Numerical Approximation of Incompressible Fluids

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Open Problem (I)

Equation: $u_t + D\mathcal{W}(u) = f$

Pivot Space: $U \hookrightarrow\!\!\!\rightarrow H \hookrightarrow U'$ (e.g. $H^1(\Omega) \hookrightarrow\!\!\!\rightarrow L^2(\Omega)$)

Energy: $\mathcal{W} : U \rightarrow \mathbb{R}$ coercive but not convex.

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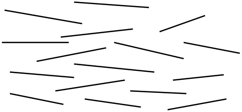
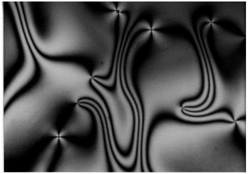
Example: Oseen Frank elastic energy for liquid crystals.



Nematic $|u| \simeq 1$



$$\mathcal{W}(u, \nabla u) = (1/2)\left(k_1\, div(u)^2 + k_2\,(u.curl(u) + q)^2 + k_3\,|u \times curl(u)|^2\right)$$

Notation: $\nabla u = [\partial u_i / \partial x_j] \in \mathbb{R}^{d \times d}$.

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Estimate: $u_t \in L^2[0, T; H]$ and $u \in L^\infty[0, T; U]$

$$\int_0^t \|u_t\|_H^2 + \mathcal{W}(u(t)) = \mathcal{W}(u(0)) + \int_0^t (f, u_t)_H$$

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Compactness:

- Solutions are compact in $L^p[0 T; H]$ for all $1 \leq p < \infty$.
- Solutions are compact in $L^p[0, T; U]$ when $\{u \in U \mid D\mathcal{W}(u) \in H\} \hookrightarrow\!\!\!\rightarrow U \hookrightarrow H$

$$D\mathcal{W}(u) = f - u_t \in L^2[0, T; H]$$

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Problem: Construct a time stepping scheme which inherits both stability and compactness in $L^p[0, T; U]$.

Open Problem (II)

Harmonic Mapping: $u : \Omega \to \mathbb{S}^{d-1}$

$$I(u) = (1/2) \int_{\Omega} |\nabla u|^2 - f.u$$

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$$\int_\Omega (\nabla u) : (\nabla v) - pu.v = \int_\Omega f.v \qquad v \in U$$

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Numerical Approximation: $0 < \epsilon << 1$

$$\int_\Omega (\nabla u) : (\nabla v) + (1/\epsilon)(|u|^2 - 1)u.v = \int_\Omega f.v$$

Stationary Stokes' Equations

Strong Form: $-\Delta u + \nabla p = f$ and $div(u) = 0$

Weak Statement: $(u, p) \in H_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$

$$\int_\Omega (\nabla u) : (\nabla v) - p\ div(u) = \int_\Omega f.v \qquad v \in H_0^1(\Omega)$$

$$\int_\Omega q\, div(v) = 0 \qquad q \in L^2(\Omega)/\mathbb{R}$$

Bilinear Forms:

$$a(u, v) = \int_\Omega (\nabla u) : (\nabla v) \quad \text{and} \quad b(p, v) = \int_\Omega p\, div(v)$$

Babuska–Brezzi Theory

Saddle Point: $a : U \times U \to \mathbb{R}$ and $b : P \times U \to \mathbb{R}$ are continuous

$$\begin{aligned} a(u,v) - b(p,v) &= f(v) & \qquad v \in U \\ b(q,u) &= g(q) & \qquad q \in P \end{aligned}$$

Kernel: $Z = \{u \in U \mid b(q,u) = 0,\ q \in P\}$

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Lax–Milgram Theorem: $\|(u,p)\|_{U\times P} \le C\|(f,g)\|_{U'\times P'}$ iff

- $a : U \times U \to \mathbb{R}$ is coercive on $Z \times Z$

$$\sup_{v\in Z} \frac{a(u,v)}{\|v\|_U} \ge c\|u\|_U,\ u \in Z \quad \text{and} \quad \sup_{u\in Z} a(u,v) > 0,\ v \in Z \setminus \{0\}$$

- $b : P \times U \to \mathbb{R}$ satisfies the inf–sup condition

$$\sup_{v\in U} \frac{b(p,v)}{\|v\|_U} \ge c\|p\|_P$$

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Kernel: $Z = \{u \in U \mid b(q, u) = 0, \ q \in P\}$

Classical Situation: $\|(u, p)\|_{U \times P} \leq C\|(f, g)\|_{U' \times P'}$ if

- $a : U \times U \to \mathbb{R}$ is coercive

$$a(u, u) \geq c\|u\|_U^2, \qquad u \in U$$

- $b : P \times U \to \mathbb{R}$ satisfies the inf–sup condition

$$\sup_{v \in U} \frac{b(p, v)}{\|v\|_U} \geq c\|p\|_P$$

Evolutionary Stokes' Equations

Strong Form: $u_t - \Delta u + \nabla p = f$ and $div(u) = 0$

Problem:

$$u_t + \nabla p = f + \Delta u \in L^2[0, T; H^{-1}(\Omega)] \quad \not\Rightarrow \quad u_t \in L^2[0, T; H^{-1}(\Omega)]$$

Evolutionary Stokes' Equations

Strong Form: $u_t - \Delta u + \nabla p = f$ and $div(u) = 0$

Weak Statement:

$$\int_0^T (u_t, v) + (\nabla u, \nabla v) - (p, div(u)) = \int_0^T f(v)$$

$$\int_0^T (q, div(u)) = 0$$

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Saddle Point:

$$\begin{aligned} A(u, v) - B(p, v) &= F(v) \\ B(q, u) &= G(q) \end{aligned}$$

Bilinear Forms:

$$A(u, v) = \int_0^T (u_t, v) + (\nabla u, \nabla v) \quad \text{and} \quad B(q, u) = \int_0^T (q, div(u))$$

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Continuity: $A : \mathbb{U} \times \mathbb{V} \to \mathbb{R}$

$$\mathbb{U} = L^2[0, T; U] \cap H^1[0, T; U'] \quad \text{and} \quad \mathbb{V} = L^2[0, T; U]$$

$$A(u, v) = \int_0^T (u_t, v) + a(u, v)$$

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$$A(u, v) = \int_0^T (u_t, v) + a(u, v)$$

$$B_1(q, u) = \int_0^T b(q, u)$$
$$B_2(p, v) = \int_0^T b(p, v)$$

Babuska–Brezzi Theory

Saddle Point: $(u, p) \in \mathbb{U} \times \mathbb{P}$

$$\begin{aligned} A(u, v) - B_2(p, v) &= F(v) \qquad && v \in \mathbb{V} \\ B_1(q, u) &= G(q) \qquad && q \in \mathbb{Q} \end{aligned}$$

Kernels:

$$\begin{aligned} \mathbb{Z}_1 &= \{u \in \mathbb{U} \mid B_1(q, u) = 0, \;\; q \in \mathbb{Q}\} \\ \mathbb{Z}_2 &= \{v \in \mathbb{V} \mid B_2(p, v) = 0, \;\; p \in \mathbb{P}\} \end{aligned}$$

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Lax–Milgram Theorem: $\|(u,p)\|_{\mathbb{U}\times\mathbb{P}} \le C\|(F,G)\|_{\mathbb{V}'\times\mathbb{Q}'}$ iff

- $a : \mathbb{U} \times \mathbb{V} \to \mathbb{R}$ is coercive over $\mathbb{Z}_1 \times \mathbb{Z}_2$

$$\sup_{v\in\mathbb{Z}_2} \frac{A(u,v)}{\|v\|_{\mathbb{V}}} \ge c\|u\|_{\mathbb{U}},\ u \in \mathbb{Z}_1 \quad \text{and} \quad \sup_{u\in\mathbb{Z}_1} a(u,v) > 0,\ v \in \mathbb{Z}_2 \setminus \{0\}$$

- $B_1 : \mathbb{Q} \times \mathbb{U} \to \mathbb{R}$ and $B_2 : \mathbb{P} \times \mathbb{V} \to \mathbb{R}$ satisfy the inf–sup conditions

$$\sup_{u\in\mathbb{U}} \frac{B_1(q,u)}{\|v\|_{\mathbb{U}}} \ge c\|q\|_{\mathbb{Q}} \quad \text{and} \quad \sup_{v\in\mathbb{V}} \frac{B_2(p,v)}{\|v\|_{\mathbb{V}}} \ge c\|p\|_{\mathbb{P}}$$

Babuska–Brezzi Theory

$$\mathbb{U} = L^2[0, T; U] \cap H^1[0, T; U'] \quad \text{and} \quad \mathbb{V} = L^2[0, T; U]$$

(1) If $\mathbb{P} = L^2[0, T; P]$ then $B_2(p, v)$ inherits the inf-sup condition from $b(p, v)$ and $\mathbb{Z}_2 = L^2[0, T; Z]$

$$\sup_{v \in U} \frac{b(p, v)}{\|v\|_U} \geq c\|p\|_P \quad \Leftrightarrow \quad \sup_{v \in \mathbb{V}} \frac{B_2(p, v)}{\|v\|_{\mathbb{V}}} \geq c\|p\|_{\mathbb{P}}$$

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(2) The inf–sup condition for $B_1(q, u)$ determines $\|.\|_{\mathbb{Q}}$

$$\|q\|_{\mathbb{Q}} \simeq \sup_{u \in \mathbb{U}} \frac{\int_0^T (q, div(u))}{\|u\|_{L^2[0,T;U]} + \|u_t\|_{L^2[0,T;U']}}, \qquad \mathbb{Z}_1 = ?$$

Example: $q(t, x) = q_0(x)\delta(t - t_0) \in \mathbb{Q}$ if $q_0 \in H^1(\Omega)/\mathbb{R}$

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(3) Let $\mathbb{U}_0 = \{u \in \mathbb{U} \mid u(0) = 0\}$ then

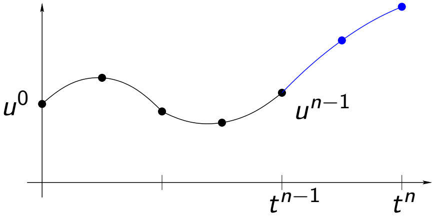
- A is coercive on $\mathbb{U}_0 \times \mathbb{V}$

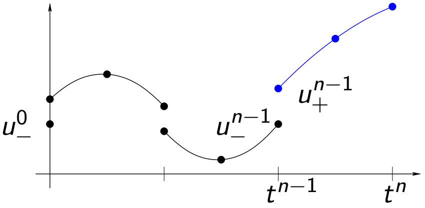
$$\sup_{v \in \mathbb{V}} \frac{A(u, v)}{\|v\|_{\mathbb{V}}} \geq c\|u\|_{\mathbb{U}} \;\; u \in \mathbb{U}_0 \quad \text{and} \quad \sup_{u \in \mathbb{U}_0} A(u, v) > 0, \;\; v \in \mathbb{V}\backslash\{0\}$$

- Except for special cases (eg. $U = H^1_{\#}(\Omega)$) coercivity on $\mathbb{Z}_1 \times \mathbb{Z}_2$ is open.

Time Stepping Schemes

Time Partition: $0 = t^0 < t^1 < \ldots < t^N = T$





CG Time Stepping Scheme DG Time Stepping Scheme

Strong Form: $u_t + Au = F$

Weak Statement: $\displaystyle\int_0^T (u_t, v) + a(u, v) = \int_0^T F(v)$

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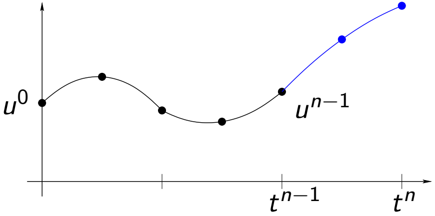

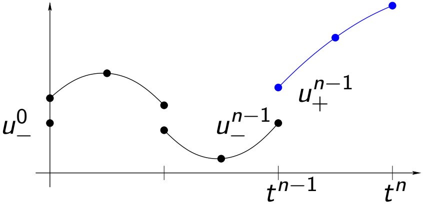

CG Time Stepping Scheme DG Time Stepping Scheme

CG Scheme: $u_h \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h]$ satisfies $u_h(t^{n-1}_+) = u_h(t^{n-1}_-)$ and

$$\int_{t^{n-1}}^{t^n} \Big((u_{ht}, v_h)_H + a(u_h, v_h) \Big) = \int_{t^{n-1}}^{t^n} \langle F, v_h \rangle$$

for all $v_h \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]$

Stability: Set $v_h = u_{ht} \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]$ ...

Time Stepping Schemes

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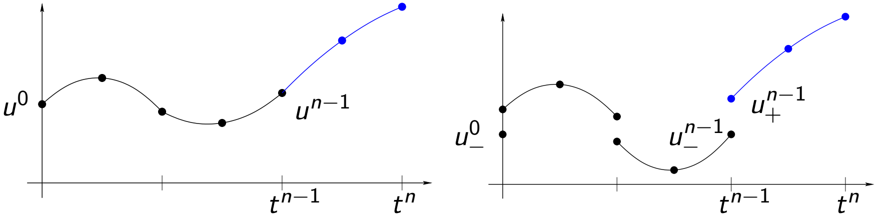

CG Time Stepping Scheme   DG Time Stepping Scheme

DG Scheme: $u_h \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h]$

$$\int_{t^{n-1}}^{t^n} \Big((u_{ht}, v_h)_H + a(u_h, v_h) \Big) + ([u^{n-1}], v_+^{n-1})_H = \int_{t^{n-1}}^{t^n} \langle F, v_h \rangle$$

for all $v_h \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h]$

Notation: Jump term $[u^m] = u_+^m - u_-^m$

Parabolic Problem

Discrete Problem: $U \hookrightarrow H \hookrightarrow U'$ and $u^0_- \in H$

$$A(u,v) = \sum_{n=1}^{N} \int_{t^{n-1}}^{t^n} \Big((u_t, v)_H + a(u,v) \Big) + ([u], v_+)_H^{n-1}$$

Discrete Space: $\mathbb{U}^\ell = \{ u \in L^2[0,T;U] \mid u_h|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1},t^n;U] \}$

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Discrete Norm: $\|u\|_h^2 = \|u\|_{L^2[0,T;U]}^2 + \|D_t u\|_{(\mathbb{U}^\ell)'}^2$

$$\|D_t u\|_{(\mathbb{U}^\ell)'} \equiv \sup_{v \in \mathbb{U}^\ell} \frac{\sum_{n=1}^{N} \int_{t^{n-1}}^{t^n} (u_t, v)_H + ([u], v_+)_H^{n-1}}{\|v\|_{L^2[0,T;U]}}$$

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Discrete Norm: $\|u\|_h^2 = \|u\|^2_{L^2[0,T;U]} + \|D_t u\|^2_{(\mathbb{U}^\ell)'}$

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Lemma: *If* $u \in \mathbb{U}^\ell$ *then* $\|u\|^2_{L^\infty[0,T;H]} \le \|u^0_-\|^2_H + C_\ell \|u\|^2_h$.

cf: $L^2[0,T;U] \cap H^1[0,T;U'] \hookrightarrow C[0,T;H]$

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Discrete Problem: $U \hookrightarrow H \hookrightarrow U'$ and $u_-^0 \in H$

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Discrete Norm: $\|u\|_h^2 = \|u\|_{L^2[0,T;U]}^2 + \|D_t u\|_{(\mathbb{U}^\ell)'}^2$

Theorem: *If* u, $v \in \mathbb{U}^\ell$ *then* $A(u, v) \le C \|u\|_h \|v\|_{L^2[0,T;U]}$ *when* $u_-^0 = 0$ *and*

$$\sup_{v \in \mathbb{U}^\ell} \frac{A(u, v)}{\|v\|_{L^2[0,T;U]}} \ge c \|u\|_h \quad \textit{and} \quad \sup_{u \in \mathbb{U}^\ell} A(u, v) > 0, \ \ v \ne 0.$$

Thus $A(u, v) = F(v)$ *has a unique solution for all* $F \in (\mathbb{U}^\ell)'$ *and* $\|u\|_h \le (1/c) \|F\|_{(\mathbb{U}^\ell)'}$.

Example: Ericksen Leslie Equations

Linear Momentum Equation: $div(v) = 0$

$$\rho \dot{v} - div\left(-pI + \mu D(v) + T_v - (\nabla n)\left[\frac{\partial \mathcal{W}}{\partial \nabla n}\right]\right) = \rho f$$

Angular Momentum Equation: $|n| = 1$

$$g_v + \theta n + \left(\frac{\partial \mathcal{W}}{\partial n}\right) - div\left[\frac{\partial \mathcal{W}}{\partial \nabla n}\right] = \rho g$$

Notation: $\dot{n} = n_t + (v.\nabla)n$ and $D(v) = (1/2)(\nabla v + (\nabla v)^T)$

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Dissipation (Leslie): Most general frame indifferent expression linear in the gradients ... (six coefficients)

$$\begin{aligned} T_v &= \hat{\gamma}_1(\mathring{n} \otimes n)_{sym} \quad - \quad \gamma_1(\mathring{n} \otimes n)_{skew} \quad + \dots \\ g_v &= \qquad\qquad\qquad\qquad\quad \gamma_1 \mathring{n} \qquad\qquad\qquad + \dots \end{aligned}$$

Notation: $\mathring{n} = \dot{n} - W(v)n$ and $W(v) = (1/2)(\nabla v - (\nabla v)^T)$

Ericksen Leslie Equations

Linear Momentum: $div(v) = 0$

$$\int_\Omega \rho \dot{v}.w - (p + \mathcal{W})\, div(w) + \mu D(v) : D(w)$$

$$+ \gamma_1 (\mathring{n} \otimes n) : W(w) + (\gamma_1 \mathring{n} - \rho g)\,.(\nabla n) w = \int_\Omega \rho f.w$$

Penalized Equation: $\mathcal{W}_\epsilon(n, \nabla n) = (1/\epsilon)(|n|^2 - 1)^2 + \mathcal{W}(n, \nabla n)$

$$\int_\Omega \gamma_1 \mathring{n}.m + \left(\frac{\partial \mathcal{W}_\epsilon}{\partial n}\right).m + \left[\frac{\partial \mathcal{W}_\epsilon}{\partial \nabla n}\right] : \nabla m = \int_\Omega \rho g.m$$

Ericksen Leslie Equations

Linear Momentum: $div(v) = 0$

$$\int_\Omega \rho \dot{v}.w - (p + \mathcal{W})\, div(w) + \mu D(v) : D(w)$$

$$+ \gamma_1(\mathring{n} \otimes n) : W(w) + (\gamma_1 \mathring{n} - \rho g)\,.(\nabla n) w = \int_\Omega \rho f.w$$

Penalized Equation: $\mathcal{W}_\epsilon(n, \nabla n) = (1/\epsilon)(|n|^2 - 1)^2 + \mathcal{W}(n, \nabla n)$

$$\int_\Omega \gamma_1 \mathring{n}.m + \left(\frac{\partial \mathcal{W}_\epsilon}{\partial n}\right).m + \left[\frac{\partial \mathcal{W}_\epsilon}{\partial \nabla n}\right] : \nabla m = \int_\Omega \rho g.m$$

Energy Estimate:

- Set $w = v$ into the linear momentum equation
- Set $m = n_t$ into the angular momentum equation

Ericksen Leslie Equations

Linear Momentum: $div(v) = 0$

$$\int_\Omega \rho\dot{v}.w - (p + \mathcal{W})\, div(w) + \mu D(v) : D(w)$$

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Theorem: (njw 2011) *If $\mathcal{W}(\nabla n)$ is quadratic then numerical approximations of (v, n) computed using*

- *DG approximations of the linear momentum equation*
- *CG approximations of the angular momentum equation*

with $\ell = 1$ converge.