$$\liminf_{\varepsilon\to 0}\int_0^t\int_{B(x_0,\rho)} g(x,s,u_{\varepsilon t}(x,s))u_{\varepsilon t}(x,s)dxds = \int_0^t\int_{B(x_0,\rho)} \xi u_t(x,s)dxds,$$

so by combining (4.22) and (4.23) we obtain:

$$\int_0^t\int_{B(x_0,\rho)} (\xi(x,s) - g(x,s,\phi(x,s)))(u_t(x,s) - \phi(x,s))dxds \ge 0,$$

for all $t < \rho/2$, so by passing to limit as $t \to \rho/2$, it holds also for $t = \rho/2$. By choosing ϕ appropriately ($\phi = u_t \pm \lambda v$ for $\lambda > 0$ and any $v \in C_c^\infty(B(x_0,\rho))$ and leting $\lambda \to 0$ for both choices), we obtain the desired equality $\xi = g$.

By moving x_0, we can cover the entire space $\mathbb{R}^n$ with balls of radius $\rho/2$. For each of them, we obtain that $u(x,s) = u^*(x,s)$, if $|x - x_0| < \dfrac{\rho}{2} + t - s$.

Thus we obtain existence of a weak solution u up to time $\rho/2$.

□

Remark: This method will not provide a uniqueness result, since the existence result was obtained by passing to the limit in the inequalities which yielded bounds for the approximate solutions.

5. Generalizations and open problems

This proof works in the variable coefficient case, i.e. for the equation:

$$u_{tt} - \sum_{i,j=1}^n a_{ij}(x)u_{x_i,x_j} + f(x,t,u) + g(x,t,u_t) = 0, \tag{5.1}$$

where to the assumptions (A0)-(A6), we add the following assumptions concerning the coefficients a_{ij}. For every $1 \leq i, j \leq n$, we impose that a_{ij} are:

(1) bounded: $a_{ij} \in L^{\infty}(\mathbb{R}^n)$;
(2) symmetric: $a_{ij} = a_{ji}$;
(3) elliptic: $\sum_{i,j=1}^{n} a_{ij}(x) x_i x_j \geq 0$, for every $x \in \mathbb{R}^n$ with components x_i.

This generalization is mainly possible due to the fact that the arguments used in the proof of Theorem 3.1 and in the finite propagation speed property do not critically rely the fact that the coefficients are constant. The rest of the proof can be easily adjusted.

This remains a local in time existence result. In order to obtain global existence, it would suffice if one were able to choose the value ρ like in (4.10), but independent of time, so one can reiterate the local result and obtain that the solution exits up to $T = \infty$.

It is our belief that the result should hold with no polynomial growth assumptions on g, since they they were not needed in the auxiliary Theorem 3.1, but it is still under investigation if the monotonicity alone will suffice. It would also be interesting to see if these techniques could be adapted in order to allow more general nonlinearities.

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