Quasistatic evolution of cavities in nonlinear elasticity

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Nonlinear Elasticity – Calculus of Variations approach

Body $\Omega \subset \mathbb{R}^3$, deformation $\mathbf{u} : \Omega \to \mathbb{R}^3$.

Elastic (bulk) energy: $\displaystyle\int_\Omega W(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$,

Body forces: $\displaystyle\int_\Omega F(\mathbf{x}, \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$,

Surface forces: $\displaystyle\int_{\Gamma_N} G(\mathbf{x}, \mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathcal{H}^2(\mathbf{x})$,

Boundary condition: $\mathbf{u} = \mathbf{u}_0$ on Γ_D.

($\partial\Omega = \Gamma_D \cup \Gamma_N$ disjoint).

An equilibrium solution $\mathbf{u}$ (Statics) is a solution of

$$\min_{\mathbf{u} \in W^{1,p}_{\mathbf{u}_0}} \int_\Omega W(\mathbf{x}, D\mathbf{u})\,\mathrm{d}\mathbf{x} - \int_\Omega F(\mathbf{x}, \mathbf{u})\,\mathrm{d}\mathbf{x} - \int_{\Gamma_D} G(\mathbf{x}, \mathbf{u})\,\mathrm{d}\mathcal{H}^2.$$

Apart from solving a minimization problem, every physically realistic solution $\mathbf{u}$ must:

- preserve the orientation: $\det D\mathbf{u} > 0$,
- be one-to-one (no interpenetration of matter).

Cavitation is the phenomenon of sudden formation of voids in near-incompressible solids subject to large triaxial tension. It is typical in elastomers and ductile metals.

(A.N. Gent & P.B. Lindley 1959)

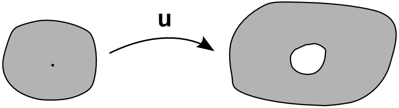
For the well-posedness of the model (and also to be physically more realistic) an extra **surface energy due to cavitation** is needed.

Existence theories (Statics: minimization of energy):

- S. Müller & S.J. Spector 95.
- J. Sivaloganathan & S.J. Spector 00.
- D. Henao & C.M.-C. 10–11.

(Idea: Surface energy $\Longrightarrow$ det is w-continuous $\Longrightarrow$ energy is swlsc.)

We will use the theory of Henao & M.-C.: The cavitation energy is proportional to the surface created. Orientation-preserving and non-interpenetration are also taken into account.



Irreversibility

Once a cavity is formed, the shape and size of the cavity surface can change in time (even dissapear macroscopically), but the cavity point in the reference configuration will remain as a flaw point.

A *configuration* is a pair $(\mathbf{u}, S)$ where $\mathbf{u}$ is a deformation, and S is a subset of Ω containing the cavity points $C(\mathbf{u})$ of $\mathbf{u}$. Intuitively,

$$S = \{\text{cavity or flaw points}\} = \{\text{past or present cavity points}\}.$$

The cavitation energy $\mathcal{S}(\mathbf{u}, S) := \mathcal{S}_1(S) + \mathcal{S}_2(\mathbf{u})$ is the sum of a fixed amount accounting for the mere process of cavity formation

$$\mathcal{S}_1(S) := \sum_{\mathbf{a} \in S} \kappa_1(\mathbf{a})$$

plus a term proportional to the area of the surface created

$$\mathcal{S}_2(\mathbf{u}) := \sum_{\mathbf{a} \in C(\mathbf{u})} \kappa_2(\mathbf{a})\, \mathcal{H}^2(C(\mathbf{u}, \mathbf{a})).$$

Quasistatic evolution

In a quasistatic theory,

- the interaction of the system with its environment is infinitely slow
- the system is always in equilibrium
- the system does not have its own dynamics, but rather the dynamics respond to changes in the external conditions
- evolution is considered as a family of minimization problems parametrized by the time variable
- at each instant of time, the energy is minimized
- an energy balance holds (taking into account dissipation).

Quasistatic evolution of cavitation

Total energy

$$\mathcal{I}(t)(\mathbf{u},S) := \underbrace{\mathcal{W}(D\mathbf{u})}_{\text{bulk}} + \underbrace{\mathcal{S}(\mathbf{u},S)}_{\text{cavitation}} - \underbrace{\mathcal{F}(t)(\mathbf{u})}_{\text{body force}} - \underbrace{\mathcal{G}(t)(\mathbf{u})}_{\text{surface force}}$$

Boundary condition: $\mathbf{u} = \mathbf{u}_0(t)$ on Γ_D.

Given a family $\{\mathbf{u}(t)\}_{t\in[0,1]}$ of deformations, define $S(t) := \bigcup_{s\in[0,t]} C(\mathbf{u}(s))$. It constitutes a *quasistatic evolution* if

a) *Global stability:* For each t, the pair $(\mathbf{u}(t), S(t))$ minimizes $\mathcal{I}(t)$ over $(\mathbf{u}, S)$ satisfying $S \supset \bigcup_{s<t} S(s)$ and b.c.

b) *Energy balance:* Increment in stored energy plus energy spent in cavities equals increase of the work of external forces:

$$\mathcal{I}(t)(\mathbf{u}(t), S(t)) = \mathcal{I}(0)(\mathbf{u}(0), S(0)) + \int_0^t [\cdots]\,\mathrm{d}s$$

Method of proof

(A. Mielke & F. Theil 99, G. Francfort & C. Larsen 03,

G. Dal Maso, G. Francfort & R. Toader 05…)

1. *Time discretization:* For each $k \in \mathbb{N}$,

   $$0 = t_k^0 < t_k^1 < \cdots < t_k^k = 1 \quad \text{with} \quad \lim_{k\to\infty} \max_{1\le i\le k} (t_k^i - t_k^{i-1}) = 0.$$

   Let $(\mathbf{u}_k^0, S_k^0) = (\mathbf{u}^0, S^0)$ (given) and for $1 \le i \le k$, let $(\mathbf{u}_k^i, S_k^i)$ be a minimizer of $\mathcal{I}(t_k^i)$ with b.c. $\mathbf{u}_0(t_k^i)$ and $S_k^i \supset S_k^{i-1}$.

2. *Constant interpolation:* For $k \in \mathbb{N}$ and $t \in [0,1]$ let $0 \le i_k \le k$ be such that $t \in [t_k^{i_k}, t_k^{i_k+1})$.

3. *Passage to the limit:* For each $t \in [0,1]$, let

   $$\mathbf{u}(t) := \lim_{k\to\infty} \mathbf{u}_k^{i_k}, \qquad S(t) := \lim_{k\to\infty} S_k^{i_k}.$$

About the limit passage

$\mathbf{u}_k^{i_k}$ being a minimizer satisfies some a priori bounds that, up to a subsequence, allow to take limits $\mathbf{u}_k^{i_k} \to \mathbf{u}(t)$ as $k \to \infty$ in the sense of the theorem of existence of minimizers. Moreover,

$$\mathcal{W}(D\mathbf{u}(t)) \leq \liminf_{k\to\infty} \mathcal{W}(D\mathbf{u}_k^{i_k}), \quad \mathcal{S}_2(\mathbf{u}(t)) \leq \liminf_{k\to\infty} \mathcal{S}_2(\mathbf{u}_k^{i_k}),$$

$$\mathcal{F}(t)(\mathbf{u}(t)) = \lim_{k\to\infty} \mathcal{F}(t_k^{i_k})(\mathbf{u}_k^{i_k}), \quad \mathcal{G}(t)(\mathbf{u}(t)) = \lim_{k\to\infty} \mathcal{G}(t_k^{i_k})(\mathbf{u}_k^{i_k}).$$

$\mathbf{u}_k^{i_k}$ being a minimizer implies that $\mathcal{H}^0(S_k^{i_k})$ is bounded. Hence up to a subsequence $\mathcal{H}^0(S_k^{i_k})$ is constant and $S_k^{i_k}$ converges componentwise to an $S(t)$. Moreover,

$$\mathcal{S}_1(S(t)) \leq \liminf_{k\to\infty} \mathcal{S}_1(S_k^{i_k}).$$

We show that $(\mathbf{u}(t), S(t))$ is admissible, i.e., $S(t) \supset C(\mathbf{u}(t))$.

Main difficulty: stability of minimizers

We have to show that $(\mathbf{u}(t), S(t))$ is a minimizer. Let $(\mathbf{u}, S)$ be admissible with $S \supset \bigcup_{s<t} S(s)$. If we construct $(\tilde{\mathbf{u}}_k, \tilde{S}_k)$ satisfying b.c. $\mathbf{u}_0(t_k^{i_k})$ and such that $\tilde{S}_k \supset S_k^{i_k - 1}$ and

$$\limsup_{k\to\infty} \mathcal{I}(t_k^{i_k})(\tilde{\mathbf{u}}_k, \tilde{S}_k) \leq \mathcal{I}(t)(\mathbf{u}, S)$$

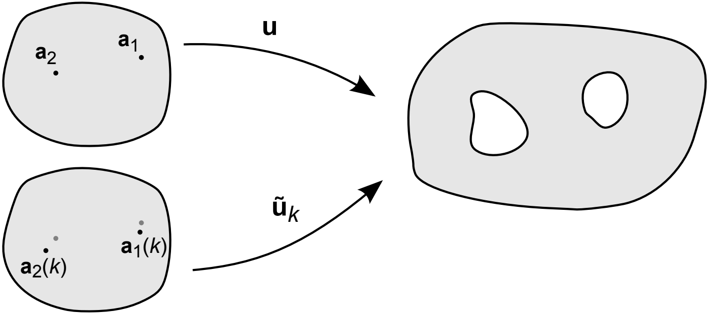
then

$$\begin{aligned} \mathcal{I}(t)(\mathbf{u}(t), S(t)) &\leq \liminf_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k}) \\ &\leq \liminf_{k\to\infty} \mathcal{I}(t_k^{i_k})(\tilde{\mathbf{u}}_k, \tilde{S}_k) \leq \mathcal{I}(t)(\mathbf{u}, S). \end{aligned}$$

(cf. 'transfer lemma' Larsen & Francfort 03. Think also of a recovery sequence in Γ-convergence)

Stability of minimizers. First attempt

Modify the position of the cavity points of $\mathbf{u}$ to coincide with those of $\mathbf{u}_k^{i_k}$. So construct $(\tilde{\mathbf{u}}_k, \tilde{S}_k)$ with $\tilde{\mathbf{u}}_k \simeq \mathbf{u}$ but with $\tilde{S}_k \supset S_k^{i_k - 1}$.



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An abstract consequence of the quasistatic theory is that

$$\mathcal{I}(t)(\mathbf{u}(t), S(t)) = \lim_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k}).$$

(cf. 'minimizers go to minimizers' and 'energy of minimizers go to energy of minimizers' in Γ-convergence)

We always have

$$\mathcal{I}(t)(\mathbf{u}(t), S(t)) \leq \liminf_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k})$$

and the recovery sequence provides

$$\mathcal{I}(t)(\mathbf{u}(t), S(t)) \geq \limsup_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k}).$$

Stability of minimizers. Second attempt

Independently of the recovery sequence, the only reason for which we may have

$$\mathcal{I}(t)(\mathbf{u}(t), S(t)) < \liminf_{k\to\infty} \mathcal{I}(t_k^{i_k})(\mathbf{u}_k^{i_k}, S_k^{i_k})$$

is that when we take the limit $S_k^{i_k} \to S(t)$ we have

$$\mathcal{H}^0(S(t)) < \liminf_{k\to\infty} \mathcal{H}^0(S_k^{i_k}). \tag{*}$$

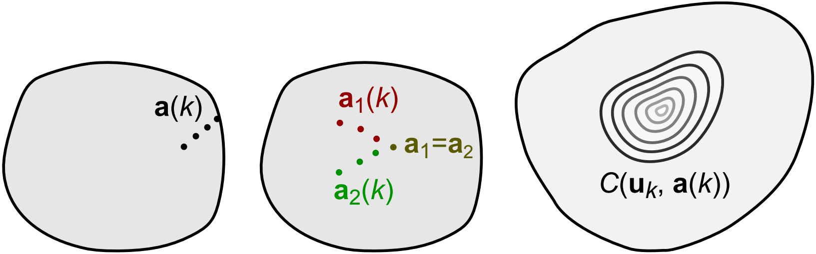
There are only three reasons for which (*) can happen: in the limit passage,

(1) Cavities scape to the boundary.

(2) Cavities collapse (coalesce).

(3) Cavities close up (heal).

A quasistatic evolution exists iff none of (1)–(3) hold.



Scape to boundary | Collapse | Close up

Stability of minimizers. Solution

Independently of the recovery sequence, we have to avoid that cavities

(1) scape to the boundary; (2) collapse; (3) close up.

Condition (1) is avoided by prohibiting *cavitation at the boundary* in the static model (as in J. Sivaloganathan & S.J. Spector 00, D. Henao 09: also prohibits pathological behaviour).

Conditions (2)–(3) may well happen *but not for minimizers.* Qualitative (not quantitative) result:

- Minimizers cannot have two cavities very close to each other (it is better to have only one).
- Minimizers cannot have a cavity enclosing a very small volume (it is better not to have any).

Conclusion

$(\mathbf{u}(t), S(t))$ is a minimizer and $S(t) = \bigcup_{s\in[0,t]} C(\mathbf{u}(s))$.

Energy balance and remaining properties of quasistatic evolution follow the lines of G. Dal Maso, G. Francfort & R. Toader 05.

Theorem: For every initial data, there exists a quasistatic evolution starting at the initial data, and satisfying global stability, irreversibility and energy balance.