FRACTAL BEHAVIOR OF THE FIBONOMIAL TRIANGLE MODULO PRIME $p$, WHERE THE RANK OF APPARITION OF $p$ IS $p + 1$

MICHAEL DEBELLEVUE AND EKATERINA KRYUCHKHOVA

Abstract. Pascal’s triangle is known to exhibit fractal behavior modulo prime numbers. We tackle the analogous notion in the fibonomial triangle modulo prime $p$ with the rank of apparition $p^* = p + 1$, proving that these objects form a structure similar to the Sierpinski Gasket. Within a large triangle of $p^{*}p^m + 1$ many rows, in the $i^{th}$ triangle from the top and the $j^{th}$ triangle from the left, $\binom{n+ip^m}{k+jp^m} \equiv_p 1$ if and only if $\binom{n}{k} \equiv_p 1$. This proves the existence of the recurring triangles of zeroes which are the principal component of the Sierpinski Gasket. The exact congruence classes follow the relationship $\binom{n+ip^m}{k+jp^m} \equiv_p (-1)^{ik-jn} \binom{n}{k}$. The periodic nature of the Fibonacci sequence exhibits a number of interesting properties, among them the divisibility property, regular divisibility by a prime, and the shifting property. The following lemmas can be found in a variety of sources, including \[8\].

1. Introduction

Pascal’s triangle is known to exhibit fractal behavior modulo prime numbers. This can be proven by using Lucas’ Theorem:

Theorem 1.1. Write $n$ and $k$ in base $p$ with digits $n_0, n_1, \ldots, n_m$ and $k_0, k_1, \ldots, k_m$. Then,

$$\binom{n}{k} \equiv_p \binom{n_0}{k_0} \binom{n_1}{k_1} \cdots \binom{n_m}{k_m}.$$  

Consider the Fibonacci numbers as defined by $F_0 = 0, F_1 = 1, and F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The fibonomial triangle is formed using the fibotorial $!_F$ function in place of the factorial function, where $n!_F = F_n F_{n-1} F_{n-2} \ldots F_1$. Then the fibonomial coefficient $\binom{n}{k}_F$ is defined as $\frac{n!_F}{(n-k)!_F p^k}$, where $\binom{n}{0}_F$ is defined to be 1 for $n \geq 0$, as with binomial coefficients. The fibonomial triangle appears to exhibit a fractal structure, but unfortunately Lucas’ Theorem does not apply to fibonomial coefficients [7]. Instead, we prove an analogue of Lucas’ Theorem for divisibility by $p$ in section 3 and deal with exact congruence classes in section 4.

2. Background

We define $p^*$ to be the rank of apparition of $p$ in the Fibonacci sequence. The rank of apparition is the index of the first Fibonacci number divisible by $p$.

The Fibonacci sequence exhibits a number of interesting properties, among them the divisibility property, regular divisibility by a prime, and the shifting property. The following lemmas can be found in a variety of sources, including [8].

Lemma 2.1. (Lucas [5]) For positive integers $n$ and $m$, $gcd(F_n, F_m) = F_{gcd(n,m)}$. If $n \mid m$ then $gcd(n, m) = n$, so $gcd(F_n, F_m) = F_n$, and so $F_n \mid F_m$.

Lemma 2.2. For positive integer $i$ and prime $p$, $p \mid F_{ip}$.

The periodic nature of the Fibonacci sequence modulo $p$ follows.

Lemma 2.3. For positive integers $n$ and $m$, $F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$. 


By a result of Sagan and Savage [6], the fibonomial coefficients have a combinatorial interpretation. It follows that \((\binom{n}{k})_F\) is a nonnegative integer.

The fibonomial coefficients conform to a recurrence relation analogous to the recurrence relation on binomial coefficients:

**Lemma 2.4.** For positive integers \(n\) and \(k\),
\[
\binom{n}{k}_F = F_{n-k+1} \binom{n-1}{k-1}_F + F_{k-1} \binom{n-1}{k}_F.
\]

Like the binomial coefficients, the fibonomial coefficients possess a number of useful properties, among them the negation property and the iterative property:

**Lemma 2.5.** (Gould [2]) For \(n, k \in \mathbb{Z}\),
\[
\binom{n}{n-k}_F = \binom{n}{k}_F.
\]

**Lemma 2.6.** (Gould [2]) For \(a, b, c \in \mathbb{Z}\),
\[
\binom{a}{b}_F \binom{b}{c}_F = \binom{a-c}{a-b}_F.
\]

It is a commonly known fact that the Fibonacci sequence modulo an integer is periodic. The period modulo \(p\) is called the Pisano period and is denoted \(\pi(p)\). A related notion is the Pisano semiperiod, defined as the period of the modulo \(p\) Fibonacci sequence up to a sign.

### 3. Divisibility

By a result of Harris [3], when \(p^* = p + 1, \pi(p) | 2p^\ast\), and \(p^\ast\) is the Pisano semiperiod. We rely on the notion of the semiperiod and assume for this paper that \(p\) is an odd prime and \(p^* = p + 1\).

For a nonnegative integer \(x\), \(\nu_p(x)\) denotes the \(p\)-adic valuation of \(x\), i.e. the highest power of \(p\) dividing \(x\).

We use a result of Knuth and Wilf [4], adapted for the Fibonacci sequence.

**Theorem 3.1.** (Knuth and Wilf) The highest power of an odd prime \(p\) that divides the fibonomial coefficient \((\binom{n}{k})_F\) is the number of carries that occur to the left of the radix point when \(k/p^\ast\) is added to \((n-k)/p^\ast\) in \(p\)-ary notation, plus the \(p\)-adic valuation \(\nu_p(F_{p^\ast}) = 1\) if a carry occurs across the radix point.

We require that \(p\) not be a Wall-Sun-Sun prime in order for \(\nu_p(F_{p^\ast}) = 1\) to hold.

Since we are interested in divisibility, we only require that the \(p\)-adic valuation is at least one, so it suffices to show that a carry occurs.

As in [1] and [7], we consider the base \(\mathcal{F}_{p^\ast} = (1, p^\ast, p^p p^2, \ldots)\). So \(n = n_0 + n_1 p^* + n_2 p^* p + \cdots + n_m p^* p^{m-1} = (n_0, n_1, n_2, \ldots, n_m)_{\mathcal{F}_{p^*}}\). In this base, division by \(p^\ast\) results in a number \((n_1, n_2, n_3, \ldots, n_m)_p\), with fractional part \(\frac{a}{p^\ast}\) only, which simplifies the counting of the carries.

Generalizing Southwick’s proof in [7], we prove the following:

**Theorem 3.2.** Given integers \(n, k > 0\),
\[
p | \binom{n}{k}_F \iff p | \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F.
\]

**Proof.** By Theorem 3.1, \(p | \binom{n}{k}_F\) if and only if a carry occurs in the addition of \((\frac{a}{p^*})\) and \((\frac{n-k}{p^*})\) in base \(p\). The first carry occurs either across the radix point or to the left of the radix point. Let \(q = n-k = (q_0, q_1, \ldots, q_m)_{\mathcal{F}_{p^*}}\).
(1) First consider the conditions necessary for the carry across the radix point.

If \( n_0 \geq k_0 \), then \( q_0 = n_0 - k_0 \), \( k_0 + (q_0) = n_0 < p^* \). In this case, there will be no carry.

Alternatively, if \( k_0 > n_0 \), then a borrow occurs, so \( q_0 = n_0 - k_0 + p^* \). The addition of \( \frac{k_0}{p^*} \) and \( \frac{q_0}{p^*} \) produces:

\[
\frac{k_0 + n_0 - k_0 + p^*}{p^*} = \frac{n_0 + p^*}{p^*} \geq 1.
\]

Thus a carry occurs across the radix point if and only if \( k_0 > n_0 \).

(2) If a carry across the radix point does not occur, then let the first carry occur in the \((j + 1)^{th}\) digit, that is, in the addition of \( k_j \) with \( q_j \) (note that the \((j + 1)^{st}\) digit of \( n \) in base \( F_{p^*} \) is \( n_j p^* p^{j-1} \)). The division by \( p^* \) moves the digits to the right by one, so the carry occurs at the \( j^{th} \) digit in base \( p \).

If \( n_j \geq k_j \), then \( k_j + q_j = k_j + (n_j - k_j) < p \) since we assume there was no previous carry.

If \( k_j > n_j \), the subtraction \( n_j - k_j \) results in a borrow, so \( q_j = n_j - k_j + p \), and so

\[
k_j + q_j = k_j + (n_j - k_j + p) = n_j + p \geq p.
\]

This case is the only one in which a carry occurs.

Therefore if a carry occurs in the \( j^{th} \) position, then \( n_j < k_j \), and so \( p \mid (\binom{n}{k})_F \) since \( (\binom{n}{k}) = 0 \), and so \( p \mid (\binom{n}{k})_F (\binom{n-1}{k-1})_F \cdots (\binom{n-m}{k-m})_F \). Using the above result and Theorem 3.1, we conclude that if \( p \mid (\binom{n}{k})_F \) then \( p \mid (\binom{n}{k})_F (\binom{n-1}{k-1})_F \cdots (\binom{n-m}{k-m})_F \).

For the reverse direction we note that all these steps and Theorem 3.1 are reversible.

Note that for \( n < k \) the statement follows trivially.

Therefore \( p \mid (\binom{n}{k})_F \iff p \mid (\binom{n}{k})_F (\binom{n-1}{k-1})_F \cdots (\binom{n-m}{k-m})_F \). \( \Box \)

**Corollary 3.3.** Given \( 0 \leq m \) and \( 0 \leq n, k < p^* p^m \), \( \forall i, j \in \mathbb{Z} \) such that \( 0 \leq j < i < p \),

\[
p \mid \left( \binom{n + ip^* p^m}{k + jp^* p^m} \right)_F \iff p \mid \left( \binom{n}{k} \right)_F.
\]

**Proof.** By Theorem 3.2, since \( p \mid \left( \binom{1}{j} \right)_F \),

\[
p \mid \left( \binom{n}{k} \right)_F \iff p \mid (\binom{n}{k})_F (\binom{n-1}{k-1})_F \cdots (\binom{n-m}{k-m})_F \iff p \mid (\binom{n}{k})_F (\binom{n-1}{k-1})_F \cdots (\binom{n-m}{k-m})_F (\binom{1}{j})_F \iff p \mid (\binom{n+ip^* p^m}{k+jp^* p^m})_F. \qquad \Box
\]

### 4. Exact Nonzero Congruence Classes

We begin with a number of necessary Lemmas.

**Lemma 4.1.** If \( a \equiv b \pmod{c} \) and \( a \equiv b' \pmod{c} \), then \( a \equiv b' \pmod{c} \) and \( b' \equiv a' \pmod{c} \).

**Proof.** Since \( c \neq 0 \), we can multiply the fraction \( \frac{a}{c} \) by \( \frac{1}{c} \). Since the resulting fraction is an integer, it can be reduced modulo \( p \) to \( a'(b')^{-1} \). Note that \((b')^{-1}\) exists because \( p \) is prime and \( b' \neq 0 \). \( \Box \)

**Lemma 4.2.** For \( 0 \leq n < p^* p^m \), \( F_{n+p^* p^m} \equiv_p -F_n \).

**Proof.** Since \( p^* = \frac{1}{2} \pi(p) \) is the semiperiod, \( F_{n+p^*} \equiv_p -F_n \).

Then since \((p^m - 1)\) is even and \( (\pi(p) = 2p^* \), \( F_{n+p^*} \equiv_p -F_n \) implies \( F_{n-p^* p^m} \equiv_p -F_n \). \( \Box \)

**Lemma 4.3.** For \( i > 0 \),

\[
\frac{F_{i p^* p^m}}{F_{p^* p^m}} \equiv_p i(-1)^{i-1}.
\]

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Proof. We prove this by induction.

First, let $i = 1$. Then the statement follows trivially.

Now, assume the inductive hypothesis:

\[
\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p (-1)^{i-1}. 
\]

Consider

\[
\frac{F_{(1+i)p^*p^m}}{F_{p^*p^m}} = \frac{F_{ip^*p^m+1} + F_{ip^*p^m-1}}{F_{p^*p^m}}. 
\]

We apply the shifting property of the Fibonacci sequence to obtain:

\[
\frac{F_{ip^*p^m+1} + F_{p^*p^m-1}(-1)^{i-1}(i) \equiv_p (-1)^i + (-1)^i(i) \equiv_p (-1)^{(i+1)-1}(i+1).}
\]

□

Lemma 4.4. For $i > 0$,

\[
\binom{ip^*p^m}{p^*p^m} \equiv_p i. 
\]

Proof. By definition of the fibonomial coefficient,

\[
\binom{ip^*p^m}{p^*p^m} \equiv_p \frac{F_{ip^*p^m} F_{ip^*p^m-1} \cdots F_{(i-1)p^*p^m} F_{ip^*p^m-1} F_{ip^*p^m-1} \cdots F_1}{(F_{(i-1)p^*p^m} F_{(i-1)p^*p^m-1} \cdots F_1) F_{p^*p^m} F_{p^*p^m-1} \cdots F_1}. 
\]

Cancelling like terms gives

\[
\frac{F_{ip^*p^m} F_{ip^*p^m-1} \cdots F_{ip^*p^m-(p^*p^m-1)}}{F_{p^*p^m} F_{p^*p^m-1} \cdots F_1}. 
\]

The terms in the above expression take three forms, which we represent separately for clarity. Note that all reduction modulo $p$ happens term-wise, and thus the result is an integer.

1. We first consider terms of the form $F_{ip^*p^m-a}$, where $p^* \nmid a$. Write

\[
\prod_{a=1}^{p^*p^m-1} \frac{F_{ip^*p^m-a}}{F_{p^*p^m-a}}. 
\]

We apply Lemma 4.2 to the top $p^*p^m-1$ many times so that we can cancel the top and bottom, resulting in $(-1)^{(i-1)(p^*p^m-1)} \equiv_p (-1)^{i-1}$, because $p^*p^m-1$ is odd.

2. Next we consider terms of the form $F_{(ip^m-a)p^*}$:

\[
\prod_{a=1}^{p^m-1} \frac{F_{(ip^m-a)p^*}}{F_{(p^m-a)p^*}}. 
\]
By Lemma 4.1 and Lemma 4.3,
\[
\left( \prod_{a=1}^{p^m-1} \frac{F_{(ip^m-a)p^m}}{F_{p^m-a}} \right) \left( \frac{1}{p} \prod_{a=1}^{p^m-1} \frac{1}{p} \right) \equiv_p \prod_{a=1}^{p^m-1} \frac{-(1)^{i(p^m-a-1)(ip^m-a)}(p^m-a)}{-(a)} \equiv_p \prod_{a=1}^{p^m-1} \frac{-(1)^{(i-1)p^m(-a)}}{(-a)}.
\]

Note that in the modular group we use division notation to represent multiplication by an inverse.

Since \(p^m\) is odd and \(p^m - 1\) is even,
\[
\prod_{a=1}^{p^m-1} \frac{-(1)^{(i-1)p^m(-a)}}{(-a)} \equiv_{p} (-1)^{(i-1)(p^m-1)} \equiv_{p} 1.
\]

(3) The only remaining term is the quotient \(\frac{F_{p^m}}{F_{p^m}} \equiv_{p} (-1)^{i(1)}\), by Lemma 4.3.

From the three cases above,
\[
\left( \frac{ip^m p^m}{p^* p^m} \right)_F \equiv_{p} i(-1)^{2(i-1)} \equiv_{p} i
\]

\[\square\]

**Lemma 4.5.** For \(0 \leq i, j < p\),
\[
\left( \frac{ip^* p^m}{jp^* p^m} \right)_F \equiv_{p} \binom{i}{j}.
\]

**Proof.** We prove this using induction. For a base case, let \(i = 0\). Then if \(j = 0\),
\[
\binom{0}{0} \equiv_{p} 1 \equiv_{p} \binom{0}{0}.
\]

If \(j > 0\), then
\[
\binom{0}{j} \equiv_{p} 0 \equiv_{p} \binom{0}{j}.
\]

Now assume \(\left( \frac{ip^* p^m}{jp^* p^m} \right)_F \equiv_{p} \binom{i}{j}\). Then we apply Lemmas 2.5 and 2.6.

We let \(a = (i + 1)p^* p^m\), \(b = (i + 1 - j)p^* p^m\), and \(c = p^* p^m\), thus yielding the following:
\[
\left( \frac{(i+1)p^* p^m}{jp^* p^m} \right)_F \left( \frac{(i+1-j)p^* p^m}{p^* p^m} \right)_F \equiv \left( \frac{(i+1)p^* p^m}{p^* p^m} \right)_F \left( \frac{ip^* p^m}{jp^* p^m} \right)_F
\]

Applying the induction hypothesis and Lemma 4.4 gives
\[
\left( \frac{(i+1)p^* p^m}{jp^* p^m} \right)_F (i+1-j) \equiv_{p} (i+1) \binom{j}{i}.
\]

We then multiply both sides by \((i+1-j)^{-1}\) to obtain
\[
\left( \frac{(i+1)p^* p^m}{jp^* p^m} \right)_F \equiv_{p} \binom{i+1}{i+1-j} \binom{i}{j}.
\]

Equivalently,
\[
\left( \frac{(i+1)p^* p^m}{jp^* p^m} \right)_F \equiv_{p} \binom{i+1}{j},
\]

as desired.

\[\square\]
We now proceed with our main theorem for the exact congruence classes of the fibonomial triangle modulo \( p \). For an visual representation of the relation, see Figure 1.

**Theorem 4.6.** For \( 0 \leq n, k < p^*p^m \), \( 0 \leq i, j < p \), \( 0 \leq m \),

\[
\binom{n + ip^*p^m}{k + jp^*p^m} \equiv_p (-1)^{ik - nj} \binom{i}{j} \binom{n}{k}.
\]

**Proof.** We proceed by induction.

First let \( n = k = 0 \). Then the statement follows directly from Lemma 4.5.

When \( n = 0 \), \( k > 0 \), by Theorem 3.3, since \( p|\binom{0}{k}_F \),

\[
\binom{ip^*p^m}{k + jp^*p^m} \equiv_p 0 \equiv_p (-1)^{ik - 0j} \binom{i}{j} \binom{0}{k}.
\]

Let \( k, n > 0 \). We assume

\[
\binom{n - 1 + ip^*p^m}{k + jp^*p^m} \equiv_p (-1)^{ik - (n-1)j} \binom{i}{j} \binom{n - 1}{k}.
\]

for all \( k \).

Using the recurrence relation for fibonomial coefficients,
\[(n + ip^*p^m)_{F_{k+jp^*p^m}} \equiv_p F_{n+(i-j)p^*p^m-k+1} F_{k-1+jp^*p^m-k+1} \]
\[\equiv_p (-1)^{i+j} F_{n-k+1}(-1)^{(i-1)-(n-1)} F_{(n-1)} \]
\[\equiv_p (-1)^{ik-nj} \left( F_{n-k+1} F_{(n-1)} \right) \]
\[\equiv_p (-1)^{ik-nj} \left( \binom{n}{k} \right) \]

This completes the proof. □

5. Further Directions

In Theorem 4.6, note that

\[ik-nj = \det \left( \begin{array}{cc} i & n \\ j & k \end{array} \right).\]

This may be a coincidence but, alternatively, it might indicate the existence of a more general relation for different types of primes.

Theorem 4.6 can be generalized to other primes by proving variants of the prerequisite lemmas. For example, in the case \(p = 5\), \(5^* = 5\), and \(F_{n+5} \equiv 3F_n [7]\).

However, for some primes, problems arise. In the case \(p = 11\), \(p^* = 10\). In this case, one would need a base other than base \(F_{p^*}\), because the divisibility theorem cannot be proven in base \(F_{p^*}\) [7]. The form of such a base remains to be investigated.

References