Essential Uses of Probability in Analysis

Part I. Brownian Couplings and Neumann Eigenfunctions

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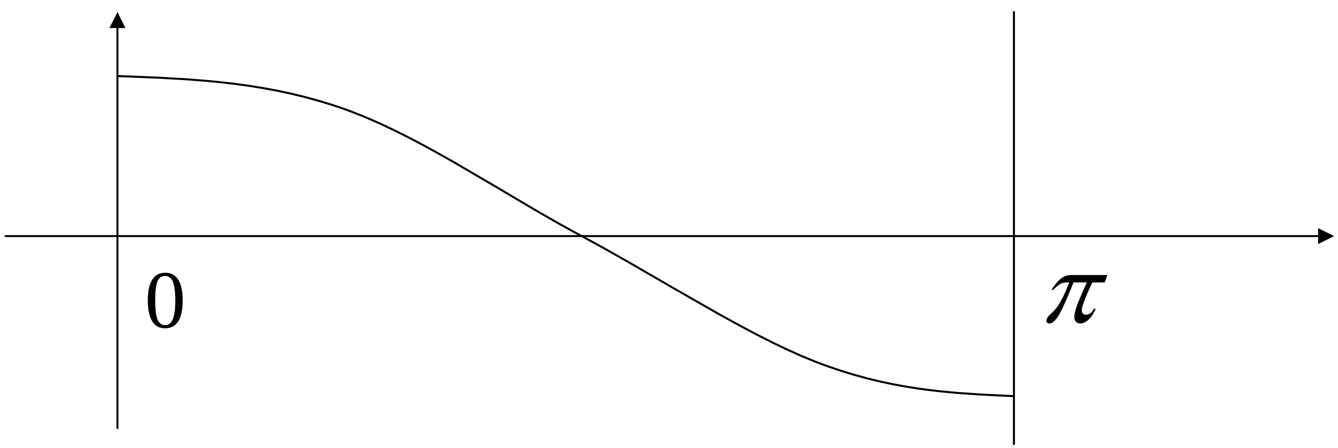
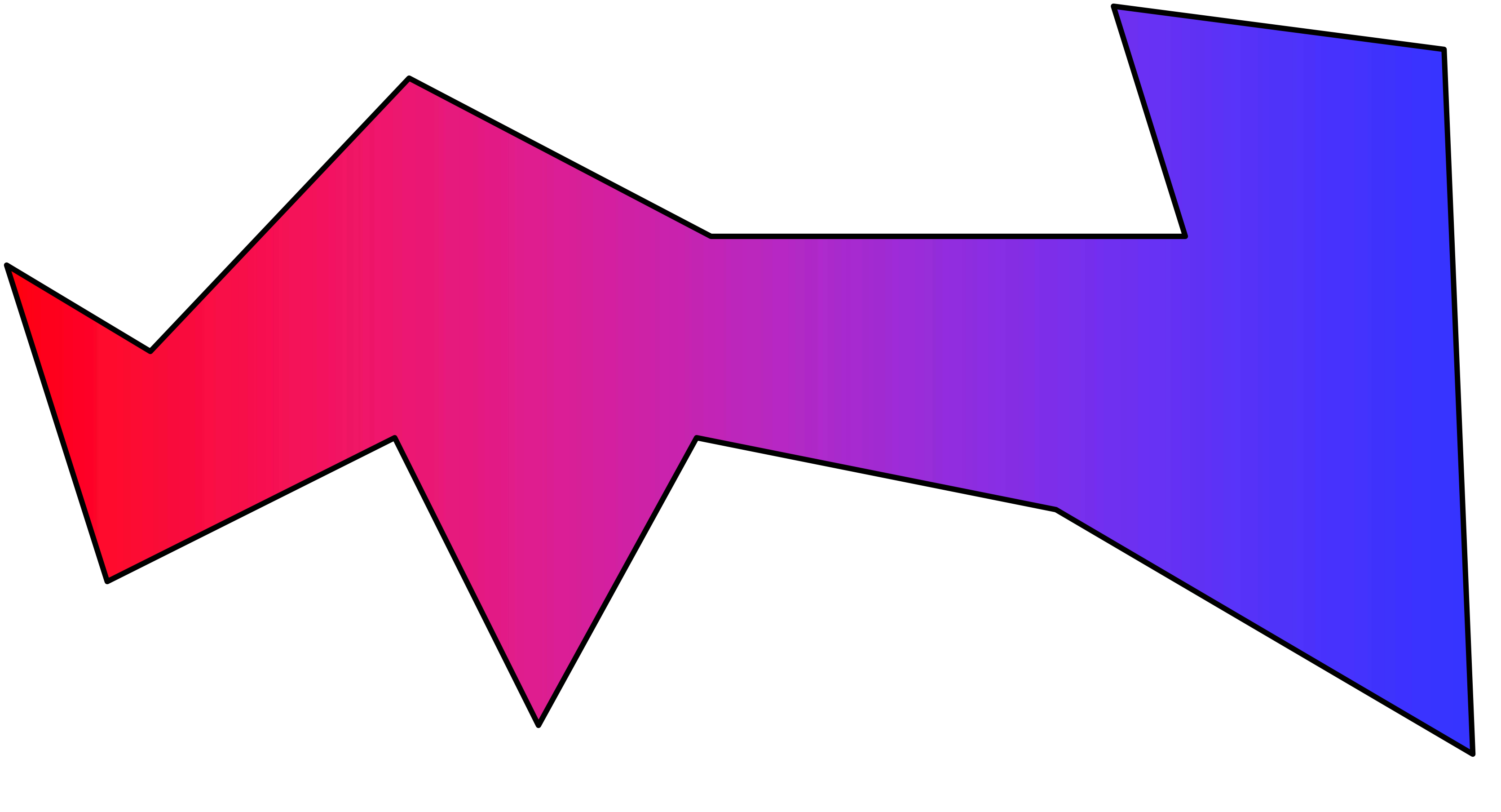
Hot Spots Conjecture

Rauch (1974) : In Euclidean domains, the second Neumann Laplacian eigenfunction attains its maximum at the boundary.



Multidimensional domains

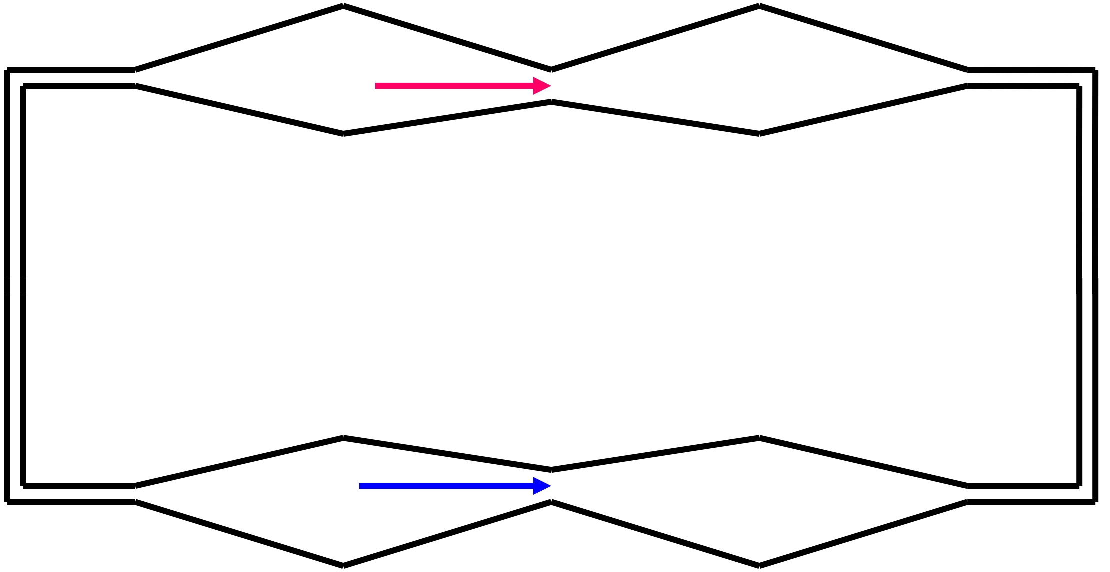


Counterexamples

B and Werner (1999)

Bass and B (2000)

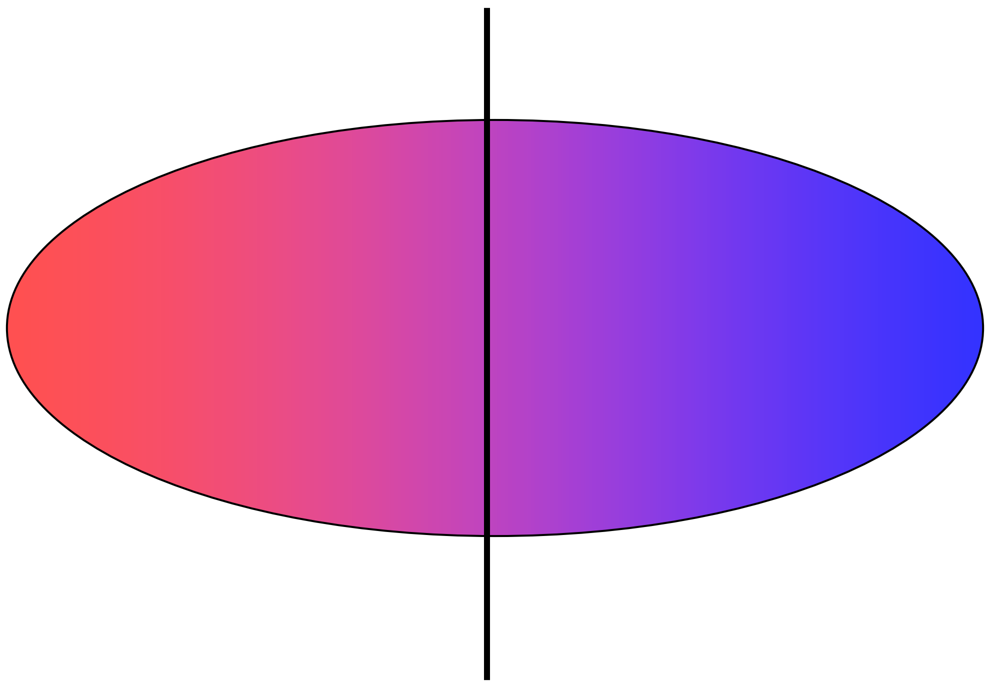
B (2005)



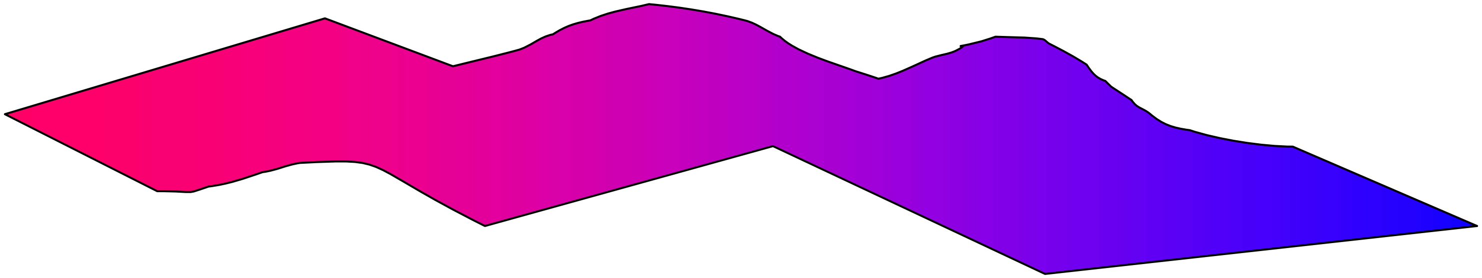
Positive direction

Kawohl (1985)
Bañuelos and B (1999)
Pascu (2002) : Conjecture holds for planar convex domains with a line of symmetry.
Atar and B (2004) : Conjecture holds for all lip domains.
Jerison and Nadirashvili (2000)
Atar (2001), …

Symmetric domains



Lip domains



We say that a set is a “lip domain” if it lies between the graphs of two Lipschitz functions with the Lipschitz constant 1.

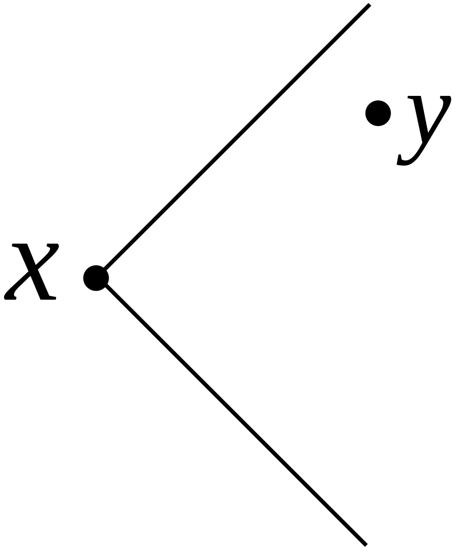
Open problems

(i) Hot spots conjecture for convex domains.

(ii) Hot spots conjecture for simply connected planar domains.

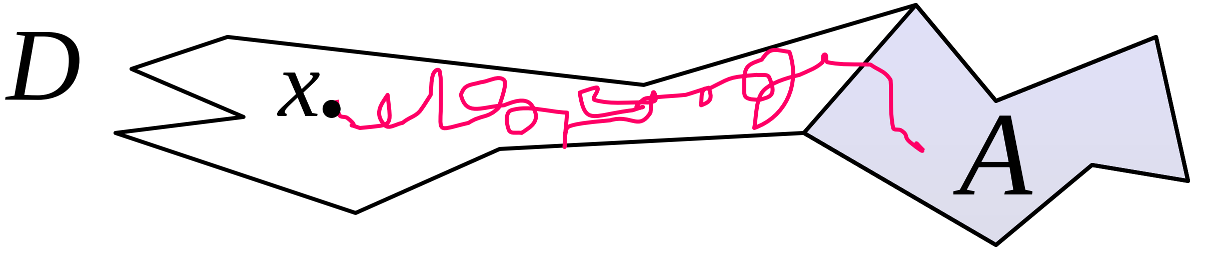
(iii) Hot spots conjecture for triangles.

A partial order



$$x \leq y$$

Heat equation and reflected Brownian motion



$u(x,t) =$ temperature at x at time t (Neumann boundary conditions)

X_t - reflected Brownian motion in D

$$u(x,0) = f(x) = 1_A(x)$$

$$u(x,t) = E^x f(X_t) = P^x(X_t \in A)$$

Neumann eigenfunctions

$$u(x,t) = c + \varphi(x)e^{-\lambda t} + \dots$$

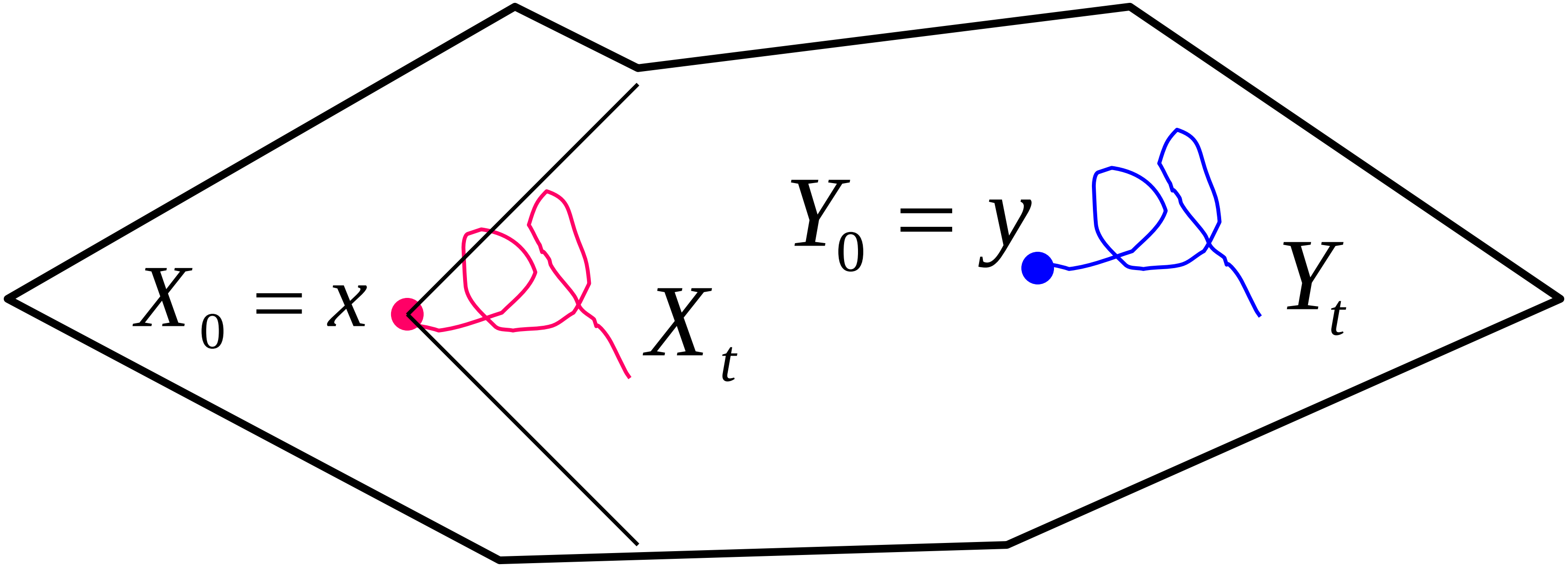
$$\varphi(x) - \varphi(y) \approx e^{\lambda t}(u(x,t) - u(y,t))$$

$$\varphi(x) - \varphi(y) \approx e^{\lambda t}(P^x(X_t \in A) - P^y(X_t \in A))$$

$$\varphi(x) - \varphi(y) \approx e^{\lambda t}(P^x(X_t \in A) - P^y(Y_t \in A))$$

Synchronous couplings

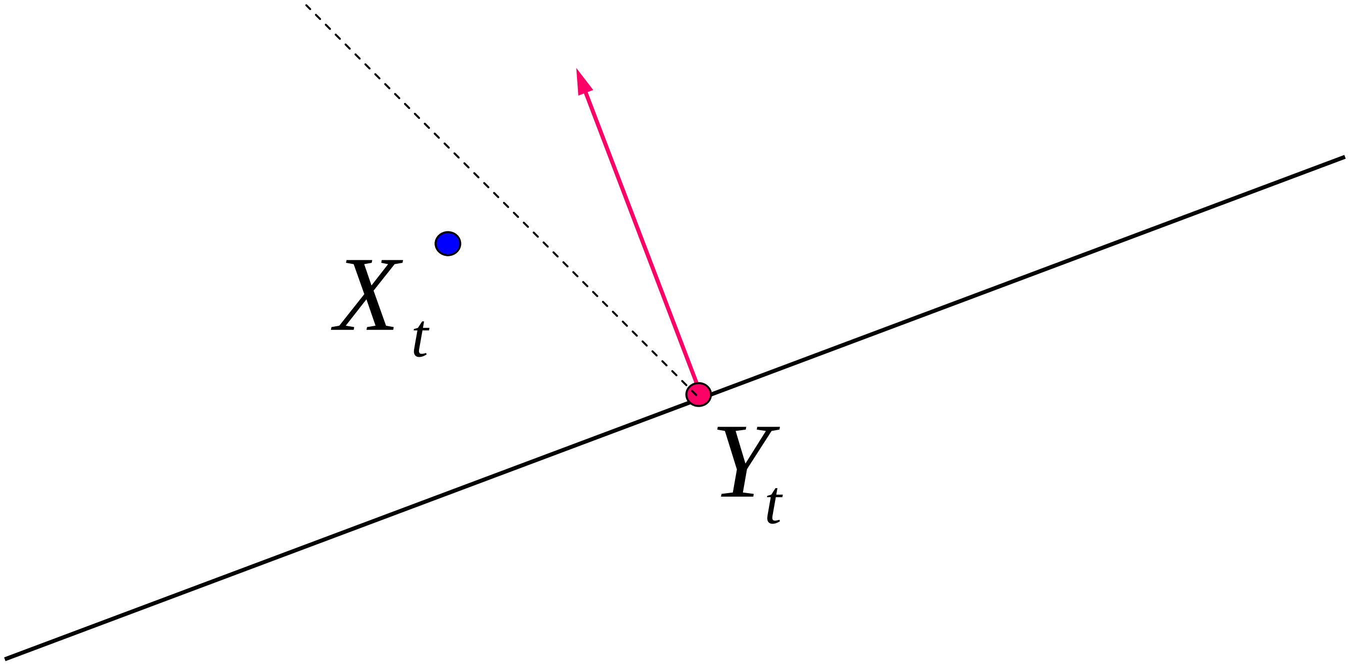
B and Kendall (2000)



$$x \leq y \Rightarrow X_t \leq Y_t \quad \forall t$$

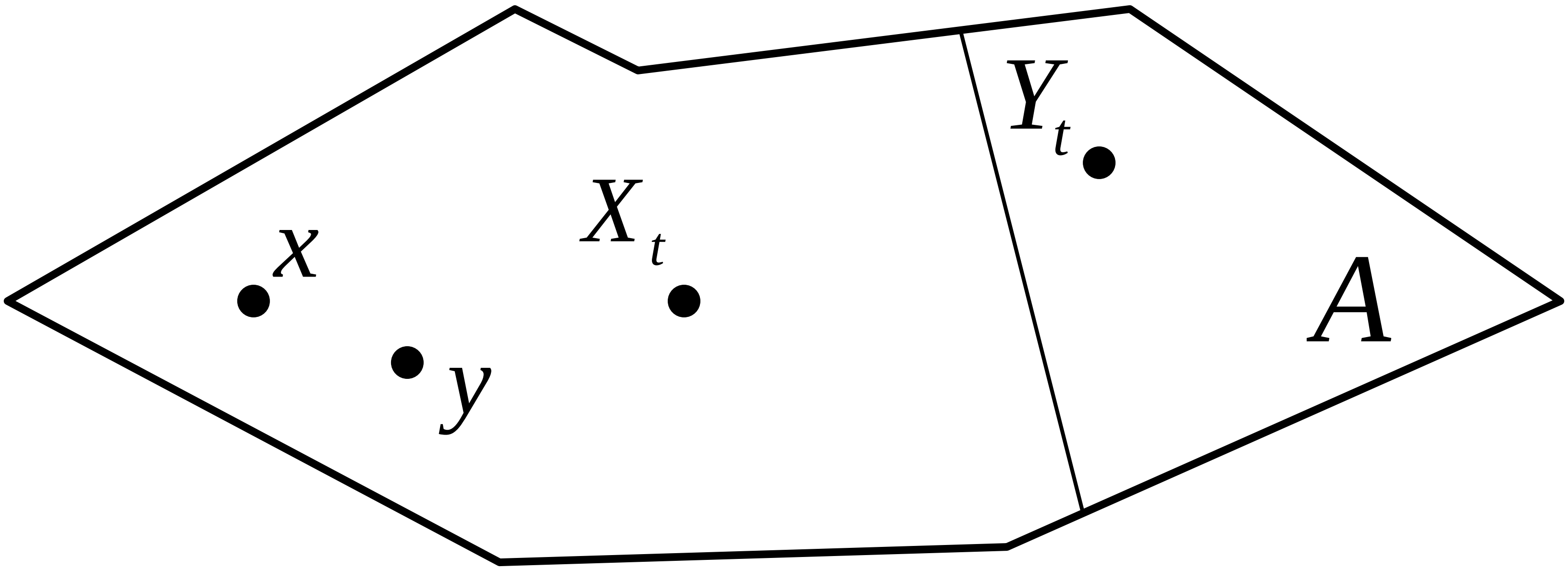
(monotonicity)

Effect of reflection



$$X_t \leq Y_t$$

Eigenfunction monotonicity

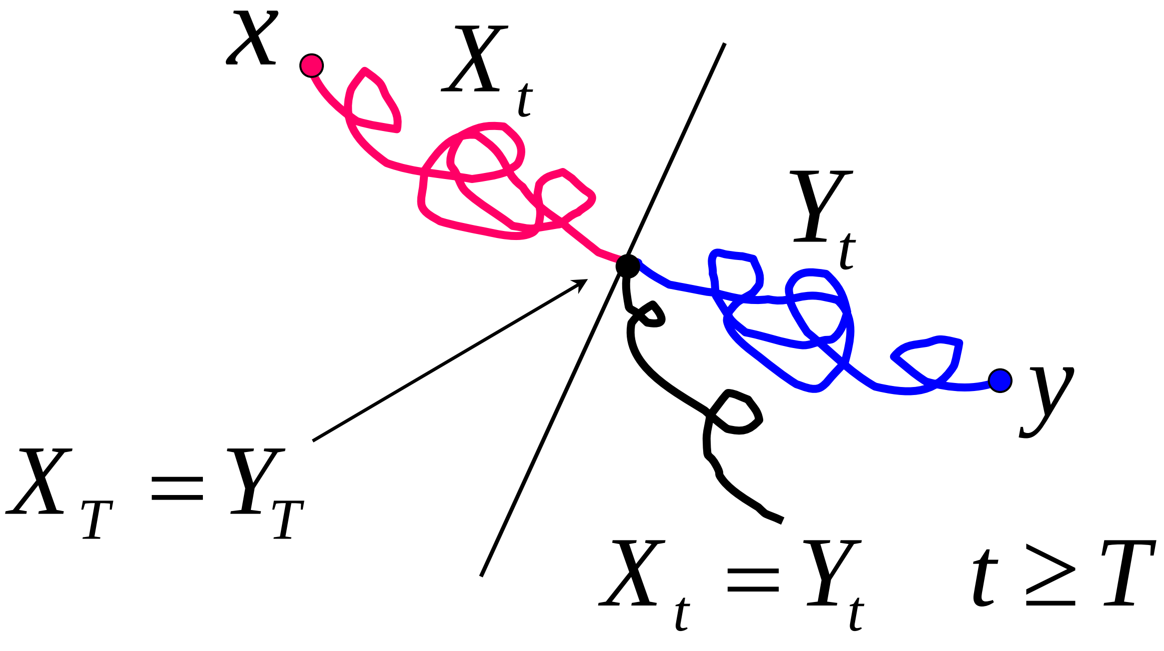


$$x \le y \Rightarrow X_t \le Y_t \quad \forall t$$

$$P^x(X_t \in A) \le P^y(Y_t \in A)$$

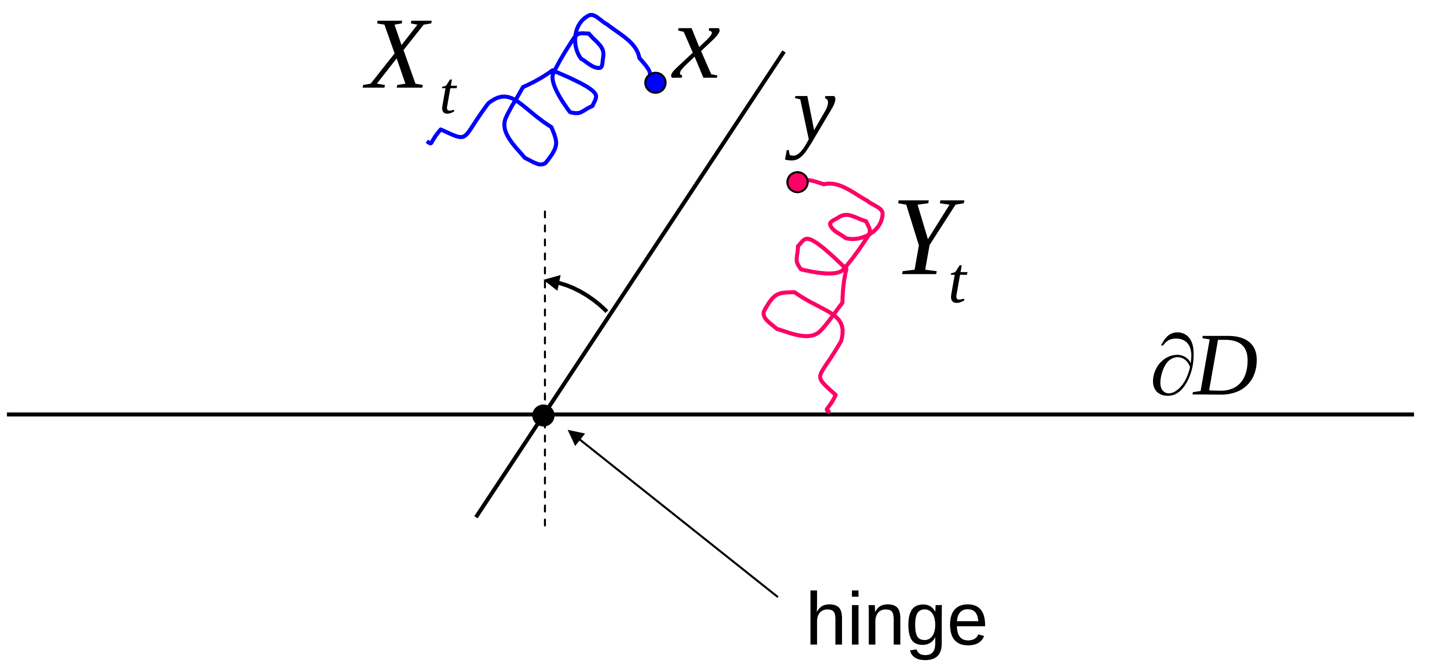
$$\varphi(x) \le \varphi(y)$$

Mirror couplings (free BM)



T - coupling time

Mirror couplings for reflected BM



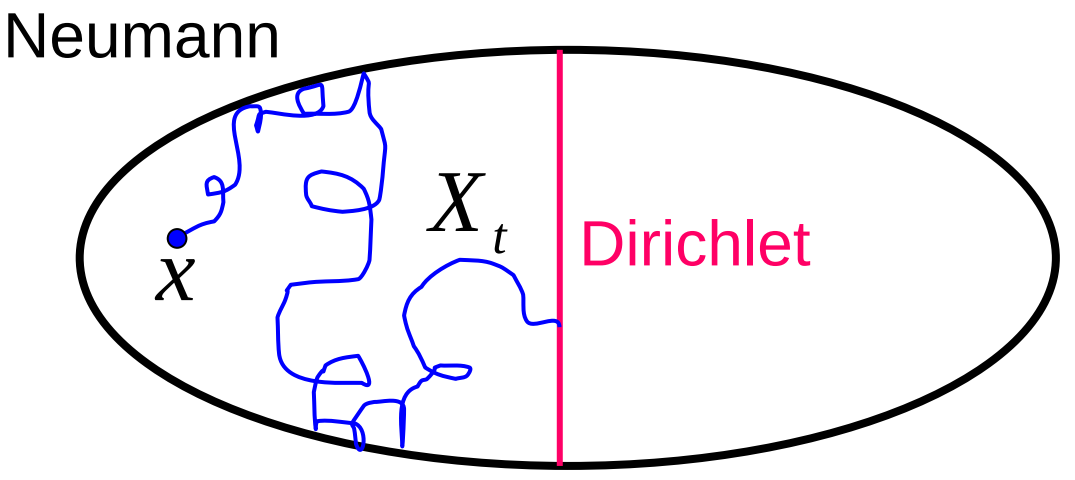
Wang (1994)

B and Kendall (2000) : reflection on a single line

Atar and B (2004) : piecewise smooth domains

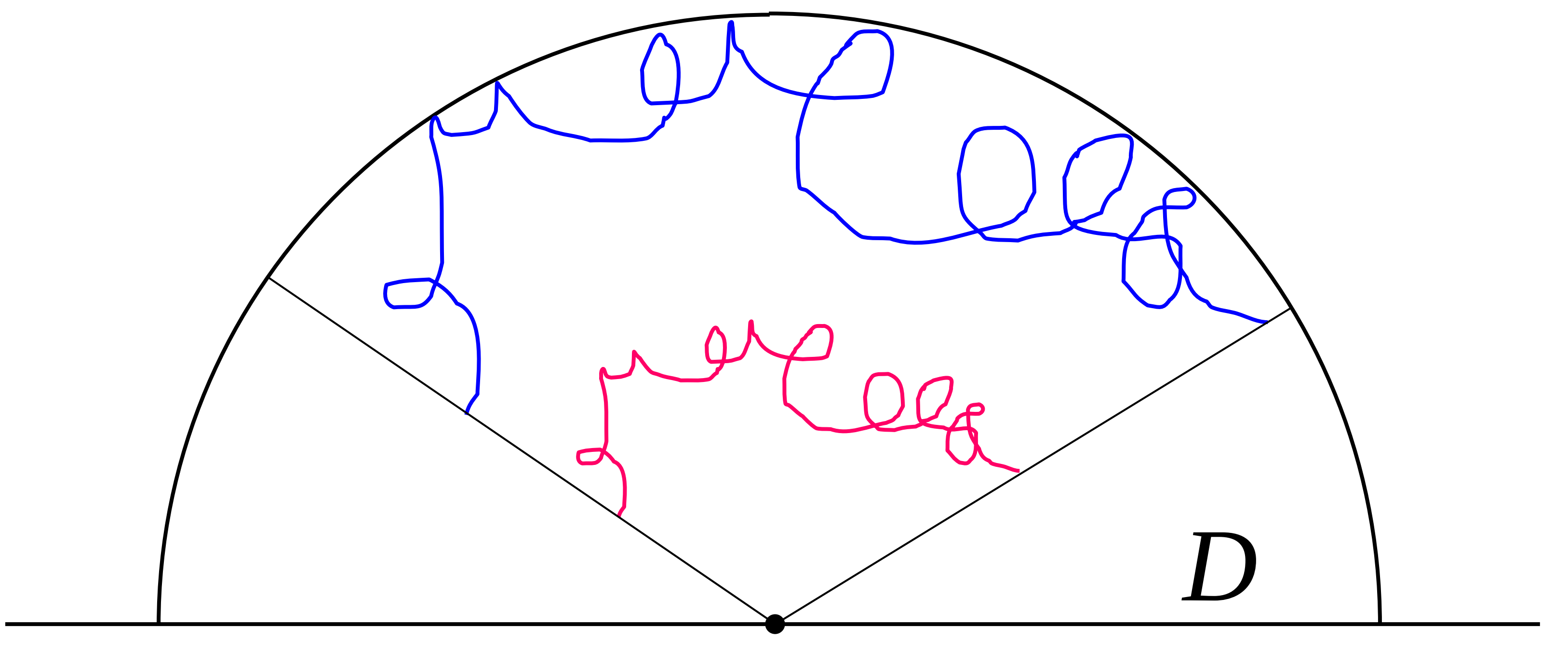
B (2005) : simultaneous reflections

Symmetric domains



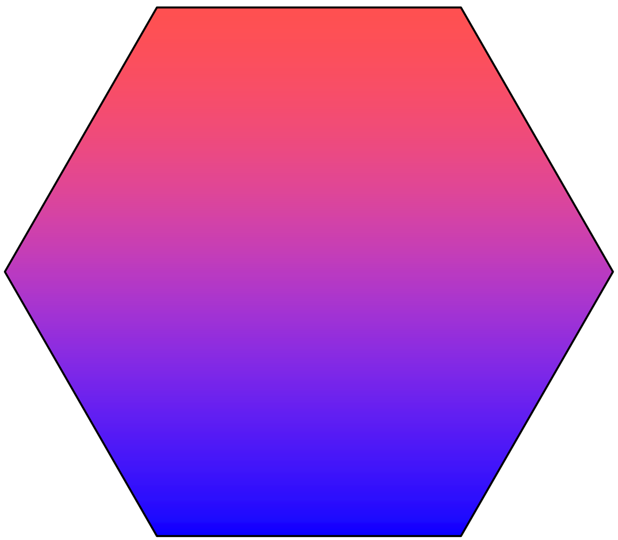
Bañuelos and B (1999)
Jerison and Nadirashvili (2000)
Pascu (2002)

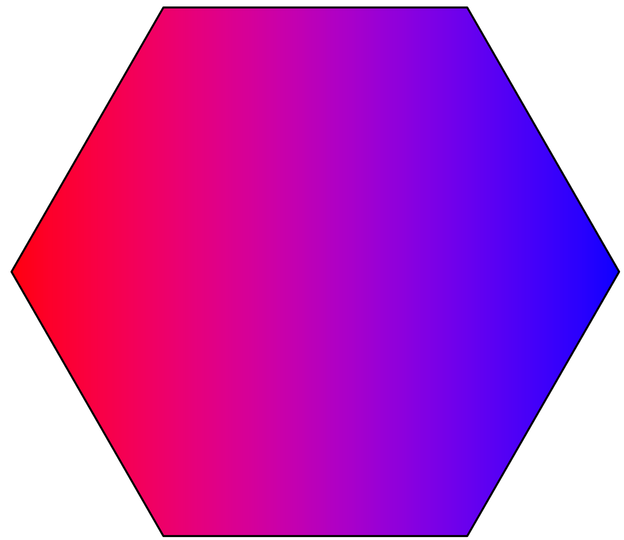
Scale couplings – Pascu (2002)



D - semidisc

Eigenvalue multiplicity



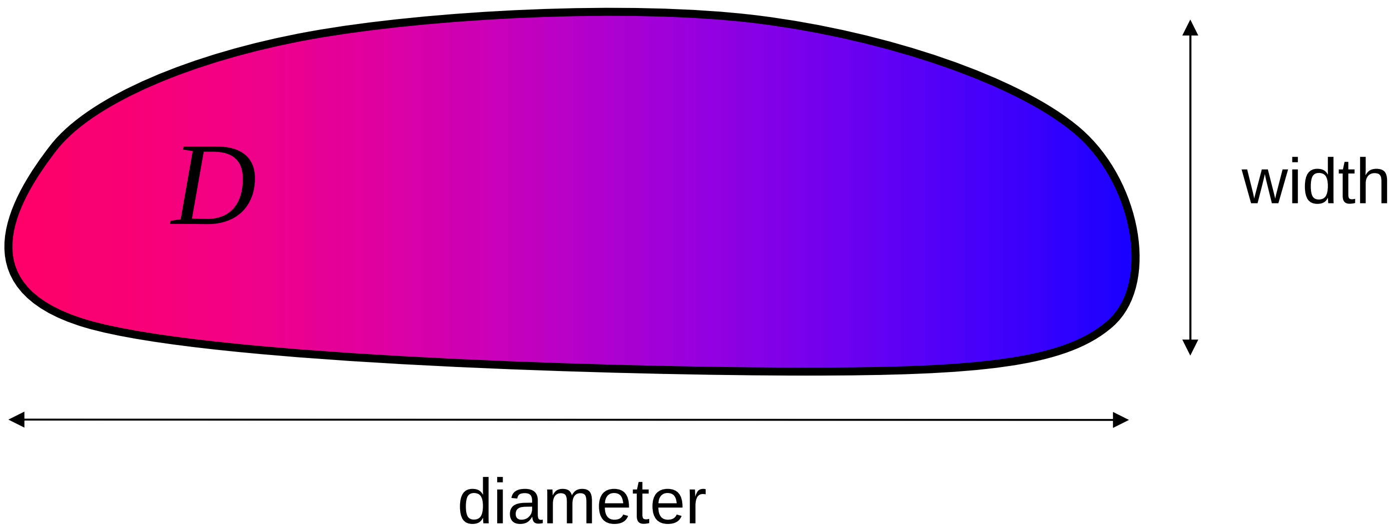


Maximum eigenvalue multiplicity

Nadirashvili (1986, 1988) :
The maximum multiplicity of the second Neumann eigenvalue for a simply connected planar domain is 2.

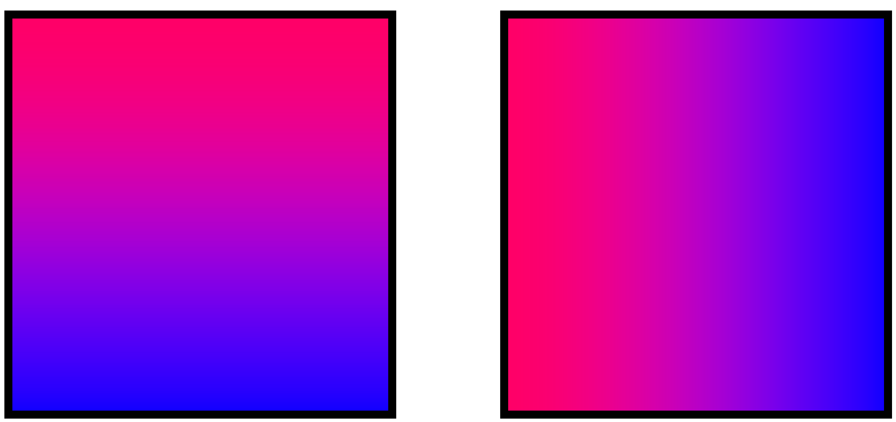
Long convex domains

Bañuelos and B (1999) : If D is convex and diameter/width is greater than 3.07 then the second eigenvalue is simple.



Convex domains - conjecture

Bañuelos and B (1999) : (Conjecture) If D is convex and diameter/width is greater than 1.41 then the second eigenvalue is simple.

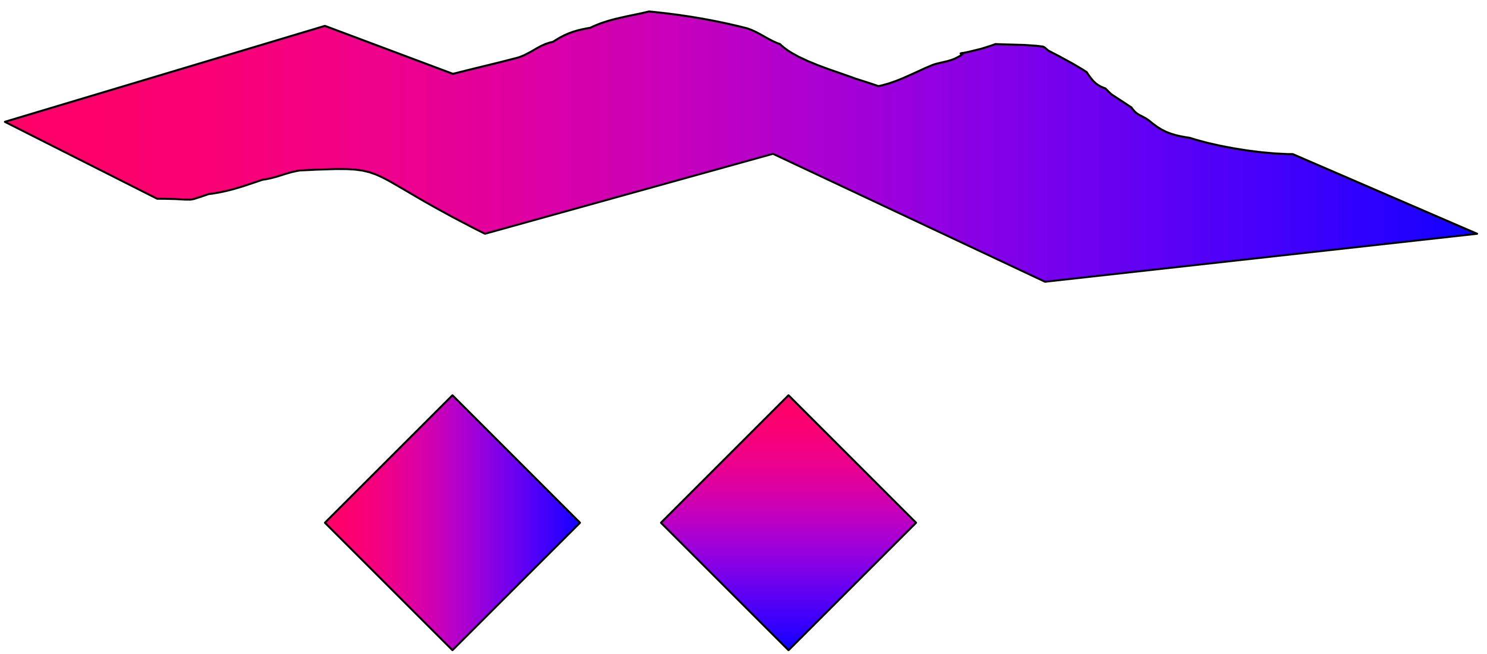


width = 1

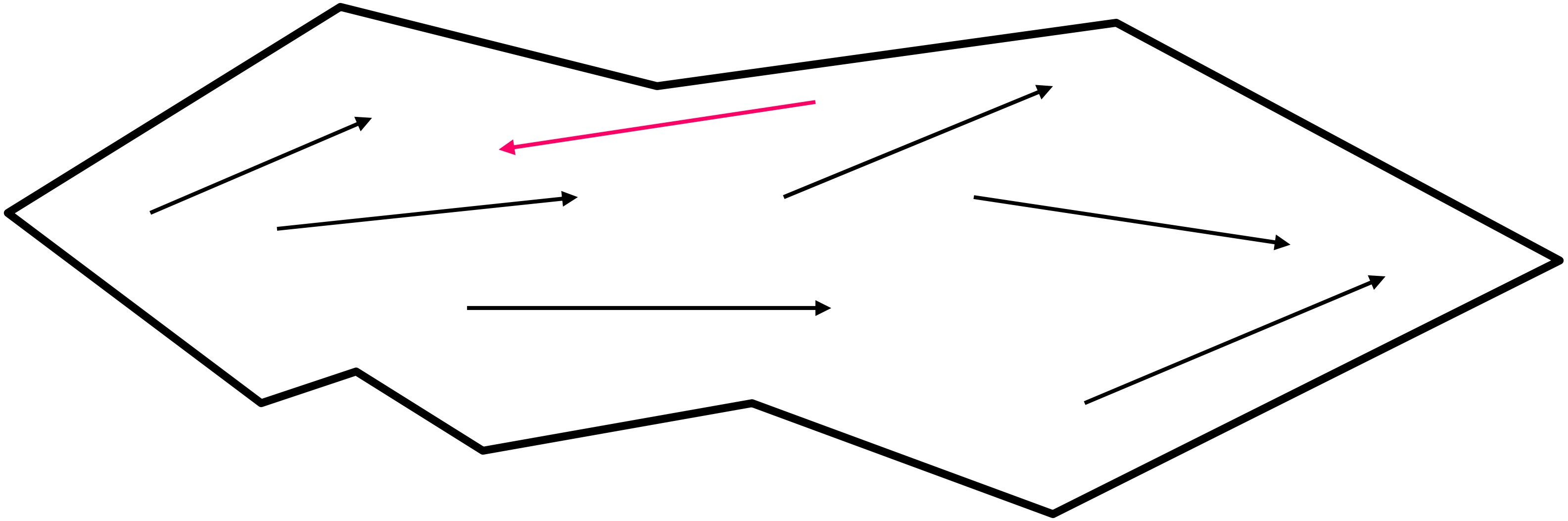
diameter = $\sqrt{2}$

Lip domains

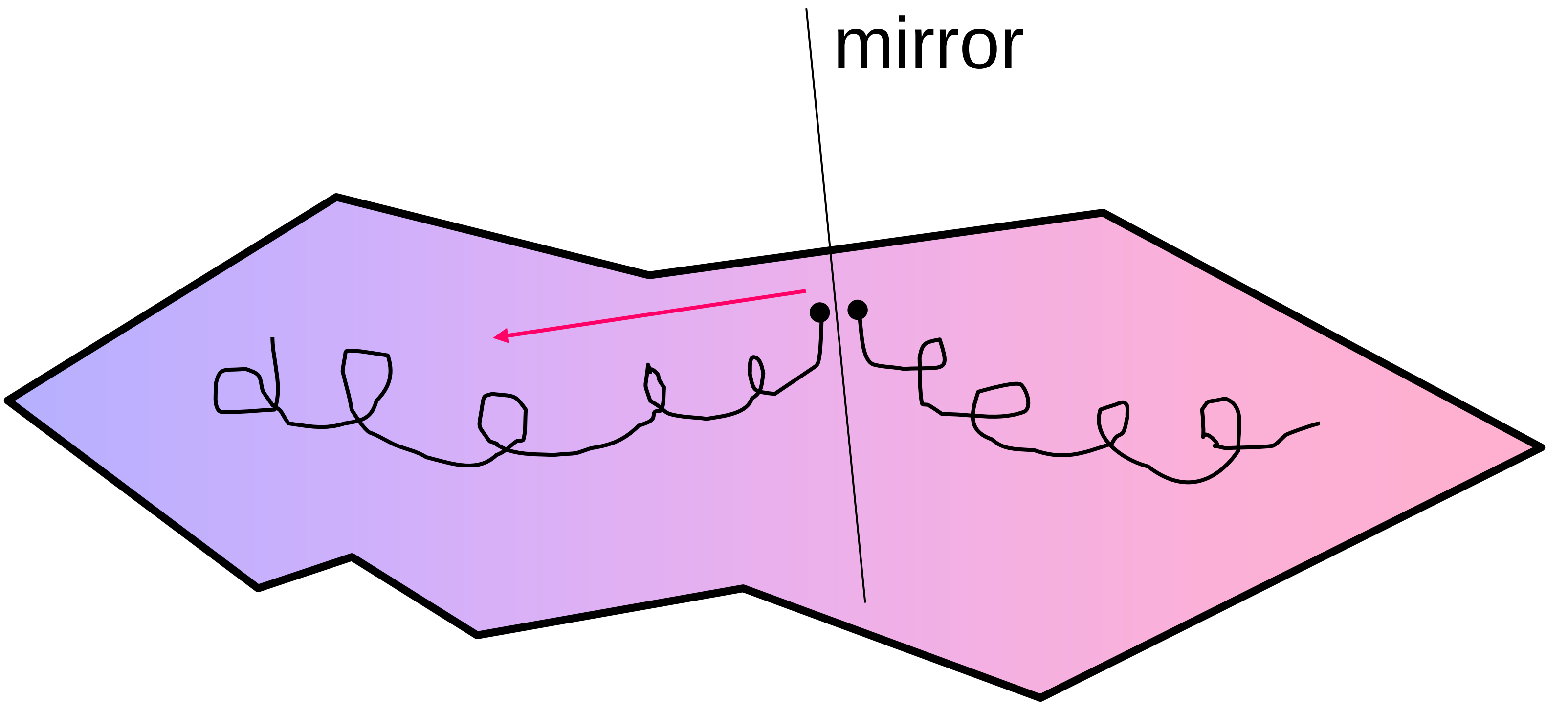
Atar and B (2004) : Second Neumann eigenvalue is simple in lip domains (except squares).



Eigenvalue multiplicity - lip domains



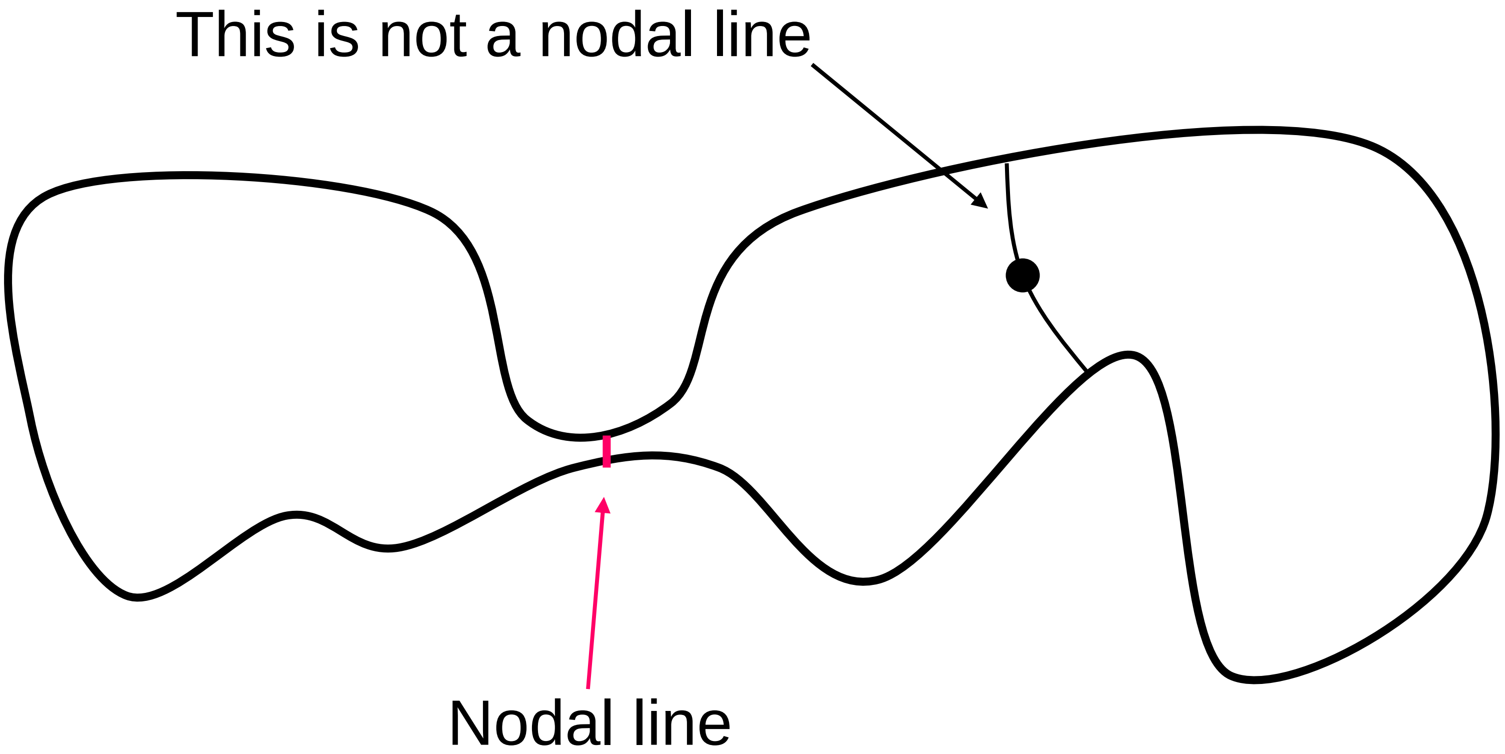
Parabolic boundary Harnack principle



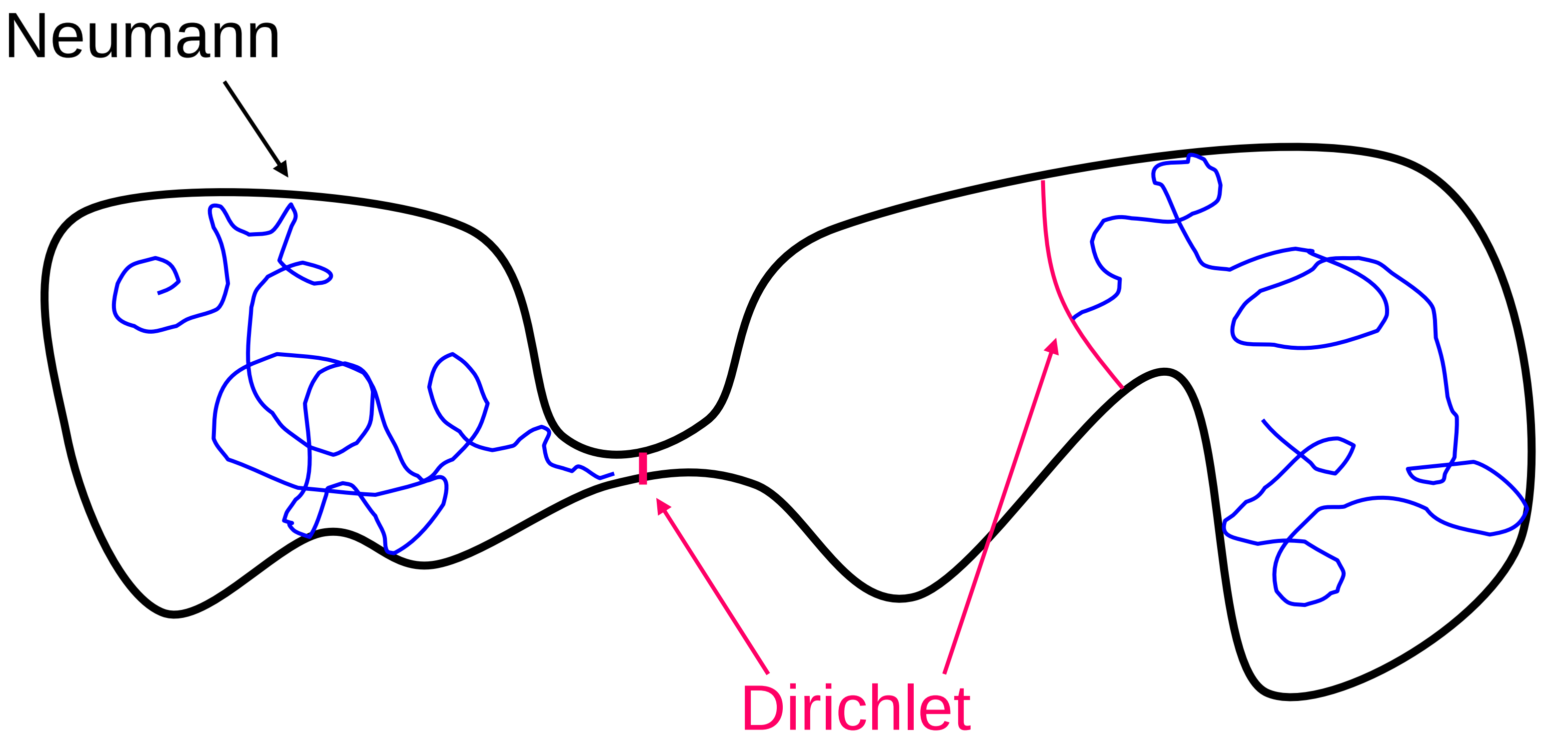
Points outside nodal lines

Bañuelos and B (1999) : If for some point the nodal (zero) line for any second eigenfunction does not pass through this point, then the second eigenvalue is simple.

Bottleneck domains



Bottleneck domains
(idea of proof)

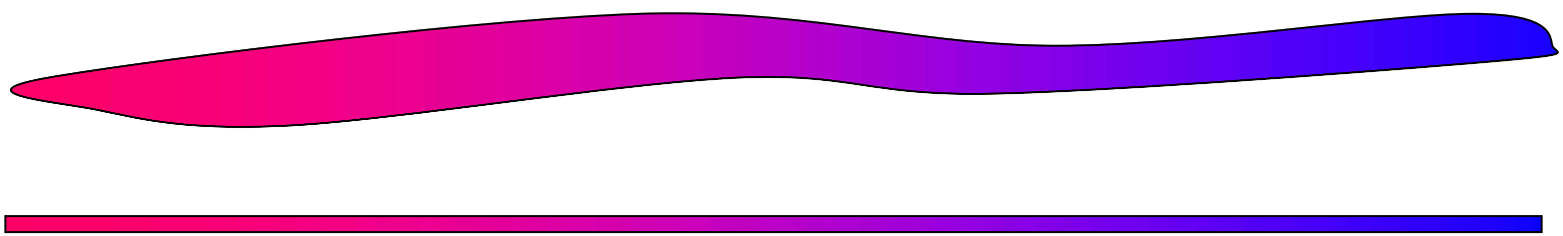


Nodal line location
Old results

(i) Rectangles, ellipses, etc.

(ii) Domains with symmetry

(iii) **Jerison (2000)** Long and thin domains



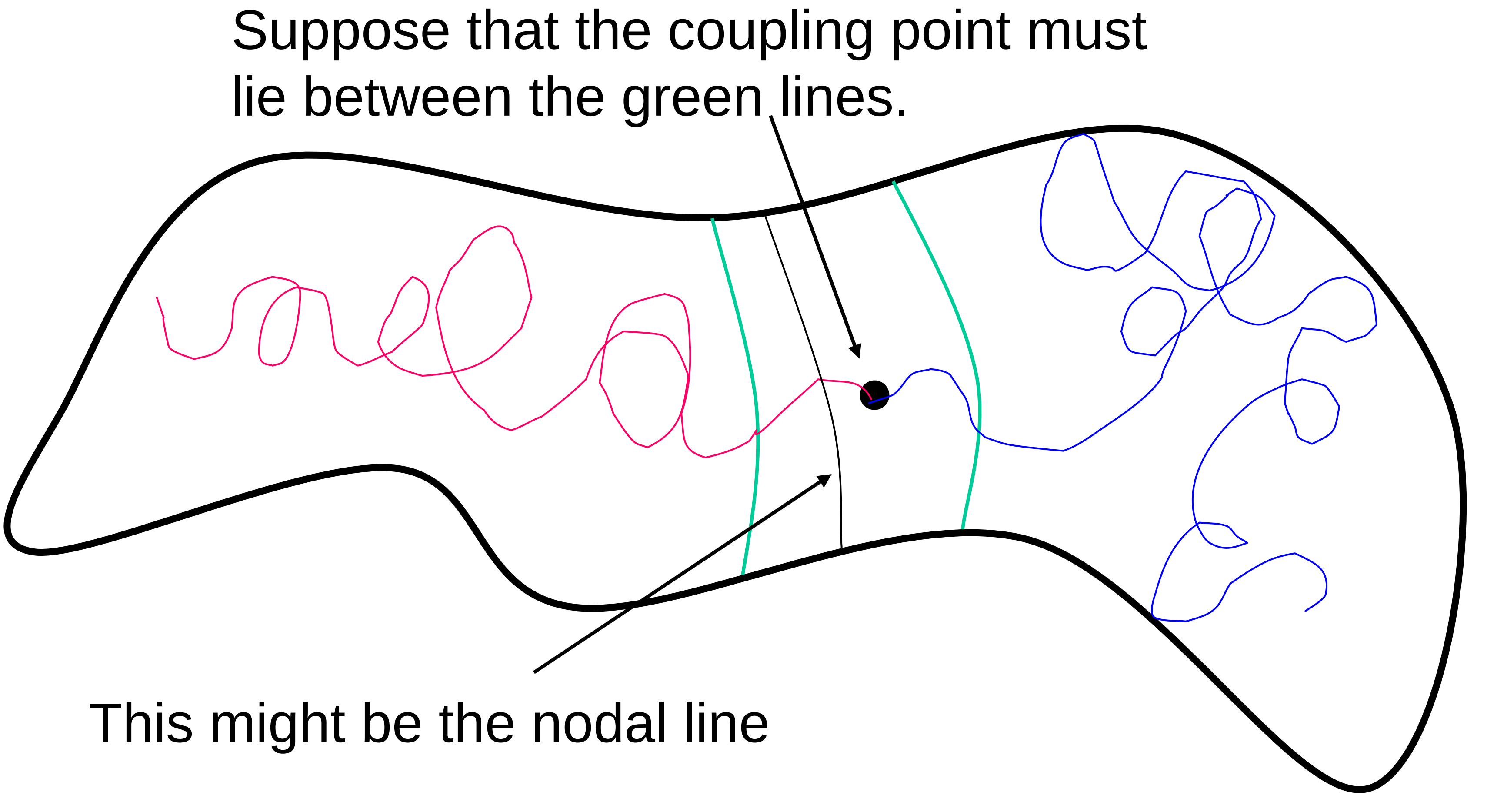
(**Melas (1992), Alessandrini (1994)** : Dirichlet nodal lines)

Nodal lines and couplings

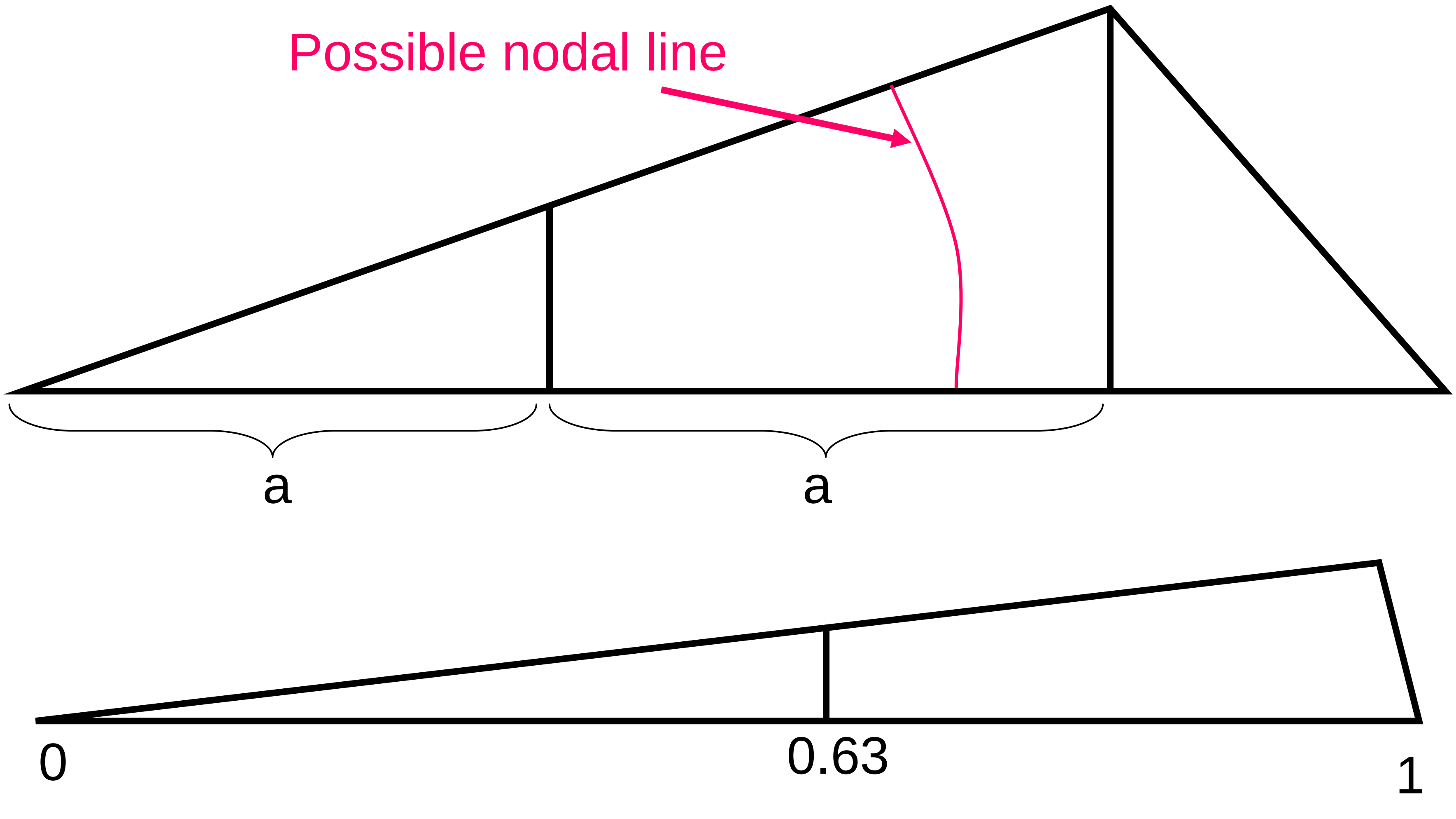
Atar and B (2002) : Consider a coupling of reflected Brownian motions. If the particles cannot couple in a subset of the domain then this subset is too small to contain a nodal domain.

Atar and B (in preparation) : If the particles can couple only in a subset of the domain then this subset must contain the nodal set.

Nodal lines and couplings (idea of proof)



Obtuse triangles



Synchronous couplings via unique strong solutions to Skorohod equation

$$X_t = x + B_t + \int_0^t N(X_s)\, dL_t^X$$

$$Y_t = y + B_t + \int_0^t N(Y_s)\, dL_t^Y$$

Lions and Sznitman (1984) :

C^2 - domains

Skorohod equation: unique strong solutions in R^2

$$X_t = x + B_t + \int_0^t N(X_s)\,dL_t^X$$

Bass, B and Chen (2004) : Unique strong solutions exist in planar Lipschitz domains with Lipschitz constant less than 1.

Bass and B (2005)

Higher dimensional domains

We say that D is a $C^{1,\gamma}$ -domain if its boundary is represented by a function Φ and

$$|\nabla\Phi(x)-\nabla\Phi(y)|\leq|x-y|^{\gamma}$$

Skorohod equation:
unique strong solutions in $R^n, n \geq 3$

$$X_t = x + B_t + \int_0^t N(X_s)\,dL_t^X$$

Bass and B (in preparation) :

(i) Unique strong solutions exist in $C^{1,\gamma}$-domains if $\gamma > 1/2$

(ii) Counterexample (?) for some $\gamma > 0$

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