The Formation and Coarsening of the Concertina Pattern

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Joint work with
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Experiment: Magnetization reversal

...in thin elongated ferromagnetic elements

magnetic field $H_{\text{ext}}$

Permalloy, width $\ell = 30\mu$m, thickness $t = 60$nm
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magnetic field $H_{\text{ext}}$

Permalloy, width $\ell = 30\,\mu\text{m}$, thickness $t = 60\,\text{nm}$

Goal: Use asymptotic analysis and numerical simulation to understand

A  Instability at critical field $H_{\text{ext}} = H_{\text{ext}}^*$
B  Formation of domain pattern
C  Coarsening
Periodic sample
\[ \Omega = \mathbb{R} \times (0, \ell) \times (0, t) \]
width \( \ell \), thickness \( t \)

External magnetic field \( H_{\text{ext}} = (-h_{\text{ext}}, 0, 0) \)

Magnetization \( m = (m_1, m_2, m_3), \ |m|^2 = 1 \)
Variational continuum model by Landau & Lifschitz (1/2)

- Periodic sample \( \Omega = \mathbb{R} \times (0, \ell) \times (0, t) \)
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- Magnetization \( m = (m_1, m_2, m_3) \), \( |m|^2 = 1 \)

Observed configuration \( \Leftrightarrow \) Minimizer of the micromagnetic energy

\[
E(m) = d^2 \int_{\Omega} |\nabla m|^2 \, d\mathbf{x} + \int_{\mathbb{R}^3} |H_{\text{stray}}|^2 \, d\mathbf{x} - 2 \int_{\Omega} H_{\text{ext}} \cdot m \, d\mathbf{x}
\]

\( E(m) \) \( \text{Exchange} \quad \text{Stray-field} \quad \text{Zeeman} \)

Material parameter: exchange length \( d = 5\text{nm} \)
Variational continuum model by Landau & Lifschitz (2/2)

**Stray field : Static Maxwell equations**

\( H_{\text{stray}} = -\nabla u \) satisfies

\[
-\Delta u = -\nabla \cdot m \quad \text{in } \Omega \quad \text{volume charges}
\]

\[
\left[ \frac{\partial u}{\partial \nu} \right] = \nu \cdot m \quad \text{on } \partial \Omega \quad \text{surface charges}
\]

\[-\Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega\]

\[
\int_{\mathbb{R}^3} |H_{\text{stray}}|^2 \, d x = \int_{\mathbb{R}^3} |\nabla u|^2 \, d x
\]
A: Linear stability analysis of uniform magnetization

**Theorem** ([CAO’06])
Oscillatory buckling instability in thin wide samples \((\ell \frac{d}{\ell} \ll \frac{t}{d} \ll (\frac{\ell}{d})^{1/2})\)

**Explanation**
- Exchange and stray-field energy enforce
  \[
  m = m'(x') = (m_1, m_2)(x_1, x_2) \quad \text{in-plane}
  \]
  \[
  m' \cdot \nu' = 0 \quad \text{edge-pinned}
  \]
- Oscillatory charges induce small stray-field energy
  \[
  w^* \approx (32\pi d^2 \ell^2 t^{-1})^{1/3} \quad \text{optimal period}
  \]
A: Linear stability analysis of uniform magnetization

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**Comparison with experiments**
- $w_{\text{exp}} \sim w^*$ but $w_{\text{exp}}$ is always larger than $w^*$
Derivation of reduced model

**Theorem ([CAOS07])** \( \Gamma \)-convergence

\[
E_0(m_2) = \int_{\Omega'} (\partial_1 m_2)^2 \, dx' + \int_{\Omega' \times \mathbb{R}} (\partial_1 u)^2 + (\partial_3 u)^2 \, dx - h_{\text{ext}} \int_{\Omega'} m_2^2 \, dx'
\]

*Exchange* + *Stray-field* - *Zeeman*

Reduced Maxwell eq.

\[
[\partial_3 u] = -\partial_1 \frac{m_2^2}{2} + \partial_2 m_2 \quad x_3 = 0
\]

\[-(\partial_1^2 + \partial_3^2)u = 0 \quad x_3 \neq 0\]

**Ingredients**

**Thin film** \((t \ll w^*)\) implies

- \(m = m'(x')\) and \(m' \cdot \nu' = 0\)
- Volume charges act as surface charges

**Wide film** \((w^* \ll \ell)\) implies \(\partial_2 m'\) and \(\partial_2 u\) negligible

**Small angles** \(m \approx m^*\) imply

\[m_1 = \sqrt{1 - m_2^2} \approx 1 - \frac{m_2^2}{2}\]
Numerical simulation of reduced energy $E_0$

**Discretization of $E_0(m_2)$**

- Finite-difference scheme
- FFT for stray-field energy
- Parallelization and implementation in PETSc

Finite-difference scheme
Numerical simulation of reduced energy $E_0$

**Discretization of $E_0(m_2)$**
- Finite-difference scheme
- FFT for stray-field energy
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**Computation of stationary points**
- Newton’s method or steepest descent
- Path following for Euler-Lagrange eq.

\[
\left( \nabla_{m_2} E_0(m_2, h_{\text{ext}}^i) \right) \cdot \left( (m_2^i, h_{\text{ext}}^i) - p^i \cdot t^{i-1} \right) = 0
\]

- Bifurcation detection and branch switching
A/B: Numerical path following

Sections of the energy landscape close to critical field $h_{\text{ext}}^*$

$h_{\text{ext}} < h_{\text{ext}}^*$

$h_{\text{ext}} = h_{\text{ext}}^*$

$h_{\text{ext}} > h_{\text{ext}}^*$

Observations

- Sub-critical bifurcation and turning point
- Mesoscopically charge-free domain-wall pattern

$-\partial_1 \frac{m_2^2}{2} + \partial_2 m_2 = 0$

$\iff$

Shocks of Burgers eq.

Stationary points
C: Numerical bifurcation detection

Stability of $w^*$-periodic branch under $Nw^*$-periodic perturbations?

Observations
- Cascade of instabilities
- Coarsening along secondary branches
- Degenerate bifurcations due to symmetries

Rotation and reflection symmetry
C: Sharp-interface model for $h_{\text{ext}} \gg 1$

Piecewise constant **Ansatz**

- period $w$
- transverse component $m_2 = \pm A$
- weak divergence-free $[(-\frac{m_2^2}{2}, m_2) \cdot \nu] = 0$

**Energy**

$$E_\#(A, w, h_{\text{ext}}) = \int_{\text{jump set}} e([m_2]) \, d\mathcal{H}^0 \, d x_2 - h_{\text{ext}} \int_{\Omega'} m_2^2 \, d x'$$

Line energy $e([m_2]) = \frac{\pi}{8} A^4 \ln^{-1}(A^2 w)$, jump $[m_2] = 2A$
C: Sharp-interface model for $h_{\text{ext}} \gg 1$

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**Optimal period**

Minimization of energy per length $\frac{E_{\#}}{w}$ in $A$ and $w$

$$w \sim h_{\text{ext}} \ln h_{\text{ext}}$$
Concavity of minimal energy per period, i.e.,

$$\min_A E_\#(A, w),$$

in $w \Rightarrow$ modulation instability
**C: Geometric criterion for the secondary instabilities**

**Concavity** of minimal energy per period, i.e.,

\[ \min_A E^\#(A, w), \]

in \( w \) ⇒ modulation instability

\[ 2E^\#(w) > E^\#(w + \epsilon) + E^\#(w - \epsilon) \]

\[ E^\#: \text{optimal and marginally stable period} \]
C: Geometric criterion for the secondary instabilities

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in \( w \Rightarrow \) modulation instability

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**Caution!**

- Sharp-interface model is rigid
- Reduced energy is non-local

\[ E_0 \left( \begin{array}{c}
\end{array} \right) \ ? \ E_0 \left( \begin{array}{c}
\end{array} \right) + E_0 \left( \begin{array}{c}
\end{array} \right) \]

\( h^*_{ext} \): optimal and marginally stable period
Theorem

For infinitesimal **Bloch**-perturbations, i.e.,

\[ \delta m_2 = e^{i\epsilon x_1} g(x_1, x_2) + c.c. \]

with \( w \)-periodic \( g \), it holds

\[ \lambda_{min}^{\epsilon} \leq \inf_{\delta m_2} \frac{\text{Hess} \ E_0(m_2)(\delta m_2, \delta m_2)}{\int \int \delta m_2^2 \ dx_1 \ dx_2} \ll \epsilon^2 \frac{w^2 \frac{d^2}{dw^2} E_0(w)}{\int (\partial_1 m_2)^2 \ dx_1 \ dx_2} \]
C: Instability criterion in the reduced model

Theorem

For infinitesimal Bloch-perturbations, i.e.,

$$\delta m_2 = e^{i\epsilon x_1} g(x_1, x_2) + c.c.$$ with w-periodic $g$,

it holds

$$\lambda_\epsilon^{\min} \leq \inf_{\delta m_2} \frac{\text{Hess } E_0(m_2)(\delta m_2, \delta m_2)}{\int \int \delta m_2^2 \, dx_1 \, dx_2} \leq \epsilon^2 \frac{w^2 \frac{d^2}{dw^2} E_0(w)}{\int (\partial_1 m_2)^2 \, dx_1 \, dx_2}$$
Theorem ([OS10])

Let $h_{\text{ext}} \gg 1$ and $L \gg h_{\text{ext}} \ln h_{\text{ext}}$.

a) **Optimal period** (lower bound) for almost optimal $m_2$, i.e.,

$$L^{-1} \int_0^L \int_0^1 (m_2(x_1 + w, x_2) - m_2(x_1, x_2))^2 \, dx_2 \, dx_1$$

$$\lesssim \left( \frac{w}{h_{\text{ext}} \ln h_{\text{ext}}} \right)^{\alpha} L^{-1} \int_0^L \int_0^1 m_2^2 \, dx_2 \, dx_1$$

for all $w > 0$ and $\alpha \in \left[0, \frac{2}{5}\right)$

b) **Minimal energy per length**

$$\min_{m_2} L^{-1} E_0 \sim -h_{\text{ext}}^3 \ln^2 h_{\text{ext}}$$
Theorem ([OS10])

Let $h_{\text{ext}} \gg 1$ and $L \sim h_{\text{ext}} \ln h_{\text{ext}}$. Let $m_2$ be almost optimal, i.e., $L^{-1} E_0(m_2) \sim \min_{m_2} L^{-1} E_0$. Then there exists $m_2^*$ with

$$L^{-1} \int_0^L \int_0^1 (m_2 - m_2^*)^2 \, dx_2 \, dx_1 \ll L^{-1} \int_0^L \int_0^1 (m_2)^2 \, dx_2 \, dx_1$$

such that

$$-\partial_1 \frac{(m_2^*)^2}{2} + \partial_2 m_2^* = 0$$

distributionally.
A: Deviation of $w^*$ and $w_{\text{exp}}$ and stability close to $h^*_{\text{ext}}$

Numerical simulation of reduced energy

- Pattern of period $w^*$ is unstable
- Minimal stable period
A: Extended bifurcation analysis of $E_0$ close to $h_{ext}^*$

- Perturbations of critical field and critical wave number:

$$h_{ext} = h_{ext}^* + \delta h_{ext} \quad \text{and} \quad k = \frac{2\pi}{w} = k^* + \delta k$$

- Perturbations of unstable mode:

$$m_2 \approx A m_2^* + A^2 m_2^{**} + A^3 m_2^{***}$$

yield asymptotic expansion of the reduced energy

$$E_0(A, \delta k, \delta h_{ext}) \approx (-c_2 \delta h_{ext} + \tilde{c}_2 \delta k^2) A^2 + (-c_4 + \tilde{c}_4 \delta k) A^4 + c_6 A^6$$

- $|c_4| \ll 1 \iff$ Asymmetric side-band instability
- Eckhaus instability in convective systems

optimal and marginally stable period
Polycrystalline anisotropy (1/2)

...causes ripple and perturbed concertina in the experiments

Permalloy $t = 30$nm, $\ell = 70$µm

position-dependent easy axis $e(x_1, x_2)$ of strength $Q$

$$-Q \int_{\Omega} (m \cdot e(x_1, x_2))^2 \, d\mathbf{x}$$

transverse stochastic field $h_{\text{ripple}} = Qe_1 e_2$

$$-2t \int_{\Omega'} h_{\text{ripple}} m_2 \, d\mathbf{x}'$$
Polycrystalline anisotropy (2/2)

Ripple and concertina in experiment (o.) and simulation (u.)

Linearization of the energy
Confirmation of smooth transition of ripple to concertina

\[ \lim_{h_{\text{ext}} \uparrow h_{\text{ext}}^{*}} \langle | k_{1}^{\text{ripple}} | \rangle = k_{1}^{*} \]
Summary

- Understanding of observations by exploration of the energy landscape
  
  Simulation (path following, ...)
  
  $\Leftrightarrow$
  
  Asymptotic analysis
  (bifurcation analysis, Bloch-waves, ...)

- Extension of reduced model
  (polycrystalline anisotropy)

Further aspects

- Anisotropy and large angles
- Finite sample size
- Refining and hysteresis
- Amplitude functional
Finite sample size: Van den Berg’s explanation

| Geometry | Length $L \gg \ell$ | Length $L \sim \ell$
|----------|---------------------|---------------------|
| External field | Critical field $h^*_\text{ext} \approx 3\left(\frac{\pi}{2}\right)^{4/3}(d^2\ell^{-4}t^2)^{1/3} - Q$ | $h^\text{begin}_{\text{ext}} \sim \frac{t}{\ell}$
| | | $h^\text{end}_{\text{ext}} \sim t\ell L^{-2} \ln t\ell^{-1}$
| length scale | $w^* \sim d^{2/3}\ell^{2/3}t^{-1/3}$ | $w \sim \ell$
| Formation | Simultaneously along sample | Successive in-growth from tips

Example: Permalloy of transverse anisotropy $Q = 5.0 \times 10^{-4}$ and dimension $\ell = 30\mu\text{m}$, $t = 30\text{nm}$

$$h^\text{begin}_{\text{ext}} \sim 1 \times 10^{-3} \gg h^*_{\text{ext}} \approx 3.4 \times 10^{-4} \gg h^\text{end}_{\text{ext}} \sim 1.6 \times 10^{-6}$$
Experiment vs. Analysis: The period of the instability

Measurements for Permalloy samples of length $\ell = 10 \ldots 50\mu$m and thickness $t = 10 \ldots 150$nm

Range of theoretical period $\frac{w^*}{w^*}(\ell_{\text{min}}, t_{\text{max}})$

Comparison of experimental and theoretical period $\frac{w_{\text{exp}}}{w^*}$
Simulation – Hysteresis loop

\[(\bar{n} n^2)^{1/2}\]

\[h_{ext}\]