Non-Essential Uses of Probability in Analysis
Part IV
Efficient Markovian Couplings

Krzysztof Burdzy
University of Washington

Review

See B and Kendall (2000) for more details. See also the unpublished monograph of Aldous and Fill (1999).

X — positive-recurrent Markov process symmetric with respect to some reference measure m,

$p(t, x, y)$ — density relative to m,

$$\boxed{p(t, x, y) = c + g(x, y)e^{-\mu_2 t} + R(t, x, y),} \quad (1)$$

first eigenvalue $= 0$,

first eigenfunction $= c$,

$\mu_2 =$ second eigenvalue,

$g(x, y) = \sum_\varphi \varphi(x)\varphi(y)$ where the φ are orthogonal eigenfunctions with eigenvalue μ_2,

$R(t, x, y) \to 0$ faster than $e^{-\mu_2 t}$ as $t \to \infty$

A "coupling" is a pair (X^1, X^2) of (typically dependent) copies of the Markov process X. "Good" couplings are characterized by small coupling time

$$\boxed{\tau = \inf\{t \geq 0 : X_t^1 = X_t^2\}.}$$

Typically, X^1 and X^2 are constructed so that $X_t^1 = X_t^2$ for all $t \geq \tau$.

Suppose that $(X_0^1, X_0^2) = (x_1, x_2)$.

The eigenfunction representation (1) gives

$$p(t,x_1,y)-p(t,x_2,y) \tag{2}$$
$$=(g(x_1,y)-g(x_2,y))e^{-\mu_2 t}$$
$$+R(t,x_1,y)-R(t,x_2,y)$$

while the coupling yields

$$|p(t,x_1,y)dy-p(t,x_2,y)dy| \tag{3}$$
$$=|P(X_t^1\in dy \mid X_0^1=x_1)$$
$$-P(X_t^2\in dy \mid X_0^2=x_2)|$$
$$\le P(X_t^1\in dy, t<\tau \mid X_0^1=x_1)$$
$$+P(X_t^2\in dy, t<\tau \mid X_0^2=x_2).$$

Suppose that one can prove that

$$P(t<\tau \mid X_t^1=x_1, X_t^2=x_2)\approx e^{-\mu^* t}. \tag{4}$$

(2), (3) and (4) imply that $\mu^*\le\mu_2.$
We will call μ^* the coupling exponent.

Informal Definition of Efficiency

We will call a coupling (X^1, X^2) an efficient Markovian coupling if (X^1, X^2) is a Markov process and $\mu^* = \mu_2$.

Informal Efficient Coupling Heuristic

A coupling (X^1, X^2) is efficient if and only if, for all t, and given $\{t < \tau\}$, the conditional distributions of (X^1_t, X^2_t) and (X^2_t, X^1_t) are singular with respect to each other.

The above heuristic is NOT true in a rigorous sense but it works in sufficiently many circumstances to make it "almost true."

Symmetric Markov chains with finite state space

$X = \{X_t : t \geq 0\}$ — a continuous time symmetric Markov process with a finite state space D and transition probabilities

$$\begin{aligned} p(t, y, x) &= p(t, x, y) \\ &= P(X_{s+t} = y \mid X_s = x). \end{aligned}$$

(X^1, X^2) — a Markovian coupling for the process X: $\{(X_t^1, X_t^2) : t \geq 0\}$, $\{X_t^1 : t \geq 0\}$ and $\{X_t^2 : t \geq 0\}$ are Markov with respect to the filtration generated jointly by X^1 and X^2, and X^1 and X^2 have the same transition probabilities as X.

$$\tau = \inf\{t \geq 0 : X_t^1 = X_t^2\}$$

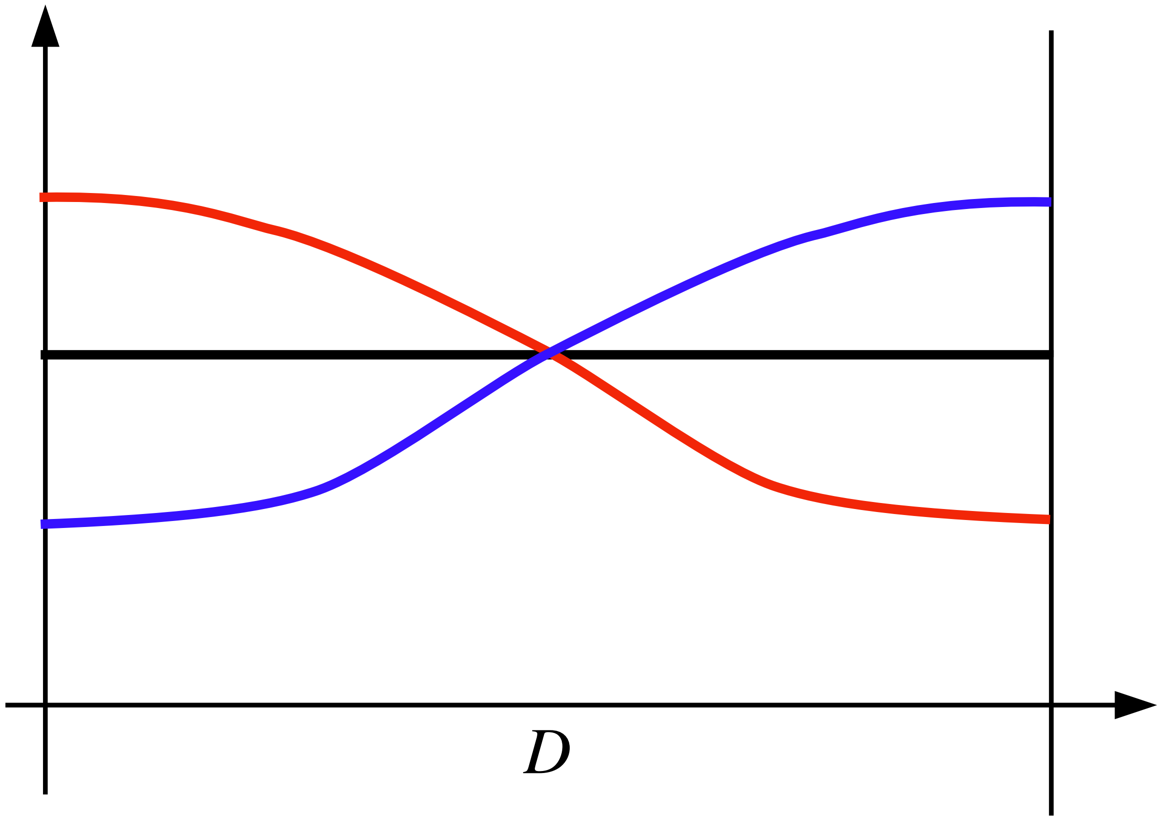
$X_t^1 = X_t^2$ for all $t \geq \tau$.

We will say that D^2 is irreducible with respect to a given coupling if every state (y_1, y_2) with $y_1 \neq y_2$ is accessible from any other state (x_1, x_2) with $x_1 \neq x_2$.

Theorem (i) If D^2 is irreducible for a coupling then this coupling is not efficient.
(ii) Suppose that for every pair of distinct points $x_1, x_2 \in D$ there exists a function $f : D \to \mathbf{R}$ with the property that

$$P^{(x_1,x_2)}(f(X_t^1) - f(X_t^2) > 0 \mid t < \tau) = 1$$

Then the coupling is efficient.



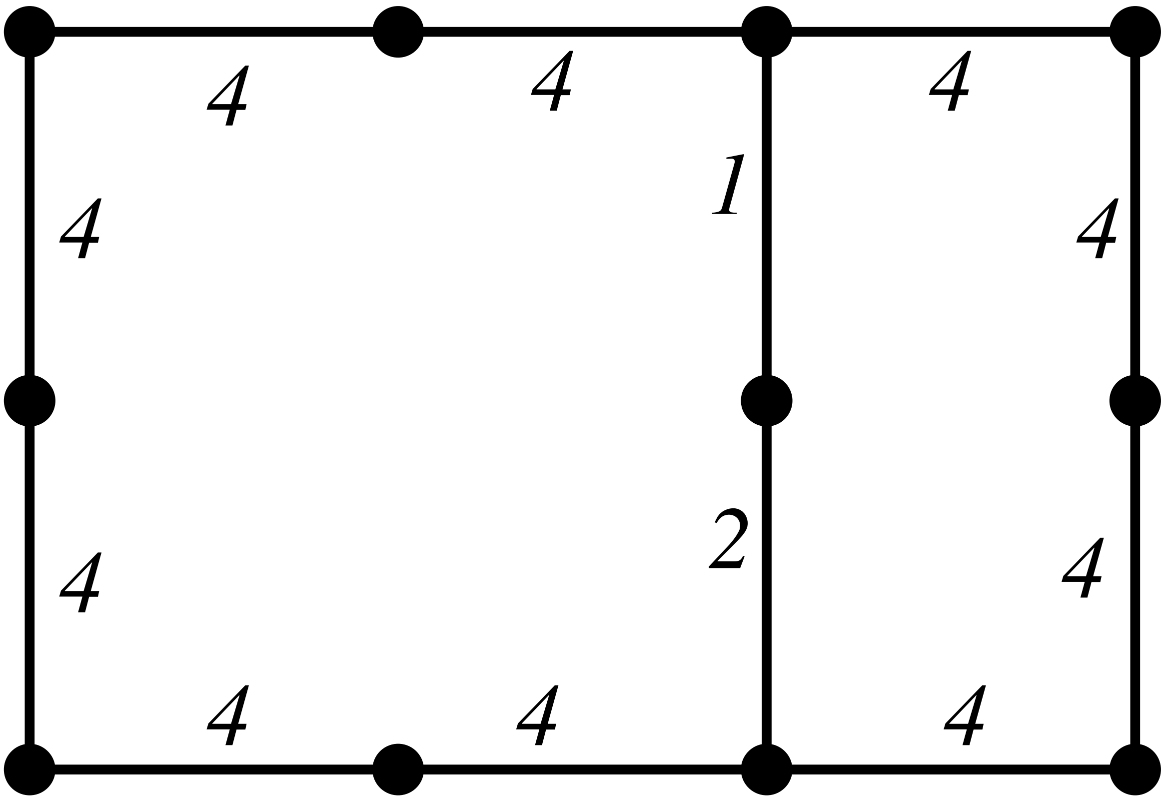
D

Example

Suppose the state space D is a finite subinterval of the integers $\mathbf{Z}$ and X can jump from x only to $x-1$ or $x+1$, for every x. Assume that (X_t^1, X_t^2) almost surely never jumps to (X_{t-}^2, X_{t-}^1). Then (X^1, X^2) is an efficient coupling. In particular, the coupling is efficient if X^1 and X^2 have independent jumps until the coupling time τ.

Example

Consider a Markov process with jump rates indicated in the figure. There are no efficient couplings for this process.



Example—"Ehrenfest model"

Consider two urns with n marked balls distributed among them. At every arrival time for a Poisson process, a ball is randomly chosen from among all balls in both urns and moved to the other urn.

$D = \{(i_1, i_2, \ldots, i_n) : i_k = 0 \text{ or } 1\}$

U_k — i.i.d. exponential, mean 1

$T_k = U_1 + \ldots + U_k$

N_k — i.i.d. uniform on $\{1, 2, \ldots, n\}$

$\{J_k\}$ — i.i.d. with

$$P(J_k = 0) = P(J_k = 1) = 1/2$$

$$X_{T_k -} = (j_1, j_2, \ldots, j_{N_k}, \ldots, j_n)$$
$$\to X_{T_k} = (j_1, j_2, \ldots, J_k, \ldots, j_n)$$

(X^1, X^2): use one family $\{T_k, N_k, J_k\}_{k\geq 1}$

$$(X^1_{T_k-}, X^2_{T_k-}) =$$

$$((j^1_1, \ldots, j^1_{N_k}, \ldots, j^1_n), (j^2_1, \ldots, j^2_{N_k}, \ldots, j^2_n))$$

$$\downarrow$$

$$(X^1_{T_k}, X^2_{T_k}) =$$

$$((j^1_1, \ldots, J_k, \ldots, j^1_n), (j^2_1, \ldots, J_k, \ldots, j^2_n))$$

Suppose that $X_0^1 = (j_1^1, j_2^1, \ldots, j_n^1)$. Let $f(i_1, i_2, \ldots, i_n)$ be the number of k such that $i_k = j_k^1$. If $X_0^1 \neq X_0^2$ then $f(X_t^1) - f(X_t^2) > 0$ for all $t < \tau$. Hence the coupling is efficient.

Elementary estimates yield $\mu^* = 1/n$, so $\mu_2 = 1/n$.

However, the mean time to coupling is not of order $1/\mu^* = n$. $E\tau \approx n \log n$.

See Liggett (1985) for the use of monotonicity in the context of particle system couplings.

Reflected Brownian motion

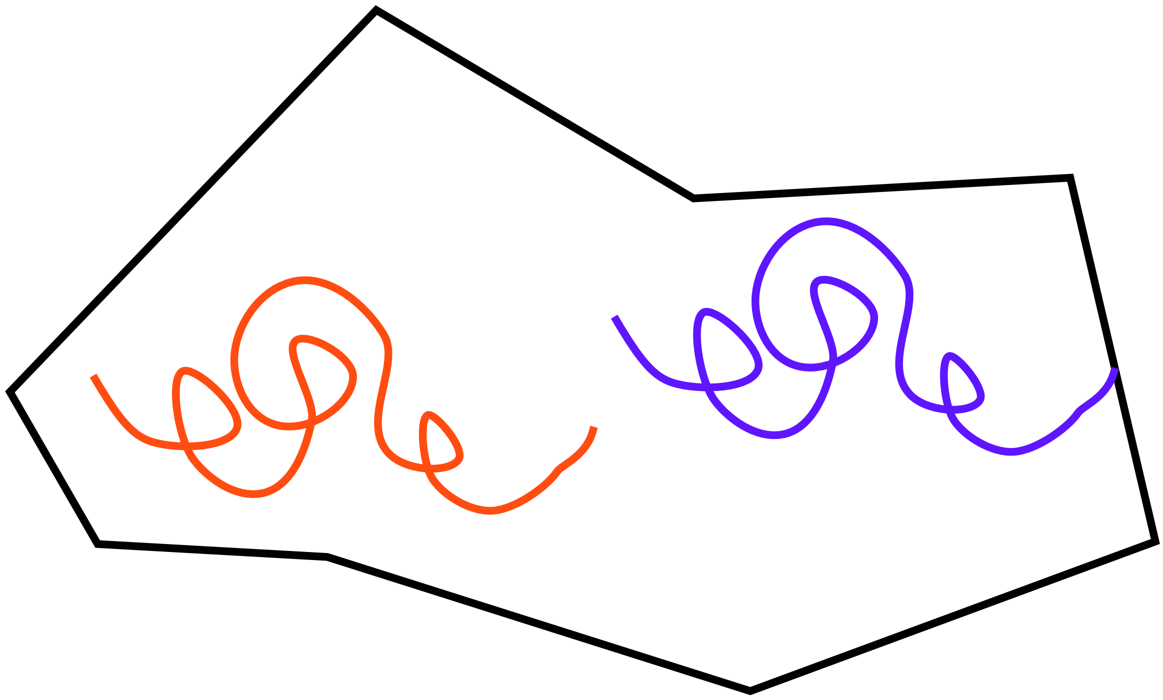
Synchronous couplings

D — Lipschitz domain in the plane

B — planar Brownian motion

$$X_t = x_0 + B_t + \int_0^t \mathbf{n}(X_s) dL_s^X,$$

$$Y_t = y_0 + B_t + \int_0^t \mathbf{n}(Y_s) dL_s^Y.$$



Cranston and Le Jan (1990): Synchronous couplings never meet in strictly convex domains.

"Generic" behavior:

$$\tau := \inf\{t \geq 0 : X_t = Y_t\} = \infty.$$

$p(t, x, y)$ — transition densities for X,

$$p(t, x, y) = c_1 + \varphi_2(x)\varphi_2(y)e^{-\mu_2 t} + R(t, x, y),$$

$R(t, x, y) \to 0$ faster than $e^{-\mu_2 t}$ as $t \to \infty$

Lemma Suppose that for some $\mu \geq 0$ and $x, y \in D$,

$$E^{(x,y)}|X_t - Y_t| \leq c(x, y)e^{-\mu t}$$

for $t \geq 0$. If $\varphi_2(x) \neq \varphi_2(y)$ then $\mu \leq \mu_2$.

Proof The function φ_2 is not identically equal to 0 so,

$$\int_D \exp(c_1 z^1 + c_2 z^2)\varphi_2(z^1, z^2)dz^1dz^2 > 0,$$

for some $c_1, c_2 \neq 0$. Since D is bounded there exists $c_3 > 0$ such that c_3^{-1} is a Lipschitz constant for $\exp(c_1x^1 + c_2x^2)$, and so

$$\begin{aligned} &E|X_t - Y_t| \\ &\geq c_3\Big(E[\exp(c_1X_t^1 + c_2X_t^2)] \\ &\quad - E[\exp(c_1Y_t^1 + c_2Y_t^2)]\Big). \end{aligned}$$

Then,

$$
\begin{aligned}
c(x,y)e^{-\mu t} &\geq E^{(x,y)}|X_t - Y_t| \\
&\geq c_3\Big(E\exp(c_1X_t^1 + c_2X_t^2) \\
&\quad - E\exp(c_1Y_t^1 + c_2Y_t^2)\Big) \\
&= c_3\int_D \exp(c_1z_1 + c_2z_2)p(t,x,z)dz \\
&\quad - c_3\int_D \exp(c_1z_1 + c_2z_2)p(t,y,z)dz \\
&= c_3\int_D [\varphi_2(x) - \varphi_2(y)]\exp(c_1z_1 + c_2z_2)\times \\
&\quad \times \varphi_2(z)e^{-\mu_2 t}dz + R(t,x,y) \\
&= c_4(x,y)e^{-\mu_2 t} + R(t,x,y).
\end{aligned}
$$

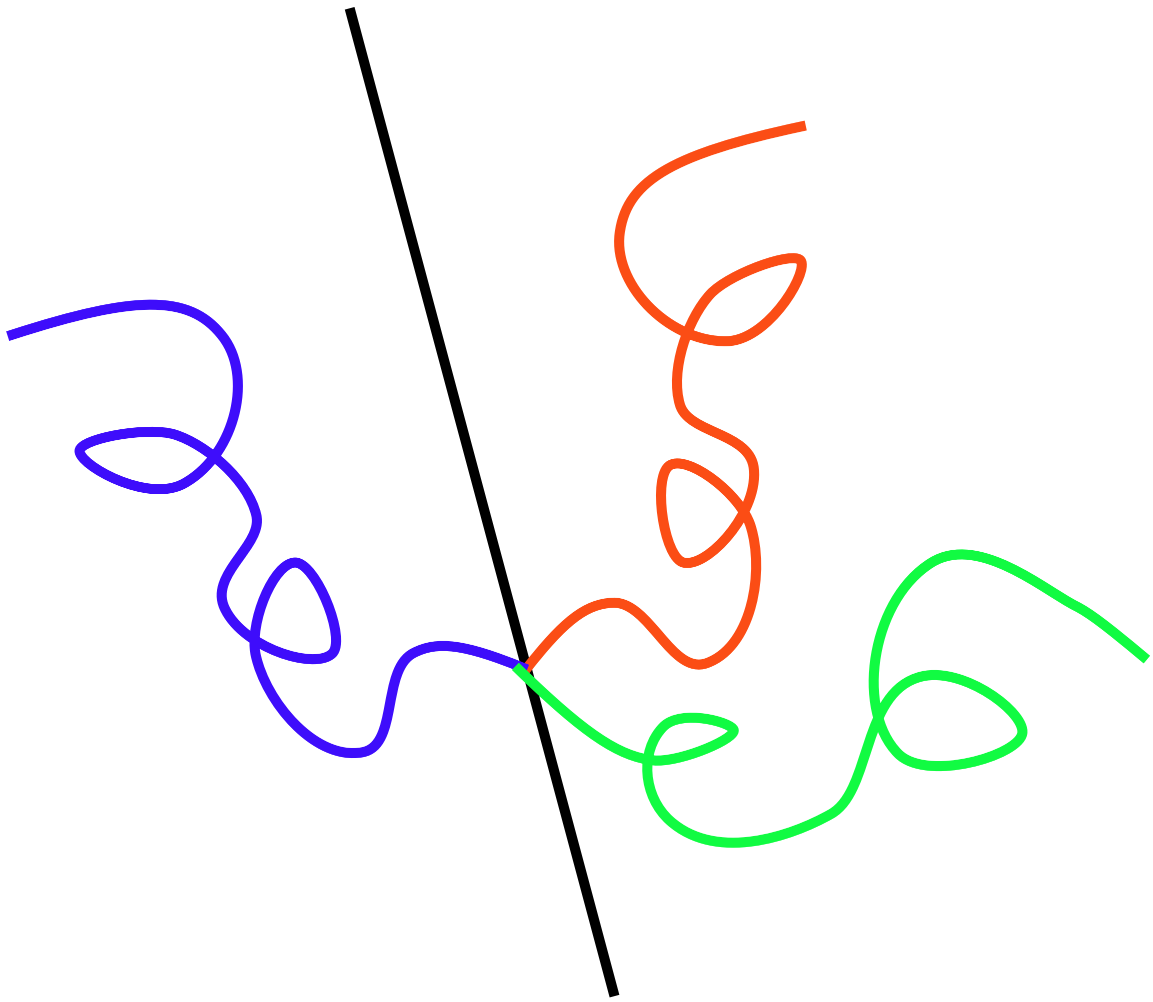
Hence $\mu \leq \mu_2$.

Definition We will call a synchronous coupling (X, Y) of reflected Brownian motions in D efficient if for some x and y with $\varphi_2(x) \neq \varphi_2(y)$, we have $\mu = \mu_2$.

Theorem If D is a triangle with an angle strictly greater than $\pi/2$ then the synchronous coupling for the reflected Brownian motion in D is efficient.

Conjecture Synchronous couplings are not efficient in triangles with all angles less than $\pi/2$.

Mirror couplings



"Generic" behavior:

$$\tau := \inf\{t \geq 0 : X_t = Y_t\} \boxed{< \infty.}$$

Lemma Suppose for some $x, y \in D$,

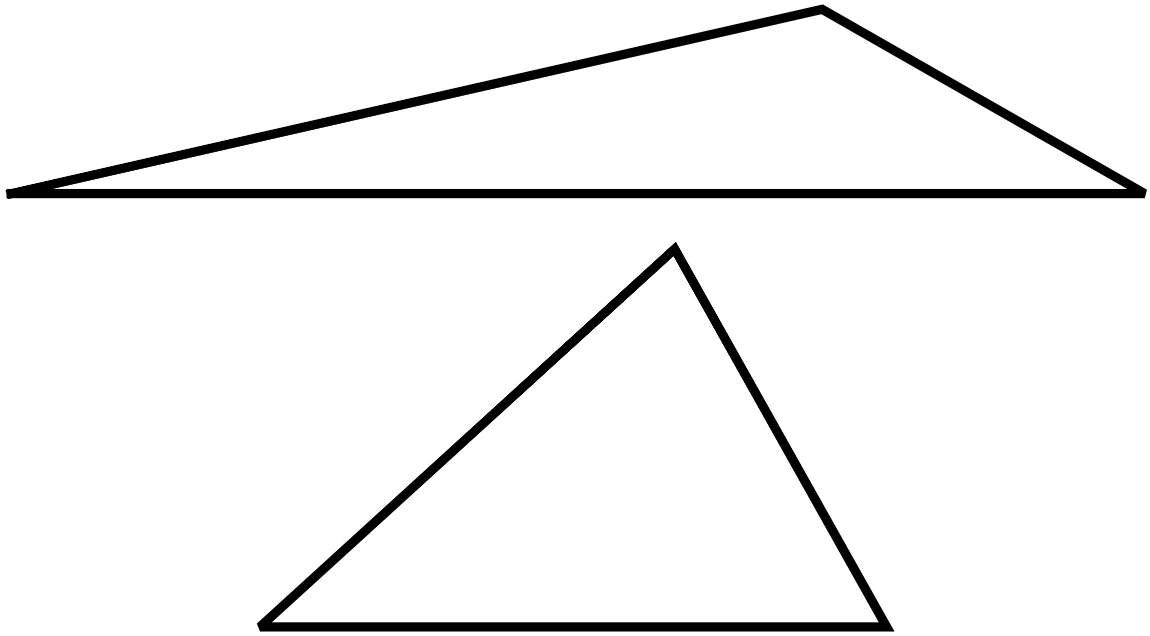
$$P^{(x,y)}(\tau > t) \leq c(x,y)e^{-\mu t}$$

for $t \geq 0$. If $\varphi_2(x) \neq \varphi_2(y)$ then $\mu \leq \mu_2$.

Definition A mirror coupling (X, Y) of reflected Brownian motions in D is said to be efficient if $\mu = \mu_2$ for some x and y with $\varphi_2(x) \neq \varphi_2(y)$.

Theorem (i) If D is a triangle with an angle strictly greater than $\pi/2$ then the mirror coupling for the reflected Brownian motion in D is efficient.
(ii) If all angles of the triangle D are distinct from each other and strictly less than $\pi/2$ then the mirror coupling for the reflected Brownian motion in D is not efficient.

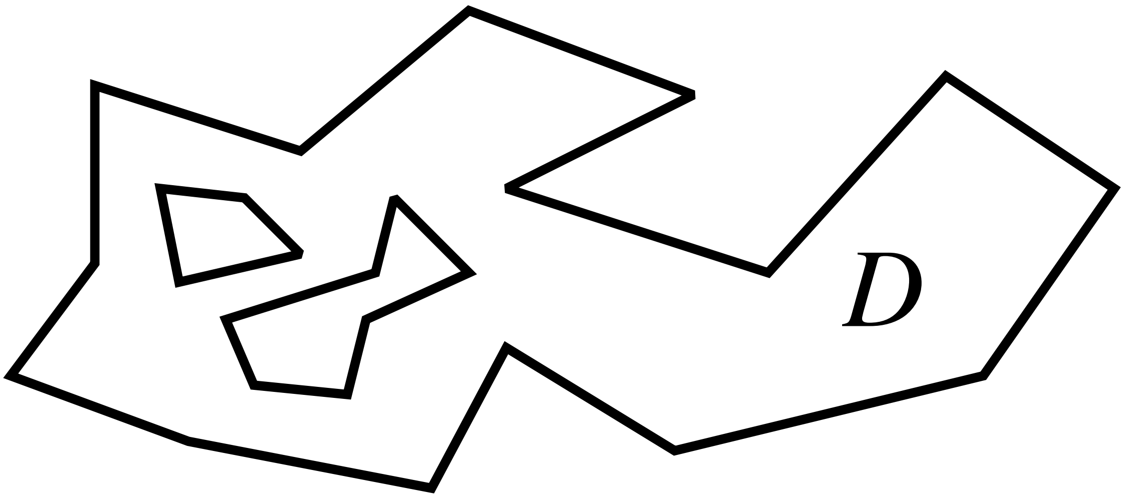
Open problem Does there exist an efficient coupling of reflected Brownian motions in a triangle with acute angles?

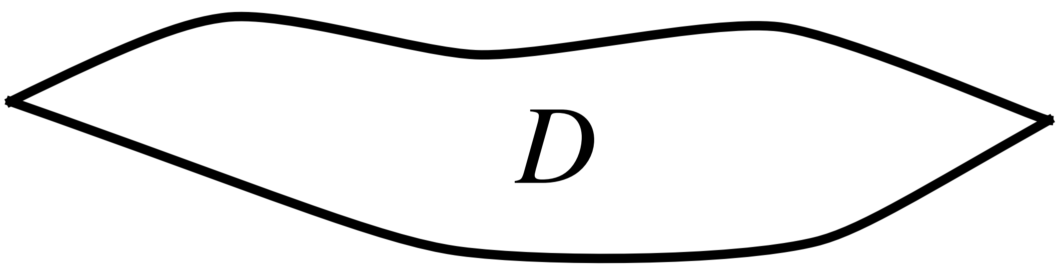


Convergence of synchronous couplings

(X, Y) — synchronous coupling of reflected Brownian motions in a planar domain D

Theorem (B and Chen 2002) $|X_t - Y_t| \to 0$ as $t \to \infty$ if (i) ∂D consists of polygonal lines or (ii) D is a lip domain.



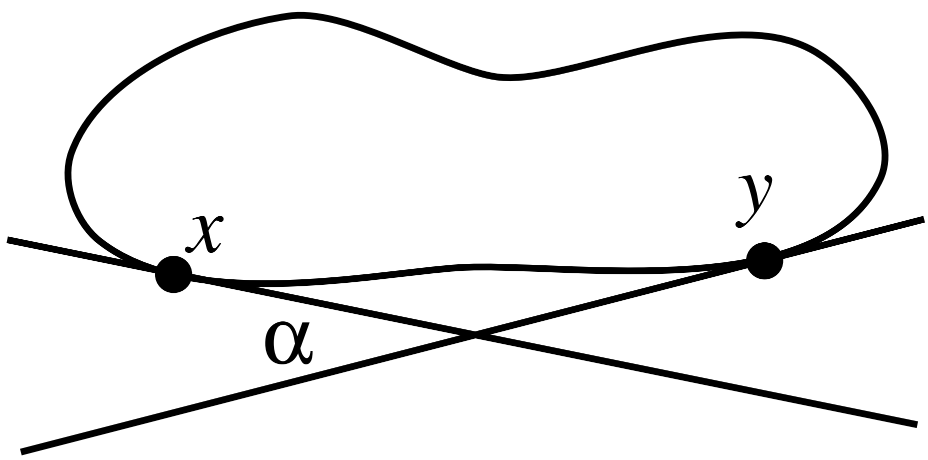


Theorem (B, Chen and Jones 2006) Suppose that D has smooth boundary. Let

$$\begin{aligned}\Lambda(D) = &\int_{\partial D} \nu(x)dx \\ &+ \int_{\partial D}\int_{\partial D} |\log\cos\alpha(x,y)|\omega_x(dy)dx \\ &= 2\pi(\underline{1 - \#\text{ of holes in }D}) \\ &+ \int_{\partial D}\int_{\partial D} |\log\cos\alpha(x,y)|\omega_x(dy)dx.\end{aligned}$$

If $\Lambda(D) > 0$ then

$$\lim_{t\to\infty} \frac{\log|X_t - Y_t|}{t} = -\frac{\Lambda(D)}{2|D|}.$$



Corollary $|X_t - Y_t| \to 0$ as $t \to \infty$ if ∂D is smooth and D has at most one hole.

$$|X_t - Y_t| \approx e^{-t\Lambda(D)/(2|D|)}$$

Open problems

(i) Does there exist a bounded planar domain D such that $|X_t - Y_t| \not\to 0$?

(ii) If D is the *exterior* of an ellipse then $\Lambda(D) = 0$. Does there exist a closed curve such that if D is its exterior then $\Lambda(D) < 0$?

REFERENCES

1. D. Aldous and J. Fill (1999) *Reversible Markov Chains and Random Walks on Graphs* (book in preparation, available online: http://www.stat.berkeley.edu /~aldous/RWG/book.html)
2. K. Burdzy and Z. Chen (2002) Coalescence of synchronous couplings *Probab. Theory Rel. Fields* **123**, 553–578.
3. K. Burdzy, Z. Chen and P. Jones (2006) Synchronous couplings of reflected Brownian motions in smooth domains *Illinois. J. Math., Doob Volume* (to appear)
4. K. Burdzy and W. Kendall (2000) Efficient Markovian couplings: examples and counterexamples *Ann. Appl. Probab.* **10**, 362–409.

5. M. Cranston and Y. Le Jan (1990) Noncoalescence for the Skorohod equation in a convex domain of $\mathbf{R}^2$ *Probability Theory and Related Fields* **87**, 241–252.
6. T.M. Liggett (1985) *Interacting Particle Systems* Springer-Verlag, New York.