The regularity of the free-boundary for the Classical Obstacle problem

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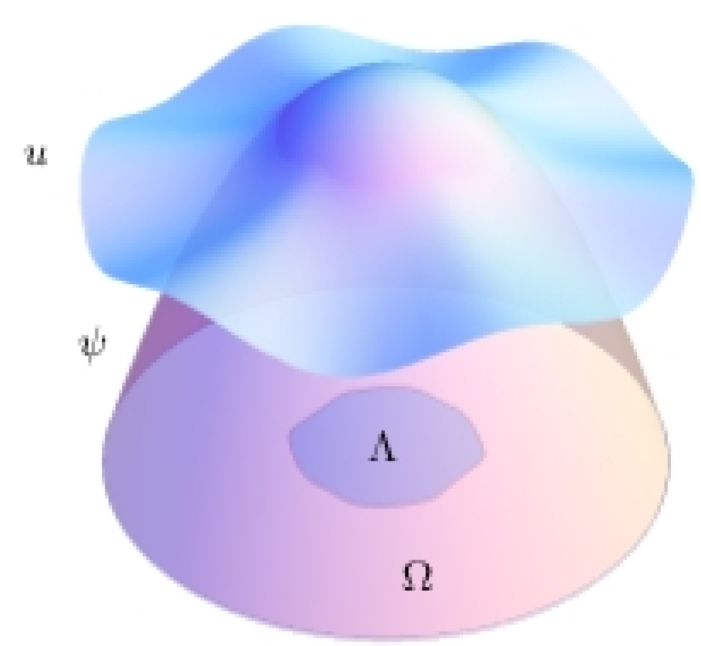
Introduction

The classical obstacle problem consists in finding u the minimizer of Dirichlet energy in a domain Ω, among all functions v, with fixed boundary data, constrained to lie above a given obstacle ψ, studying proprieties of minimizer and analysing the regularity of the boundary of the coincidence set $\Lambda_u := \{u = \psi\}$ between minimizer and obstacle, $\Gamma_u := \partial\Lambda_u \cap \Omega$.



In this context, we aim at minimizing the following energy (we are reduced to the 0 obstacle case, so $f = -\mathrm{div}(\mathbb{A}\nabla\psi)$)

$$\mathcal{E}(v) := \int_\Omega \left(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2f(x)v(x)\right) dx, \tag{1}$$

on $K_0 = \{v \in H^1(\Omega) \ : \ v \geq 0 \ \mathcal{L}^n \ a.e., \ Tr(v) = g \in H^{\frac{1}{2}}(\partial\Omega)\}$, where $\Omega \subset \mathbb{R}^n$ is a smooth, bounded and open set, $n \geq 2$, $\mathbb{A} : \Omega \to \mathbb{R}^{n\times n}$ and $f : \Omega \to \mathbb{R}$ are functions satisfying:

(H1) $\mathbb{A} \in W^{1+s,p}(\Omega; \mathbb{R}^{n\times n})$ with $s > \frac{1}{p}$, $p > \frac{n^2}{n(1+s)-1} \wedge n$ or $s = 0$ and $p = +\infty$;

(H2) $\mathbb{A}(x) = (a_{ij}(x))_{i,j=1,\dots,n}$ continuous, $a_{ij} = a_{ji}$ $\mathcal{L}^n$ a.e. Ω and $\exists \Lambda \geq 1$

$$\Lambda^{-1}|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \qquad \mathcal{L}^n \text{ a.e. } \Omega, \ \forall \xi \in \mathbb{R}^n; \tag{2}$$

(H3) f Dini-continuous

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty \tag{3}$$

where $\omega(t) = \sup_{|x-y|\leq t} |f(x) - f(y)|$, $f \geq c_0 > 0$.

(H3)$'$ Let $a > 2$ be

$$\int_0^1 \frac{\omega(r)}{r} |\log r|^a\, dr < \infty. \tag{4}$$

Result and discussions

We analyse the properties of the minimizer and, studying the properties of blow-ups, we obtain the regularity of the free-boundary. We prove that the minimizer u is a regular solution of an elliptic differential equation in divergence form:

$$\mathrm{div}(\mathbb{A}(x)\nabla u(x)) = f(x)\chi_{\{u>0\}}(x) \qquad a.e. \text{ on } \Omega \text{ and in } \mathcal{D}'(\Omega). \tag{5}$$

The lack of smoothness and homogeneity of the matrix of coefficients $\mathbb{A}$ does not permit to exploit elementary freezing arguments to locally reduce the regularity problem above to the analogous one for smooth operators, for which a complete theory has been developed by Caffarelli [3]. We fix x_0 a point of the free-boundary Γ_u and by a suitable change of variable $\mathbb{L}(x_0)$, w.l.o.g., we suppose

$$x_0 = 0 \in \Gamma_u, \qquad \mathbb{A}(0) = I_n, \qquad f(0) = 1. \tag{6}$$

Building upon the variational approach to the classical obstacle problem developed by Weiss and Monneau, the strategy to prove the regularity of free-boundary is energy-based and relies on quasi-mononicity formulas, on Weiss' epiperimetric inequality as well as on Caffarelli's fundamental blow up analysis. We proceed, as in [4], introducing the rescaled 2-homogeneous functions $u_r(x) := \frac{u(rx)}{r^2}$ and we introduce an associate energy "à la Weiss"

$$\Phi(r) := \int_{B_1} \left(\langle \mathbb{A}(rx)\nabla u_r(x), \nabla u_r(x)\rangle + 2f(rx)u_r(x)\right) dx + \int_{\partial B_1} \langle \mathbb{A}(rx)\frac{x}{|x|}, \frac{x}{|x|}\rangle u_r^2(x)\, dH^{n-1}, \tag{7}$$

The rescaled functions satisfy an appropriate PDE and an uniform estimate.

Proposition 1 (Uniform boundness $W^{2,p}$)

Assume (6) holds. Then, $\forall R > 0$ $\exists C > 0$ such that, for $r << 1$

$$\|u_r\|_{W^{2,p^*}(B_R)} \leq C. \tag{8}$$

In particular, the functions u_r are equibounded in $C^{1,\gamma'}$ for $\gamma' \leq \gamma := 1 - \frac{n}{p^*}$.

It holds a quasi-monotonicity formula that extends Weiss' formula in [9]:

Theorem 2 (Weiss' type quasi-monotonicity formula)

Assume that (H1)-(H3) and (6) are satisfied, and let $\Theta = \Theta(n, p, s)$ be an exponent such that $\Theta > n$ ($\Theta = +\infty$ if $\mathbb{A} \in W^{1,\infty}$). Then $\exists$ $\bar{C}_3, C_4 > 0$ independent from r such that

$$r \mapsto \Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt \tag{9}$$

is nondecreasing on $(0, \frac{1}{2}\mathrm{dist}(0, \partial\Omega) \wedge 1)$.

By Blanck and Hao [1] we have the following result.

Proposition 3 (Quadratic growth)

Let $x_0 \in \Gamma_u$, then $\exists \theta > 0$ such that

$$\sup_{\partial B_r(x_0)} u \geq \theta\, r^2. \tag{10}$$

By boundness estimate we have the existence of blow-ups, by quasi-monotonicity formula we prove that the blow-ups are 2-homogeneous and by quadratic growth we obtain the non degeneracy of blow-ups. Moreover thanks Γ-convergence argument we give a classification of blow-ups as in the classical case established by Caffarelli [2, 3].

Proposition 4 (Classification of blow-ups)

Every blow-up v_{x_0} at a free boundary point $x_0 \in \Gamma_u$ is of the form $v_{x_0} = w(\mathbb{L}^{-1}(x_0)y)$, where w is a non-trivial, 2-homogeneous for which one of the following two cases occurs:

(A) *$w(y) = \frac{1}{2}(\langle y, \nu\rangle \vee 0)^2$ for some $\nu \in \mathbb{S}^{n-1}$;*

(B) *$w(y) = \langle \mathbb{B}y, y\rangle$ with $\mathbb{B}$ a symmetric, positive definite matrix satisfying and $\mathrm{Tr}\mathbb{B} = \frac{1}{2}$.*

The above proposition allows us to formulate a simple criterion to distinguish between regular and singular free boundary points.

Definition 5 (Regular and Singular points of the free-boundary)

A point $x_0 \in \Gamma_u$ is a regular free boundary point, and we write $x_0 \in Reg(u)$ if there exists a blow-up of u at x_0 of type (A). Otherwise, we say that x_0 is singular and write $x_0 \in Sing(u)$.

We prove a Monneau's type quasi-monotonicity formula (see [8]) for singular free boundary points.

Theorem 6 (Monneau's type quasi-monotonicity formula)

Let $0 \in Sing(u)$. Below hypotesies of Theorem 1 $\exists C_5 > 0$ independent of r such that for some v 2-homogeneous polynomial positive function, solving $\Delta v = 1$ on $\mathbb{R}^n$, the function

$$r \longmapsto \int_{\partial B_1} (u_r - v)^2\, dH^{n-1} + C_5 \left(r^{(1-\frac{n}{\Theta})} + \omega(r)\right) \tag{11}$$

is nondecreasing on $(0, \frac{1}{2}\mathrm{dist}(0, \partial\Omega) \wedge 1)$.

The following result prove the property of uniqueness of blow-ups. We use the Theorem 6 for singular points and the Weiss' epiperimetric formula [9] for regular points.

Proposition 7 (Uniqueness of blow-ups)

(i) *Assume (H1)-(H3), then $\forall x \in Sing(u)$ there exists a unique blow-up limit $v_x(y) = w(\mathbb{L}^{-1}(x)y)$. Moreover, if $K \subset Sing(u)$ is a compact subset, then $\forall x \in K$*

$$\left| u_{\mathbb{L}(x),r} - w \right|_{C^1(B_1)} \leq \sigma_K(r) \qquad \forall r \in (0, r_K), \tag{12}$$

for some modulus of continuity $\sigma_K : \mathbb{R}^+ \to \mathbb{R}^+$ and a radius $r_K > 0$.

(ii) *Suppose (H1), (H2) and (H3)$'$ and let $x_0 \in Reg(u)$. Then, $\exists r_0 = r_0(x_0)$, $\eta_0 = \eta_0(x_0)$ such that every $x \in Reg(u) \cap B_{\eta_0}(x_0)$ and, denoting by $v_x = w(\mathbb{L}^{-1}(x)y)$ any blow-up of u in x we have*

$$\int_{\partial B_1} |u_{\mathbb{L}(x),r} - w|\, dH^{n-1}(y) \leq C_7\, \rho(r) \qquad \forall r \in (0, r_0), \tag{13}$$

where C_7 is an independent constant from r and $\rho(r)$ a growing, infinitesimal function in 0. In particular, the blow-up limit v_x is unique.

These results allow to prove the regularity of free-boundary:

Theorem 8 (Regularity of the free-boundary)

We assume the hypothesis (H1)-(H3). The free-boundary decomposes as $\Gamma_u = Reg(u) \cup Sing(u)$ with $Reg(u) \cap Sing(u) = \emptyset$.

(i) *Assume (H3)$'$. $Reg(u)$ is relatively open in $\partial\{u = 0\}$ and for every point $x_0 \in Reg(u)$ there exists $r = r(x_0) > 0$ such that $\Gamma_u \cap B_r(x_0)$ is a C^1 hypersurface with normal versor σ is absolutely continuous.*
In particular if f is Hölder continuous there exists $r = r(x_0) > 0$ such that $\Gamma_u \cap B_r(x)$ is $C^{1,\beta}$ hypersurface for some universal exponent $\beta \in (0, 1)$.

(ii) *$Sing(u) = \cup_{k=0}^{n-1} S_k$ and for all $x \in S_k$ there exists r such that $S_k \cap B_r(x)$ is contained in a regular k-dimensional submanifold of $\mathbb{R}^n$.*

Futher development

As a direct outcome of Theorem 8 we shall deduce the analogous result for u, solution of nonlinear variational problem

$$\min_{K_\psi} \int_\Omega F(x, v, \nabla v)\, dx \tag{14}$$

where $K_\psi := \{v \in H^1(\Omega) \ : \ v \geq \psi \ \mathcal{L}^n \ a.e., \ Tr(v) = g \in H^{\frac{1}{2}}(\partial\Omega)\}$, $F(x, z, \xi)$ is a nonlinear function for which $(\nabla_\xi F, \partial_z F)(x, z, \xi)$ is smooth strongly coercive vector field and ψ is a regular obstacle.

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