Essential Uses of Probability in Analysis

Part III. Robin problem in fractal domains

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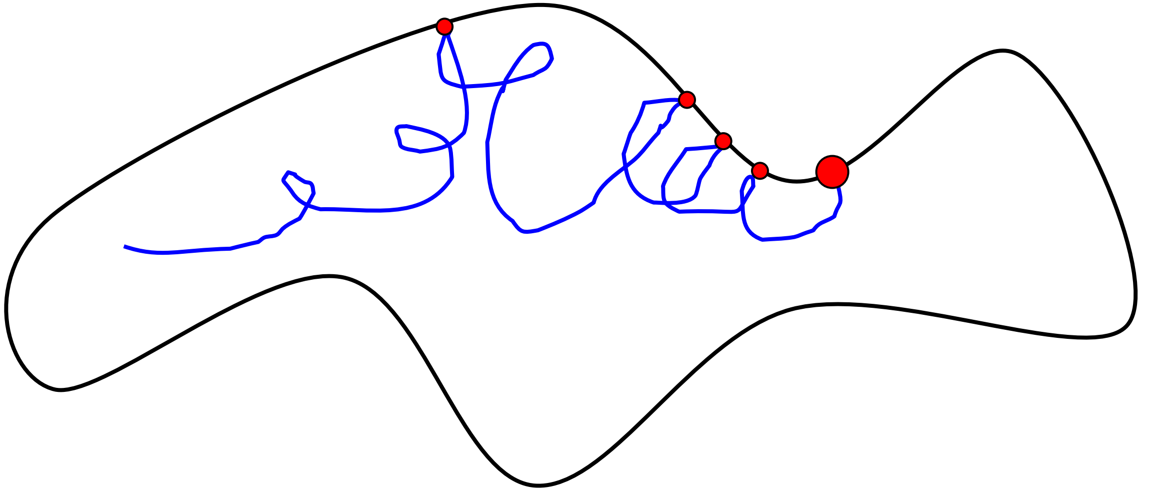
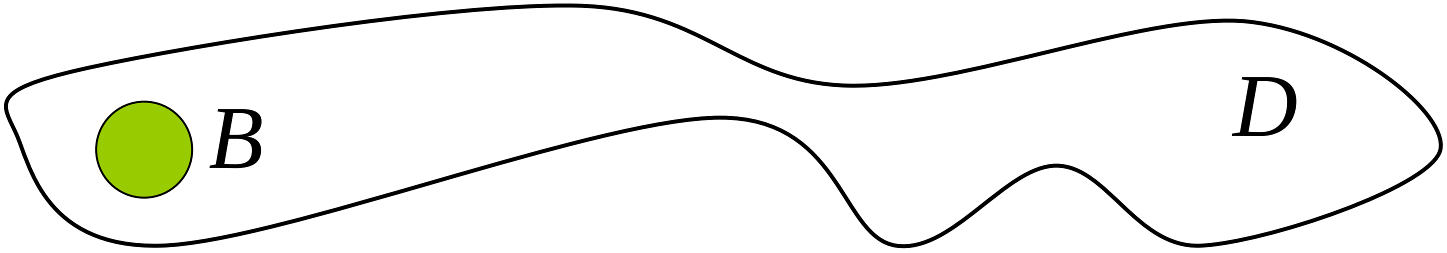
Inspiration – physical phenomena

- physiology
- heterogeneous catalysis
- electrochemistry



Partial differential equation



$$
\begin{cases}
\Delta u(x) = 0, \quad x \in D \setminus B, \\
\dfrac{\partial u}{\partial \boldsymbol{n}}(x) = c\,u(x), \quad x \in \partial D, \\
u(x) = 1, \quad x \in \partial B.
\end{cases}
$$

Robin boundary condition

Remarks

(i) Robin boundary condition is not Robin's (Gustafson and Abe (1998))

(ii) $u(x) > 0, \quad x \in D \setminus B$

(iii) In the rest of the paper $c = 1$

Robin condition:

$$\frac{\partial u}{\partial \boldsymbol{n}}(x) = u(x)$$

Is the whole surface active?

Definition. We say that the whole surface of D is active if

$$\inf_{x\in\partial D} \frac{\partial u}{\partial \boldsymbol{n}}(x) > 0$$

$$\inf_{x\in D\setminus B} u(x) > 0$$

Problem. Give a geometric characterization of domains whose whole surface is active.

Simple observations

If ∂D is smooth or even Lipschitz then the whole surface is active.

If the area of the boundary of D is infinite then it is not the case that the whole surface is active.

Domains with infinite surfaces

Suppose that $\inf_{x\in D\setminus B} u(x) > 0$ and $|\partial D| = \infty$

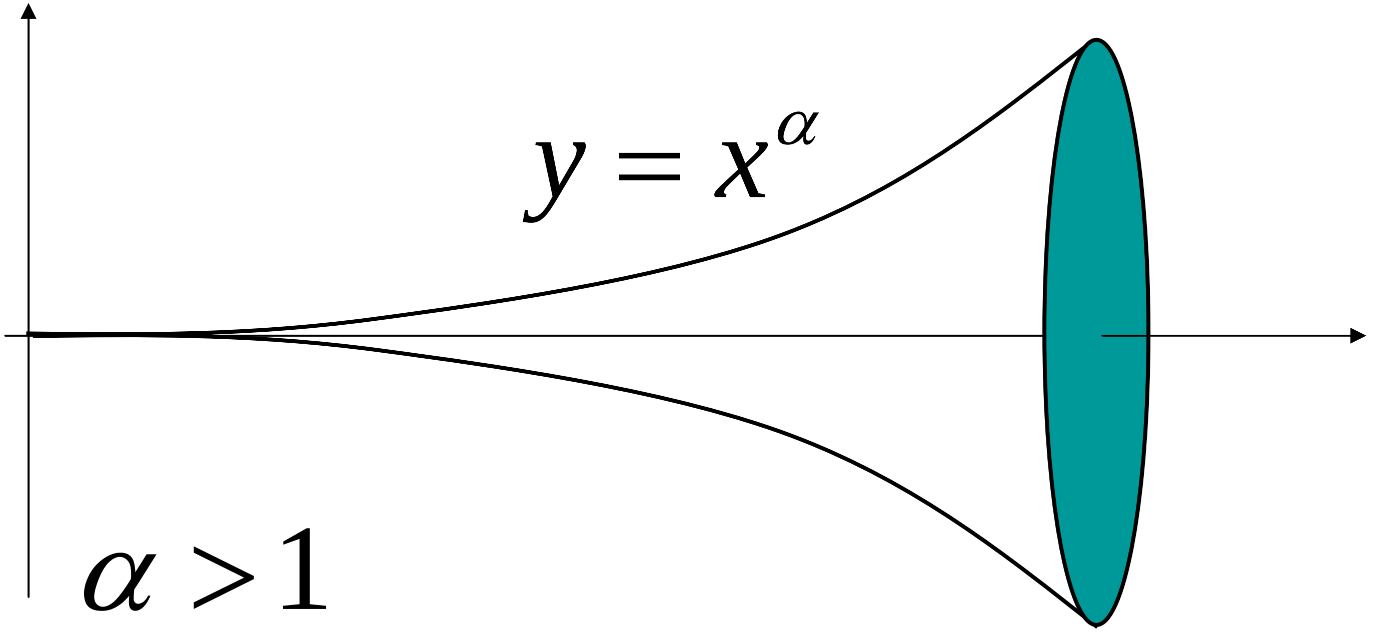
Then $\inf_{x\in\partial D} \frac{\partial u}{\partial \boldsymbol{n}}(x) > 0$

and the flow outside the domain is

$$\int_{\partial D} \frac{\partial u}{\partial \boldsymbol{n}}(x)\, d\sigma(x) = \infty$$

The flow inside the domain through ∂B is finite – a contradiction.

Example I

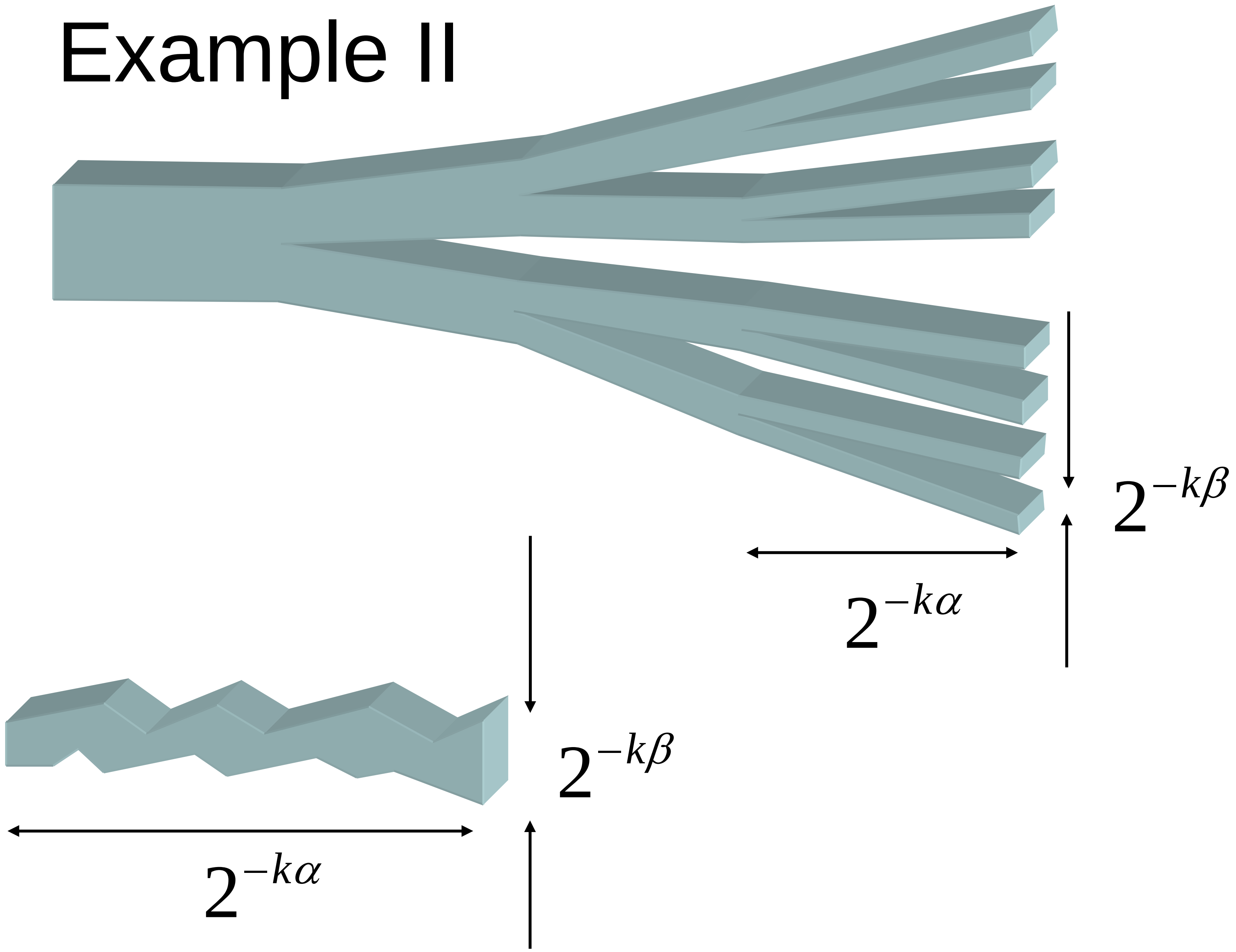


Is the whole surface active?

$$\alpha \in (1,2) \Rightarrow \text{ yes}$$

$$\alpha \geq 2 \Rightarrow \text{ no}$$

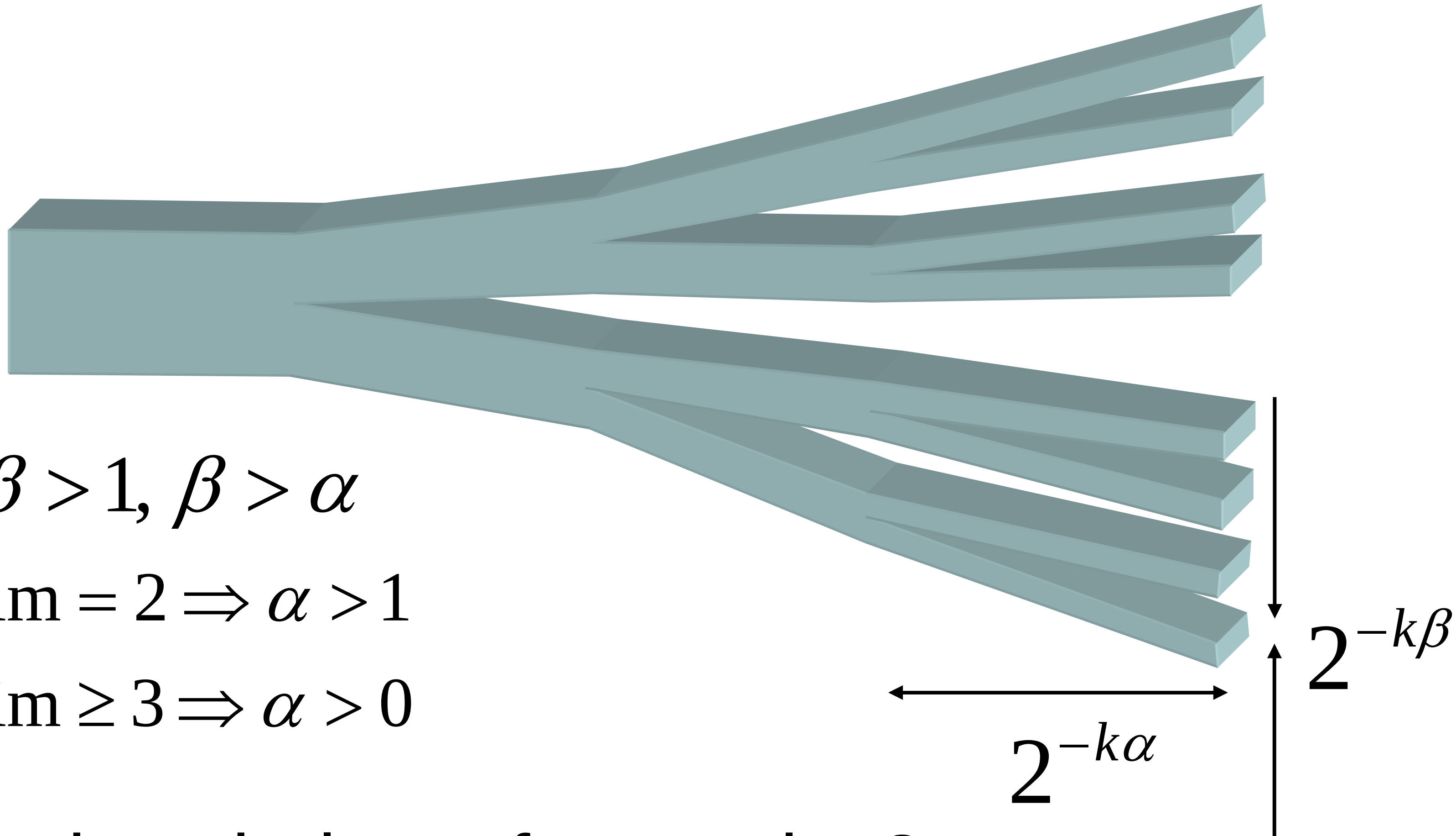
Example II



$$\beta > 1,\ \beta > \alpha$$

$$\dim = 2 \Rightarrow \alpha > 1$$

$$\dim \geq 3 \Rightarrow \alpha > 0$$

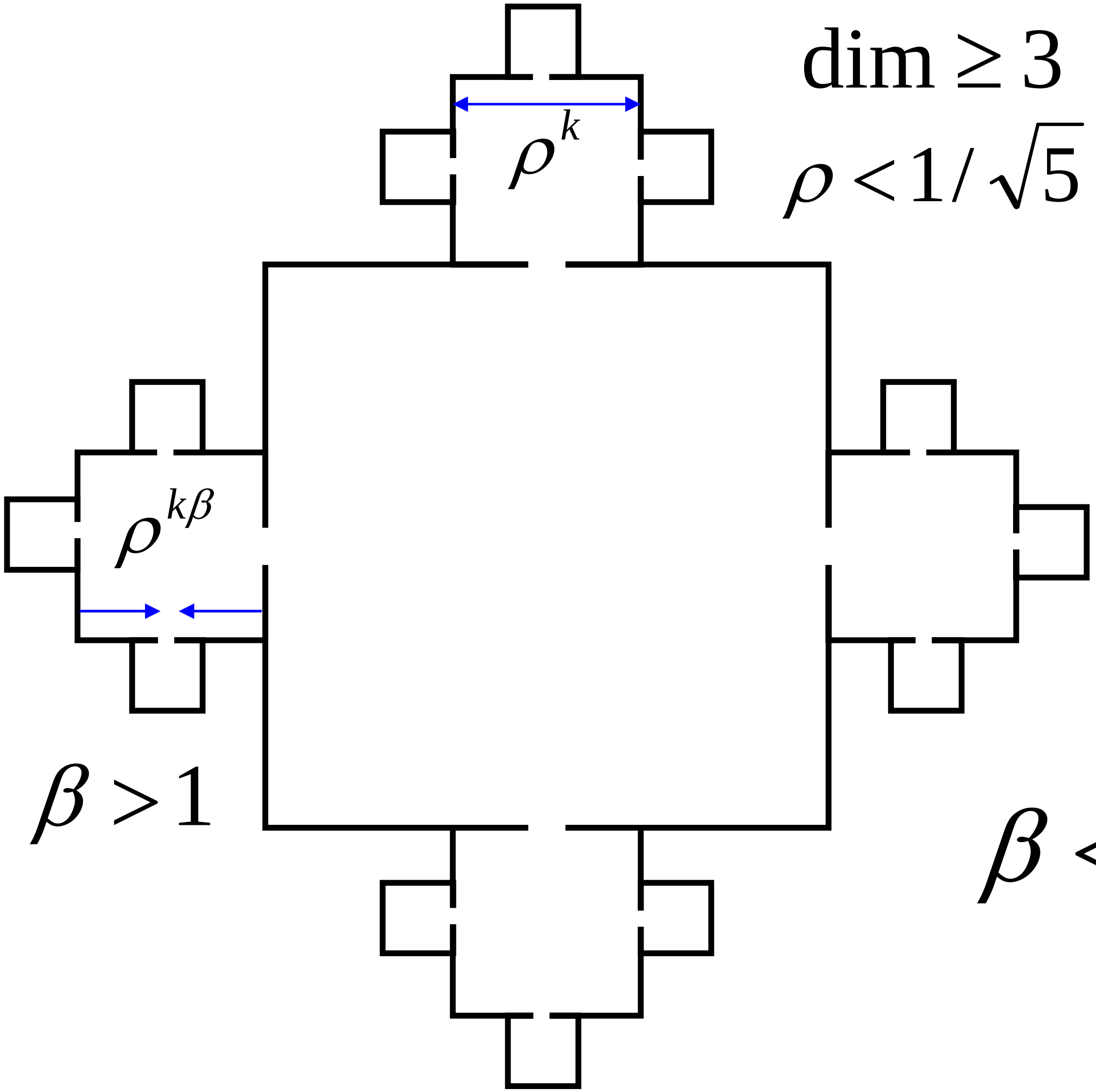


Is the whole surface active?

$$\beta < 2\alpha \Rightarrow$$ yes

$$\beta \geq 2\alpha \Rightarrow$$ no

Example III



$\beta > 1$

$\dim \geq 3$

$\rho < 1/\sqrt{5}$

Is the whole surface active?

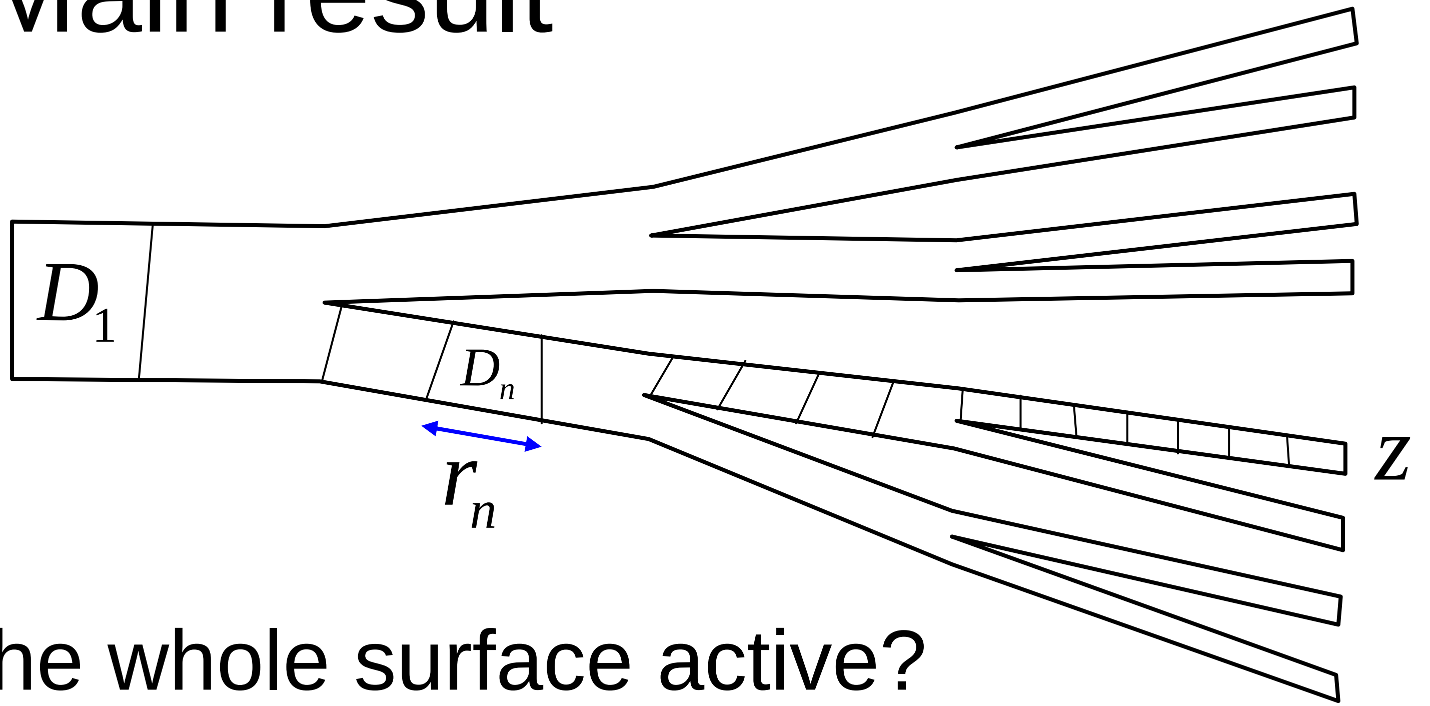
$d = \dim$

Yes, if

$$\beta < (d-1)/(d-2)$$

Otherwise no.

Main result

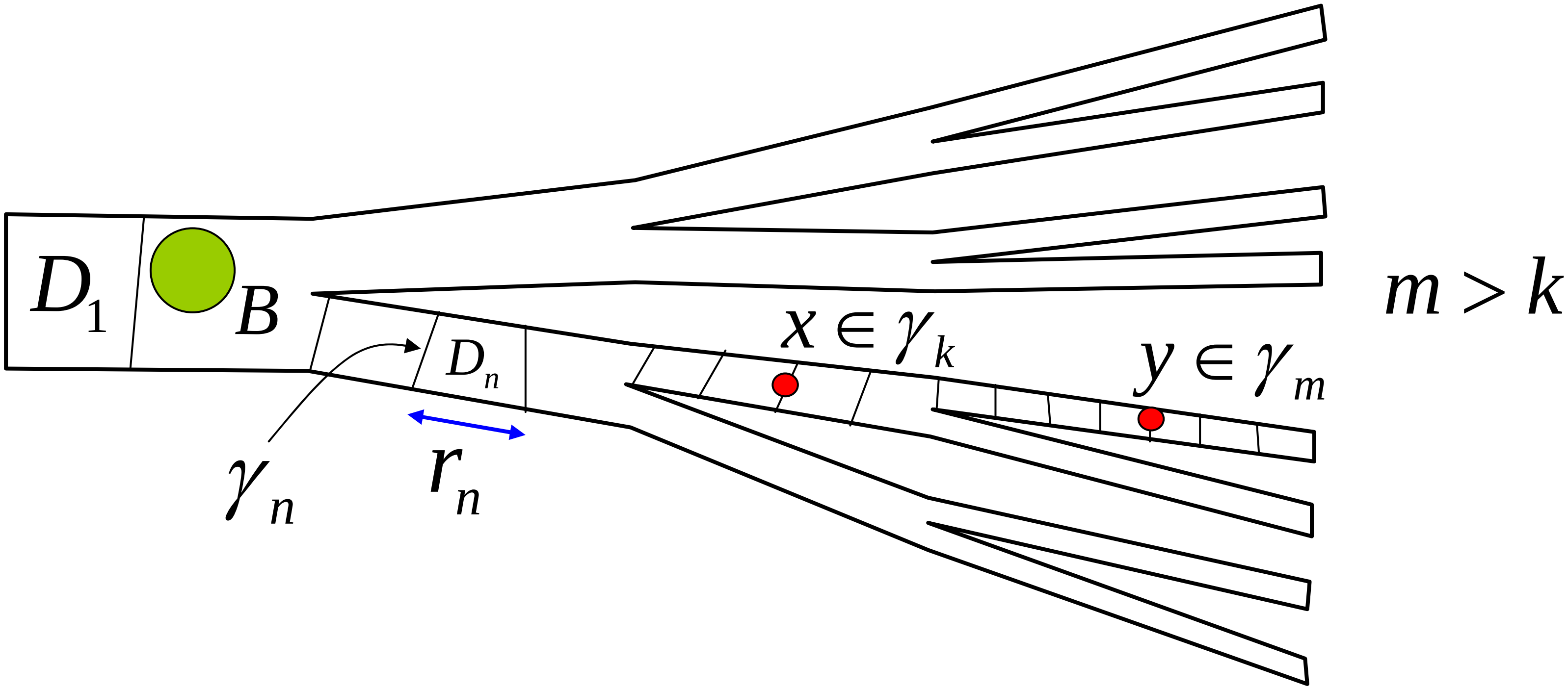


Is the whole surface active?

$$\sup_{z\in\partial D}\sum_{n\geq 1}|\partial D_n \cap \partial D|\sum_{k=1}^{n} r_k^{2-d} < \infty \Rightarrow$$ yes

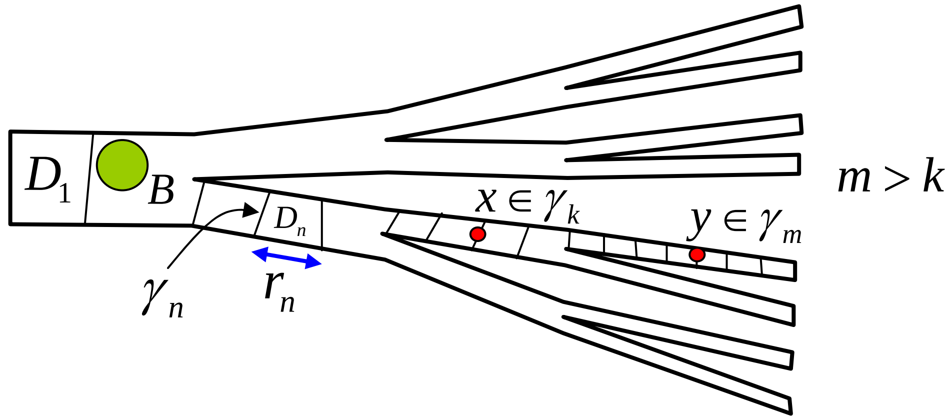
$$\exists z \in \partial D: \quad \sum_{n\geq 1} r_n^{d-1} \sum_{k=1}^{n} r_k^{2-d} = \infty \Rightarrow$$ no

Green function estimate



$G(x,y)$ - Green function with Neumann boundary conditions on ∂D and Dirichlet boundary conditions on ∂B

$$c_1 \sum_{j=1}^{k} r_j^{2-d} \leq G(x,y) \leq c_2 \sum_{j=1}^{k} r_j^{2-d}$$



$G(x, y)$ - Green function with Neumann boundary conditions on ∂D and Dirichlet boundary conditions on ∂B

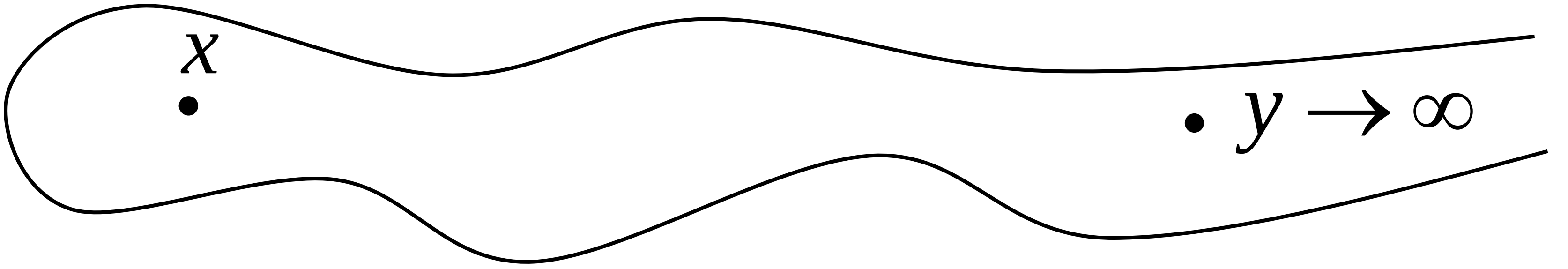
$$G(x, y) \approx \sum_{j=1}^{k} r_j^{2-d}$$

$\tilde{G}(x, y)$ - Green function with Dirichlet boundary conditions on $\partial D \cup \partial B$

Robin

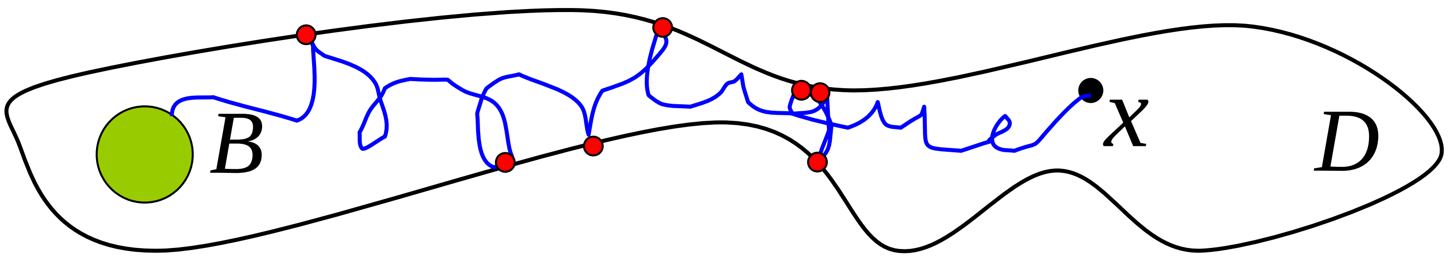
$$\log \tilde{G}(x, y) \approx f(x, y)$$

Green function asymptotics



- Neumann – linear growth of $G(x, y)$
- Dirichlet, Robin – exponential decay of $\tilde{G}(x, y)$

Feynman-Kac formula



Brownian motion reflected on ∂D and killed on ∂B at time $\tau_{\partial B}$

• $L(t)$ - Local time on ∂D

$$u(x) = E_x \exp(-L(\tau_{\partial B}))$$

Ma and Song (1990), Papanicolau (1990)

$$u(x) = E_x \exp(-L(\tau_{\partial B}))$$

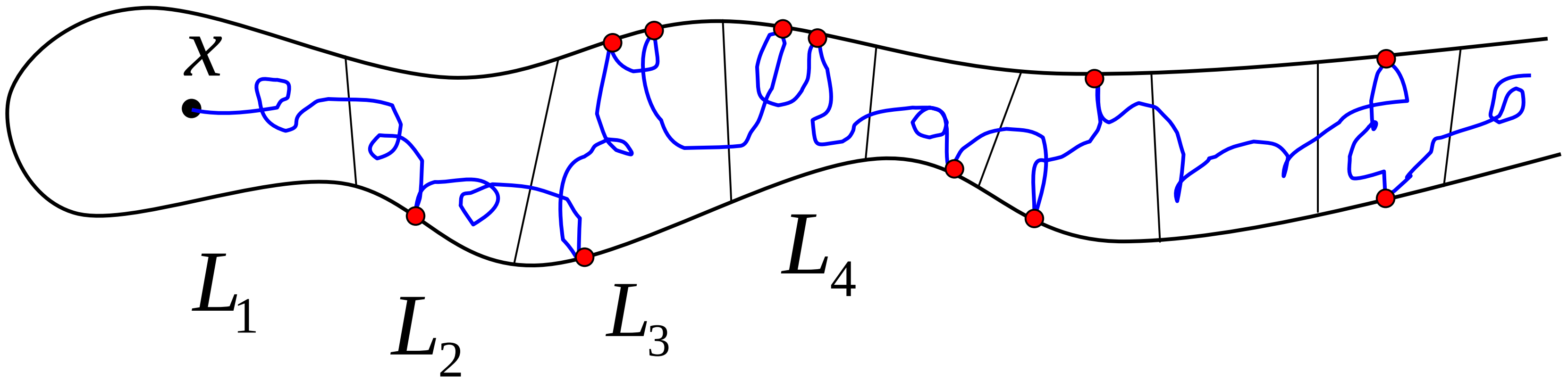
Problem: $\inf_{x \in D \setminus B} u(x) > 0$?

$u(x) = 0$?

$$u(x) = 0 \quad \Leftrightarrow \quad P(L(\tau_{\partial B}) = \infty) = 1$$

$$E_x L(\tau_{\partial B}) < \infty \quad \Rightarrow \quad P_x(L(\tau_{\partial B}) = \infty) = 0$$

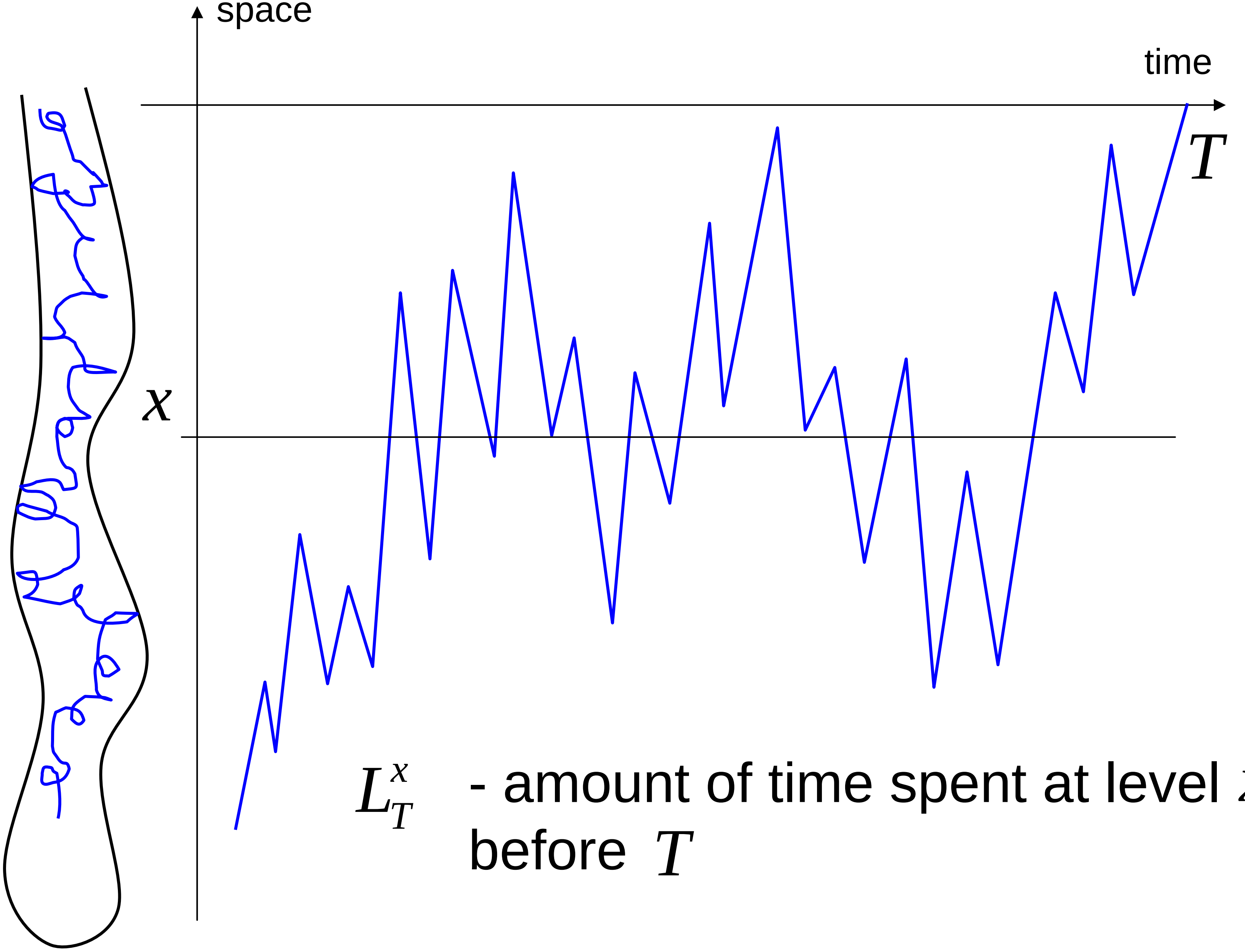
$$E_x L(\tau_{\partial B}) = \infty \quad \not\Rightarrow \quad P_x(L(\tau_{\partial B}) = \infty) = 1$$



$$L = L_1 + L_2 + \ldots + L_n$$

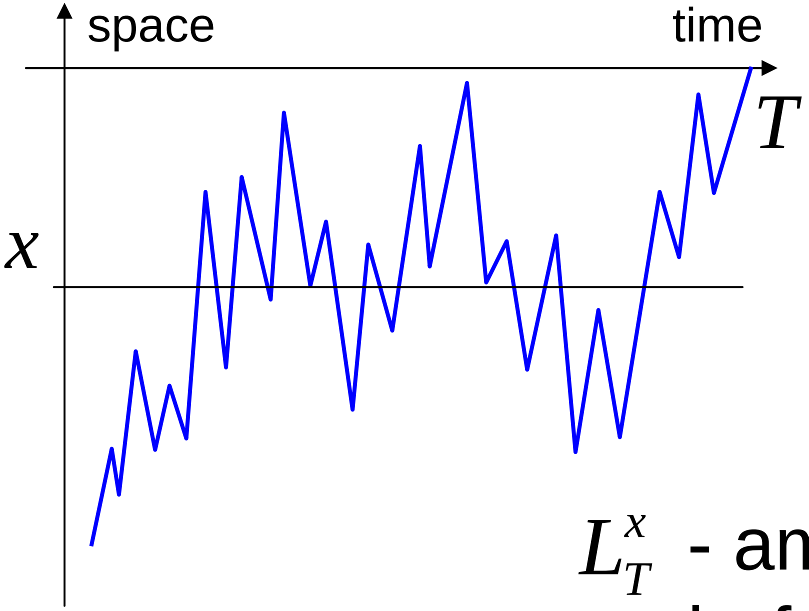
The Law of Large Numbers and the Central Limit Theorem are applicable when $L_1, L_2, L_3, \ldots$ are “nearly” independent.

In fact, $L_{k+1} = L_k (1 + o(1))$



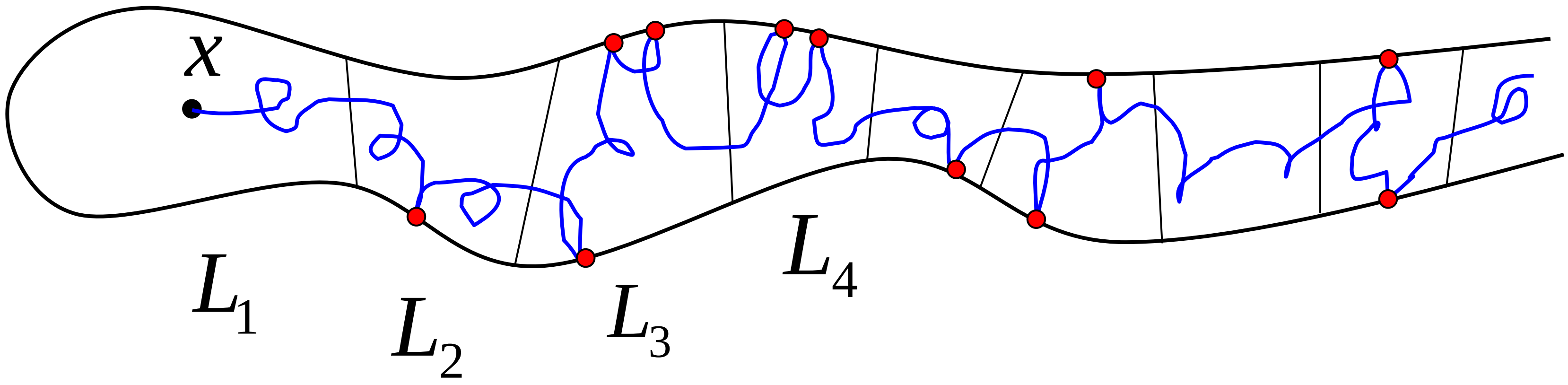
L_T^x - amount of time spent at level x before T

Ray-Knight Theorem



L_T^x - amount of time spent at level x before T

$x \to L_T^x$ is a diffusion with an explicit distribution.



$$L = L_1 + L_2 + \ldots + L_n$$

The joint distribution of $L_1, L_2, L_3, \ldots$ is given (approximately) by the Ray-Knight Theorem.

Challenge

Assume: $\omega_x(dz) < \omega_y(dz) \quad \forall z \in \gamma_2$

$$\omega_x(dv) > \omega_y(dv) \quad \forall v \in \gamma_1$$

Prove: $P^x(T_A < T_B) > P^y(T_A < T_B)$

References

- Bass, B and Chen (2005) “On the Robin problem in fractal domains” (preprint)
- Gustafson and Abe (1998) “The third boundary condition-was it Robin's?” *Math. Intelligencer* **20**, 63-71.
- Ma and Song (1990) “Probabilistic methods in Schrodinger equations” *Seminar on Stochastic Processes 1989*, 135-164, *Progr. Probab.*, **18**, Birkhauser, Boston.
- Papanicolaou (1990) “The probabilistic solution of the third boundary value problem for second order elliptic equations” *Probab. Theory Rel. Fields* **87** 27-77.