# On the well-posedness of the Prandtl boundary layer equation

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# The Euler and Navier-Stokes equations

$$\partial_t u^{NS} + u^{NS} \cdot \nabla u^{NS} + \nabla p^{NS} = \nu \Delta u^{NS}$$
$$\nabla \cdot u^{NS} = 0, \qquad \gamma u^{NS} = 0$$

$$\partial_t u^{E} + u^{E} \cdot \nabla u^{E} + \nabla p^{E} = 0$$
$$\nabla \cdot u^{E} = 0, \qquad u^{E} \cdot n = 0$$

- $(x, y) = (x_1, \ldots, x_{d-1}, y) \in \Omega \subset \mathbb{R}^d$ is smooth ($d = 2, 3$)
- $(u, w) = (u_1, \ldots, u_{d-1}, w)$ the velocity, $p$ the pressure
- $\gamma$ is the trace onto $\partial\Omega$
- $n$ is the outward unit normal to $\Omega$
- $\nu$ is the kinematic viscosity
- Q: does $u^{NS} \to u^E$ as $\nu \to 0$ ? in what space ? rates ?

# A few results on the inviscid limit

In the absence of boundaries ($\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d$)

- inviscid limit holds in $L^2$: Kato ('72), Swann ('71)
- rates of convergence which are optimal $\mathcal{O}(\sqrt{\nu})$, and attained for smooth vortex patch initial data: Constantin and Wu ('95)

In the presence of boundaries (and Dirichelt B.C.)

- Kato ('84) shows that if

$$\lim_{\nu \to 0} \nu \int_{d(y,\partial\Omega)<\nu} \|\nabla u^{NS}\|_{L^2}^2 dy = 0$$

  then the inviscid limit holds (in $L^2$)
- improved by Temam and Wang ('98), to only require the tangential part of the velocity gradient
- inviscid limit holds if $-\nu\Delta$ is replaced by anisotropic viscosity $\nu_1 \partial_{yy} - \nu_2 \partial_{xx}$, with $\nu_1/\nu_2 \to 0$: Masmoudi ('98)

Further works by: Bardos, Beirao da Veiga, Crispo, Lopes Filho, Mazzucato, Nussenzveig Lopes, Taylor, Kelliher, etc.....

# Asymptotic Expansions in the inviscid limit

- Prandtl (1904) lays foundations of boundary layer theory
- in the boundary layer (BL) the inertial and viscous forces should be comparable
- in the BL, $\gamma u$ (the tangential component of the velocity), has to jump from 0 (as prescribed by Navier-Stokes), to $\gamma u^E$ (as prescribed by Euler)
- in the BL, the viscous term $\nu \partial_{yy} u$ should be $\mathcal{O}(1)$, so that the thickness of the BL should be $\varepsilon = \sqrt{\nu}$
- hence, for $\nu \ll 1$, it is natural to consider the asymptotic expansion

  $$u^{NS} = u^{NS,0} + \varepsilon u^{NS,1} + \varepsilon^2 u^{NS,2} + \dots$$

  where as before $\varepsilon = \sqrt{\nu}$
- the idea is that $u^{NS,0}$ is approximatively $u^E$ outside of the BL
- inside the BL: let $Y = y/\varepsilon$, so that $\partial_y w = \partial_Y w/\epsilon$; hence $w$ is $\mathcal{O}(\epsilon)$
- hence, for the solution inside the BL, Prandtl makes the ansatz:

  $$u^{NS,0}(x,y) = (u^P(x,Y), \varepsilon w^P(x,Y))$$

# The Prandtl boundary layer equations

- Plug in the velocity $(u(x, y/\varepsilon), \varepsilon w(x, y/\varepsilon))$ in the Navier-Stokes equations and formally send $\varepsilon$ to 0.
- To avoid issues due to the curvature of the boundary, let $\Omega = \mathbb{H} = \{(x, Y) \in \mathbb{R}^d \colon \; Y > 0\}$ be the upper half-plane/space.
- In the limit we obtain the Prandtl boundary layer equations:

$$\partial_t u^P - \partial_{YY} u^P + u^P \cdot \nabla_x u^P + w^P \partial_Y u^P + \nabla_x p^P = 0$$
$$\partial_Y p^P = 0$$
$$\nabla_x \cdot u^P + \partial_Y w^P = 0$$

- Boundary conditions

$$\lim_{Y \to \infty} u^P = \gamma u^E$$
$$\lim_{Y \to \infty} p^P = \gamma p^E$$
$$\gamma u^P = \gamma w^P = 0$$

- study the IVP: $u^P(x, Y, 0) = u_0^P(x, Y)$

# Mathematical issues for the Prandtl equations

Well-posedness in suitable functional spaces:

- Monotonic data in $y$: Oleinik ('66)
- Analytic data: Caflisch and Sammartino ('98 - Part I) – requires analyticity w.r.t. both $x$ and $y$; improved by Cannone, Lombardo, and Sammartino ('03) to require analyticity w.r.t. only $x$
- Weak solutions for pressure of fixed sign: Xin and Zhang ('04)

Ill-posedness and blow-up:

- Sobolev data ill-posedness: Grenier ('00), Gerard-Varet and Dormy ('09)
- Sobolev data blow-up in $W_Y^{1,\infty}$: E and Engquist ('97)

Justify the formal derivation of the Prandtl equations in the inviscid limit, i.e. prove that

$$u^{NS} = u^E(1 - \chi_{BL}) + u^P \chi_{BL} + \mathcal{O}(\varepsilon)$$

- Sammartino and Caflisch ('98 - Part II)

# Our motivation

- remove the need for exponential matching at the top of the BL
- indeed, there is no physical justification for exponential matching (mathematical artifact?)
- in fact, the quantity that measures the effect of the flow inside of the BL on the underlying Euler flow outside of the BL is the so-called displacement thickness cf. Batchelor (99'), which is defined as

$$\delta_1(x) = \int_0^\infty \left(1 - \frac{u^P(x, Y)}{U(x)}\right) dY.$$

- it seems that any integrable algebraic matching is sufficient to at least define the displacement thickness

# Re-write the Prandtl equations

- For notational convenience, let $U = \gamma u^E$
- Homogenize B.C. at $Y = \infty$, and get rid of pressure, by using

$$\partial_t U + U \partial_x U + \gamma \partial_x p^E = 0,$$

and the variable change

$$v = u^p - U$$

- The Prandtl evolution for the prognostic variable become

$$(\partial_t - \partial_{YY} + Y \partial_x U \partial_Y)\, v + v \partial_x v - \partial_Y v \int_0^Y \partial_x v + U \partial_x v + v \partial_x U = 0$$

# A dynamic change of coordinates

- For simplicity of exposition set $d = 2$. Recall: $\gamma u^E(x,t) = U(x,t)$.
- Define $A(x,t)$ to be the unique real-analytic solution of the IVP

$$\partial_t A(x,t) + U(x,t)\partial_x A(x,t) = A(x,t)\partial_x U(x,t)$$
$$A(x,t)|_{t=0} = 1$$

  on $\mathbb{R} \times [0,T]$, for some $T > 0$.
- For $\theta > 1$, let $\Phi(y) = \int_0^y \phi(\zeta)d\zeta$, where

$$\phi(y) = \langle y \rangle^{-\theta} = (1 + y^2)^{-\theta/2}$$

- Change of variables: the "new" vertical variable $y$ and velocity $v$

$$y = Y\,A(x,t)$$
$$v(x,y,t) = u^P(x,Y,t) - (1 - \phi(y))U(x,t)$$

# The dynamically reformulated Prandtl equations

- Under this change of variables, the Prandtl system reads

$$\partial_t v - A^2\partial_{yy} v + N(v) + L(v) = F$$
$$N(v) = v\partial_x v - \partial_x W(v)\partial_y v + \partial_x a W(v)\partial_y v$$
$$W(v)(x,y) = \int_0^y v(x,\zeta)\, d\zeta$$
$$L(v) = \partial_x W(v)\partial_y \phi U + \partial_x v(1-\phi)U + \partial_y v\left(\Phi\partial_x U - \partial_x a \Phi U\right) - W(v)\partial_x a\partial_y\phi U + v(1-\phi)\partial_x U$$
$$F = \left(\phi(1-\phi) + \Phi\partial_y\phi\right) U\partial_x U - \partial_x a\partial_y\phi\Phi U^2 - A^2\partial_{yy}\phi U - \phi\partial_x P$$

- The system is supplemented with the boundary conditions

$$v(x,y,t)|_{y=0} = u(x,Y,t)|_{Y=0} - (1-\phi(0))U(x,t) = 0$$
$$\lim_{y\to\infty} v(x,y,t) = \lim_{Y\to\infty} u(x,Y,t) - U(x,t) = 0$$

for all $(x,t) \in \mathbb{R}\times[0,\infty)$

# Functional Setting

- Consider the $y$-weight given by

$$\rho(y) = \langle y \rangle^{\alpha}$$

for some $\alpha > 0$ to be fixed later.

- We define a norm for the set of functions which are real-analytic in $x$ and decay in $y$ by

$$\|v\|_{X_\tau}^2 = \sum_{m \geq 0} \|\rho(y)\partial_x^m v(x,y,t)\|_{L^2(\mathbb{H})}^2 \tau^{2m}(t) M_m^2,$$

where $\tau > 0$ is the analyticity radius, and we denote the analytic weights

$$M_m = \frac{(m+1)^r}{m!}$$

for $r > 0$, a Sobolev exponent.

# The Main Theorem

## Theorem (Kukavica and V. ('11))

*Fix real numbers $\alpha > 1/2, \theta > \alpha + 1/2$, and $r > 1$.*

*Assume that the initial data for the underlying Euler flow is uniformly real analytic.*

*There exists $\tau_0 > 0$ such that for all $v_0 \in X_{\tau_0}$ there exits $T_* > 0$ such that the initial value problem associated to the Prandtl boundary layer equations has a unique real-analytic solution on $[0, T_*]$.*

- Solutions may be constructed even if the initial datum $v_0$ decays only as $\langle y \rangle^{-1-\epsilon}$ for arbitrary $\epsilon > 0$.
- This improves on the previous works, which require exponential matching at $Y = \infty$.

# Setup of the proof

- By the definition $\|v\|_{X_\tau}$, we have formally have

$$\frac{1}{2}\frac{d}{dt}\|v\|_{X_\tau}^2 + (-\dot{\tau})\|v\|_{Y_\tau}^2 = \sum_{m\geq 0}\left(\frac{1}{2}\frac{d}{dt}\|\rho v\|_{\dot{H}_x^m}^2\right)\tau^{2m}M_m^2,$$

where we denoted

$$\|v\|_{Y_\tau}^2 = \sum_{m\geq 1}\|\rho v\|_{\dot{H}_x^m}^2\tau^{2m-1}mM_m^2.$$

- The heart of the matter consists of estimating the term on the right side of the above equality, via Sobolev energy estimates.
- After applying $\partial_x^m$ to the Prandtl equations, multiplying by $\rho^2\partial_x^m v$, and integrating, we get

$$\frac{1}{2}\frac{d}{dt}\|\rho\partial_x^m v\|_{L^2}^2 - \langle\partial_x^m(A^2\partial_{yy}v), \rho^2\partial_x^m v\rangle = \langle\rho\partial_x^m(F - N(v) - L(v)), \rho\partial_x^m v\rangle.$$

# Closing the a priori estimates

- combining all previous estimates, we have

$$\begin{aligned}\frac{d}{dt}\|v\|_{X_\tau}^2 + \|A\rho\partial_Y v\|_{X_\tau}^2 \\ \leq C(1+\tau^{-2})\|v\|_{X_\tau}^2 + C\tau^{-1}\|A\rho\partial_Y v\|_{X_\tau}\|v\|_{X_\tau}^2 + C\|v\|_{X_\tau} \\ + \left(\dot{\tau} + C + C_*\tau^{-1}\|A\rho\partial_Y v\|_{X_\tau}\right)\|v\|_{Y_\tau}^2\end{aligned}$$

  for some positive constant $C$.
- choose $\tau$ to solve the ODE

$$\frac{d}{dt}(\tau^2) + 4C\tau_0 + 4C\|A\rho\partial_Y v\|_{X_\tau} = 0$$

  with initial condition $\tau_0$
- as long as $\tau > 0$ this implies that

$$\dot{\tau} + 2C + 2C\tau^{-1}\|v\|_{Z_\tau} \leq 0$$