Lectures at the CNA Summer School, June 2008
(mostly) on Energy-driven Pattern Formation
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Overview: Nature is full of energy-driven patterns. Some represent local or global minimizers of a suitable free energy. Others are self-organized transients produced by energy-dissipating dynamics. Simulation can demonstrate the adequacy of a model, but it rarely explains "why" a pattern forms. Nonlinear PDE and the calculus of variations can sometimes provide a more global understanding. I'll give four independent lectures on problems of this type, followed by a fifth lecture that's a bit different.

Note: references on which I'm an author can be downloaded as pdf files from www.math.nyu.edu/faculty/kohn.

Lecture 1: Bounds on coarsening rates. Some energy-driven systems develop interesting patterns transiently (as they evolve) rather than in steady state (at local minima). An example is the coarsening of a complex initial state under motion by surface diffusion. In this setting (and many others), the "local length scale" increases with time, often with an exponent that can be guessed by dimensional analysis. I'll introduce this phenomenon, then discuss a scheme introduced with F. Otto a few years ago for proving an upper bound on the coarsening rate, focusing on one of the earliest applications: "motion by surface diffusion."

1. R. Kohn and F. Otto, *Upper bounds on coarsening rates*, Comm. Math. Phys. 229 (2002) 375-395 (focuses on diffuse-interface models; the sharp-interface setting of my lecture is a little simpler)

2. R. Kohn and X. Yan, *Upper bounds on the coarsening rate for an epitaxial growth model*, Comm. Pure App. Math. 56 (2003) 1549-1564 (this paper's introduction is shorter and more focused than the CMP paper)

3. R. Kohn and X. Yan, *Coarsening rates for models of multicomponent phase separation*, Interfaces and Free Boundaries 6 (2004) 135-149 (again, this paper's introduction is shorter and more focused than the CMP paper)

4. F. Otto, T. Rump, and D. Slepcev, *Coarsening rates for a droplet model: rigorous upper bounds*, SIAM J. Math. Anal. 38 (2006) 503-529 (besides giving an interesting new application of the method, this paper explains that there's a natural way to choose the "negative Sobolev norm" L.)

5. S. Dai and R. Pego, *Universal bounds on coarsening rates for mean-field models of phase transitions*, SIAM J. Math. Anal. 37 (2005) 347-371.

6. R. Pego, *Lectures on dynamics in models of coarsening and coagulation*, in Dynamics in Models of Coarsening, Coagulation, Condensation, and Quantization, W. Bao and J-G Liu eds, (Lect. Notes Ser. Inst. Math. Sci., Nat. Univ. Singapore) World Scientific, 2007. Available from www.math.cmu.edu/cna as preprint 06-CNA-001

7. S. Conti, F. Otto, B. Niethammer, *Coarsening rates in off-critical mixtures*, SIAM J. Math. Anal. 37 (2006) 1732-1741

8. R. Choksi, S. Conti, R. Kohn, F. Otto, *Ground state scaling laws during the onset and destruction of the intermediate state in a type-I superconductor*, Comm. Pure Appl. Math. 61 (2008) 595-626 (not about coarsening; but the end of Section 3 discusses our "interpolation inequality" in the low-volume-fraction regime, getting the optimal dependence of the "constant" on volume fraction)

9. S. Esedoglu and J. Greer, *Upper bounds on the coarsening rate of discrete, ill-posed nonlinear diffusion equations*, Comm. Pure Appl. Math, in press. Available from www.math.lsa.umich.edu/∼esedoglu

Lectures 2 and 3: The internal structure of a cross-tie wall. The cross-tie wall is a particular type of domain wall that forms in soft, thin ferromagnetic films. I'll explain its structure by identifying an associated variational problem, then showing that the pattern we see achieves its minimum. Central issues include (a) the relation between sharp-interface and diffuse-interface models, and (b) use of suitable "entropies" to prove lower bounds on the energy of a boundary value problem. In exploring these issues we'll discuss the Modica-Mortola problem and the Aviles-Giga problem as well as the Alouges-Rivière-Serfaty picture of a cross-tie wall.

1. R. Kohn, *Energy-driven pattern formation*, in Proceedings of the International Congress of Mathematicians – Madrid, August 22-30, 2006, Vol 1, M. Sanz-Solé et. al. eds., European Mathematical Society, 2007 (an expository article; Section 3 is very close to my lectures).

2. A. DeSimone, R. Kohn, S. Müller, F. Otto, *Recent analytical developments in micromagnetics*, in The Science of Hysteresis II: Physical Modeling, Micromagnetics, and Magnetization Dynamics, G. Bertotti and I. Mayergoyz eds., pp. 269–381, Elsevier 2006 (a review article; Section 6.5 is close to my lecture).

3. F. Alouges, T. Rivière, S. Serfaty, *Néel and cross-tie wall energies for planar magnetic configurations*, ESAIM:COCV 8 (2002) 31-68 (more general than my treatment; their formulation has diffuse rather than sharp walls)

4. Y. Nakatani, Y. Uesaka, N. Hayashi, *Direct solution for the Landau-Lifshitz-Gilbert equation for micromagnetics*, Jap. J. Appl. Phys. 28 (1989) 2845-2507 (includes both numerical and experimental pictures of cross-tie walls; my favorite experimental picture comes from there)

5. W. Jin and R. Kohn, *Singular perturbation and the energy of folds*, J. Nonlin. Sci. 10 (2000) 355-390 (first use of "entropy" for Aviles-Giga)

6. A. DeSimone, R. Kohn, S. Müller, F. Otto, *Repulsive interaction of Néel walls, and the internal length scale of the cross-tie wall*, Multiscale Modeling & Simulation 1 (2003) 57-104 (explains internal length scale of cross-tie wall).

7. A. DeSimone, R. Kohn, S. Müller, F. Otto, *A compactness result in the gradient theory of phase transitions*, Proc. Royal Soc. Edinburgh 131A (2001) 833-844 (uses entropies to prove compactness for Aviles-Giga).

Lecture 4: The sharp-interface limit of action minimization. Energy-driven systems typically achieve local not global minima. Thermal fluctuations lead to switching from one local minimum to another. The action functional identifies the rate and most likely pathway of switching. I'll introduce this topic, then consider the sharp-interface limit of action minimization for the Modica-Mortola functional, drawing on recent joint work with F. Otto, Y. Tonegawa, E. Vanden-Eijnden, and M. Westdickenberg.

1. R. Kohn, *Energy-driven pattern formation*, in Proceedings of the International Congress of Mathematicians – Madrid, August 22-30, 2006, Vol 1, M. Sanz-Solé et. al. eds., European Mathematical Society, 2007 (an expository article; Section 4 is very close to my lecture).

2. M. Westdickenberg, *Rare events, action minimization, and sharp interface limits*, to appear in proc. CRM Workshop on Singularities in PDE and the Calculus of Variations (an expository article, close to my lecture, goes further than my ICM piece). Available from www.math.gatech.edu/∼maria

3. R. Kohn, F. Otto, M. Reznikoff, E. Vanden-Eijnden, *Action minimization and sharp-interface limits for the stochastic Allen-Cahn equation*, Comm. Pure Appl. Math. 60 (2007) 393-438 (partly formal, partly rigorous)

4. R.V. Kohn, M. Reznikoff, Y. Tonegawa, *Sharp-interface limit of the Allen-Cahn action functional in one space dimension*, Calc. Var. PDE 25 (2006) 503-534 (fully rigorous lower bound in 1D)

5. M. Westdickenberg and Y. Tonegawa, *Higher multiplicity in the one-dimensional Allen-Cahn action functional*, Indiana Univ Math J. 56 (2007) 2935-2990 (sharp-interface Γ-limit of action functional in 1D)

6. L. Mugnai and M. Röger, *The Allen-Cahn functional in higher dimensions*, Interfaces and Free Boundaries 10 (2008) 45-78 (sharp-interface Γ-limit of action functional in 2 and 3 dimensions)

7. M. Röger and R. Schätzle, *On a modified conjecture of DeGiorgi*, Math. Zeitschrift 254 (2006) 675-714 (sharp-interface Γ-limit of action is closely related to DeGiorgi's conjecture)

8. R. Kohn, M. Reznikoff, and E. Vanden-Eijnden, *Magnetic elements at finite temperature and large deviation theory*, J. Nonlin. Sci. 15 (2005) 223-253 (includes an expository discussion of action minimization, and an application involving magnetic switching)

9. M. Heymann and E. Vanden-Eijnden, *The geometric minimum action method: a least action principle on the space of curves*, Comm. Pure Appl. Math., presently online

in “early-view” (recent progress on numerical methods for finding action-minimizing paths in non-gradient systems)

10. W. E, W. Ren, and E. Vanden-Eijnden, *Simplified and improved string method for computing the minimum energy paths in barrier-crossing events*, J. Chem. Phys. 126 (2007) 164103 (recent progress on finding saddles, thereby determining transition rates for gradient systems)

Lecture 5: Cloaking by change of variables. We say a region of space is “cloaked” with respect to electromagnetic measurements if its contents – and even the existence of the cloak – are inaccessible to such measurements. One recent proposal for achieving cloaking takes advantage of the coordinate-invariance of Maxwell’s equations. I’ll explain this scheme, including its mathematical basis and its apparent limitations, drawing on recent work with Onofrei, Shen, Vogelius, and Weinstein.

1. R. Kohn, H. Shen, M. Vogelius, and M. Weinstein, *Cloaking via change of variables in electric impedance tomography*, Inverse Problems 24 (2008) 015016 (this paper includes a long, expository introduction and is a lot like my lecture)

2. A. Greenleaf, M. Lassas, and G. Uhlmann, *On nonuniqueness for Calderon’s inverse problem*, Math. Res. Lett. 10 (2003) 685-693

3. J. Pendry, D. Schurig, and D. Smith, *Controlling electromagnetic fields,* Science 312 (2006) 1780-1782

4. A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, *Full-wave invisibility of active devices at all frequencies*, Comm. Math. Phys. 275 (2007) 749-789

5. R. Kohn, D. Onofrei, M. Vogelius, and M. Weinstein, in preparation (this work discusses the design of a change-of-variable-based “near-cloak” in the finite-frequency setting, for a system described by Helmholtz’s equation).