Integral variational problems

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..
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Incompressible Fluids, Turbulence and Mixing
In honor of Peter Constantin's 60th birthday

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General form

$$\partial_t \theta(t,x) - \int_{\mathbb{R}^N} \psi(\theta(t,y) - \theta(t,x)) K(t,x,y)\,dy = 0. \qquad (1)$$

General form

$$\partial_t\theta(t,x) - \int_{\mathbb{R}^N} \psi(\theta(t,y)-\theta(t,x))K(t,x,y)dy = 0. \qquad (1)$$

We can consider also the equation with (linear or nonlinear) right hand side terms

$$\partial_t\theta(t,x) + v\cdot\nabla\theta(t,x) - \int_{\mathbb{R}^N} \psi(\theta(t,y)-\theta(t,x))K(t,x,y)dy = f(t,x). \qquad (2)$$

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Main Hypothesis on the kernel K:

- symmetry: $K(t,x,y) = K(t,y,x)$.

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Main Hypothesis on the kernel K:

- symmetry: $K(t,x,y) = K(t,y,x)$.
- K is singular enough for $x = y$. This implies some regularization effects.

Probabilistic interpretation

$$\partial_t \theta(t,x) - \int_{\mathbb{R}^N} (\theta(t,y) - \theta(t,x)) K(x,y)\, dy = 0.$$

Corresponds to dynamic including jumps from x to y with probability $K(t,x,y)$.

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Same equation with

$$K_s(x,y) = \frac{2-s}{|x-y|^{s+N}}.$$

General comments

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- But with a singular kernel (in the flavor of the Boltzmann equation with grazing collisions).
- Because of this singular kernel, the equation behaves more like a parabolic equation (regularization effect).
- Except that the operator is **nonlocal**: The knowledge of the θ on a neighborhood of x is not enough to compute the operator at the point x.

Passive transport in a turbulent flow

Fractional Laplacian used, for instance, to model passive transport in a turbulent flow (anomalous diffusion):

$$\partial_t \theta + v \cdot \nabla \theta - \Delta^{s/2} \theta = 0.$$

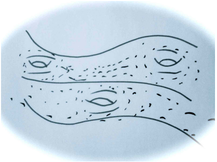


Figure: Passive transport in a turbulent flow

2D projection

The half Laplacian $\Delta^{1/2}$ corresponds to the "Dirichlet to Neumann" map.

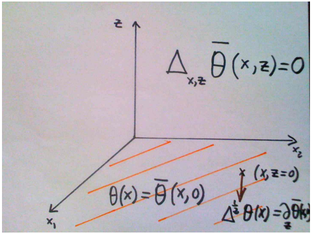


Figure: 2D projection

It is used for modeling in different situation (Dislocation dynamics, Surface Quasi-Geostrophic equation...)

Surface Quasi Geostrophic equation

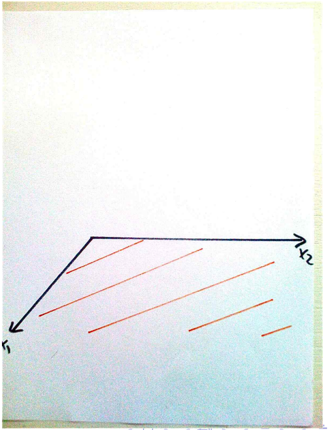
We consider the potential temperature function $\theta : \mathbb{R}^2 \to \mathbb{R}$ at the surface of the earth.

$$\partial_t \theta + u \cdot \nabla \theta = \Delta^{1/2} \theta,$$
$$u = R^\perp \theta.$$

$R^\perp \theta = (R_2 \theta, -R_1 \theta)$
where:

$$\widehat{R_i \theta} = \frac{\xi_i}{|\xi|} \hat{\theta}.$$

Note $\operatorname{div} u = 0$ (incompressibility).



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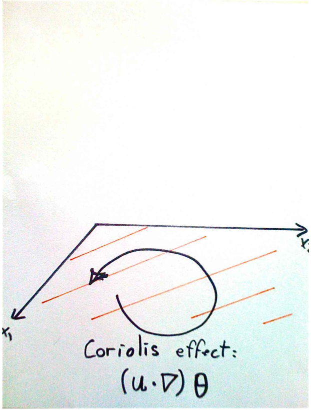

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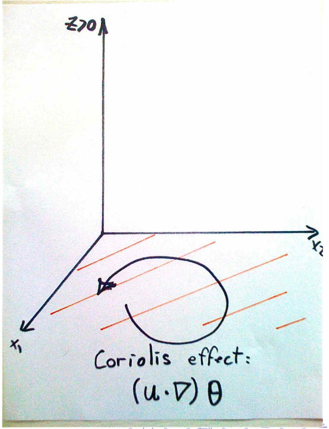

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$$\widehat{R_j \theta} = \frac{i\xi_j}{|\xi|}\hat{\theta}.$$

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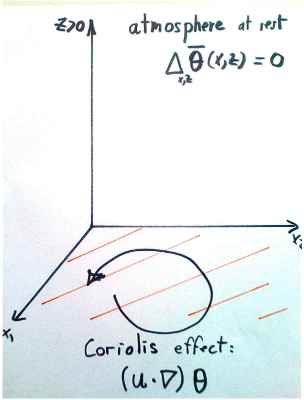

Figure: SQG equation

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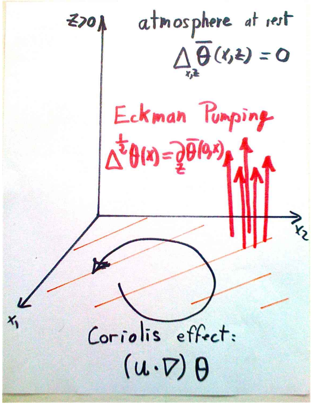

Citations

- Anomalous diffusion:
  - Fractal Burgers (Growing interfaces): Woyczynski, Meleard Jourdain,
    Karch, Kiselev Nazarov Shterenberg, Alibaud Droniou Vovelle Imbert Gallouet, Chan Czuback...
  - Fisher KPP (Biology): Cabre Roquefoffre
- Surface quasi-geostrophic: Constantin majda Tabak, Resnick, Constantin Wu,
  Cordoba Cordoba, Rodrigo, Kiselev Nazarov Volberg,
  Carillo, Schonbek & Schonbek...
- Dislocation dynamics: Monneau,
  Chasseigne Imbert, Barles...

The main result

Theorem

(Caffarelli, V.) Let $\theta_0 \in L^2(\mathbb{R}^2)$. Then for every $t_0 > 0$, θ lies in $C^\infty((t_0, \infty) \times \mathbb{R}^2)$.

- based on the De Giorgi method for regularity of solutions of elliptic equations.
- A different proof was found independently by Kiselev, Nazarov and Volberg.
- An interesting new proof proposed recently by Constantin and Vicol.

Main steps

- L^2 (bounded energy) to L^∞ (uniformly bounded).
- L^∞ (uniformly bounded) to C^α (modulus of continuity).

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We can then obtain full regularity with standard methods (potential theory and bootstrapping).

Energy inequalities

We can construct solutions which verify the energy inequality (see Resnick):

$$\partial_t \int \theta^2 \, dx + 2 \int |\Delta^{1/4}\theta|^2 \, dx \leq 0.$$

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$$\partial_t \int \theta^2 \, dx + 2 \int |\Delta^{1/4}\theta|^2 \, dx \leq 0.$$

Indeed, we can derive such a inequality for all the truncations (see Cordoba and Cordoba)

$$\theta_k = (\theta - C_k)_+.$$

We have:

$$\partial_t \int \theta_k^2 \, dx + 2 \int |\Delta^{1/4}\theta_k|^2 \, dx \leq 0.$$

L^∞ bounds

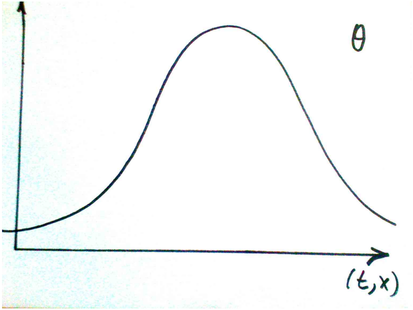
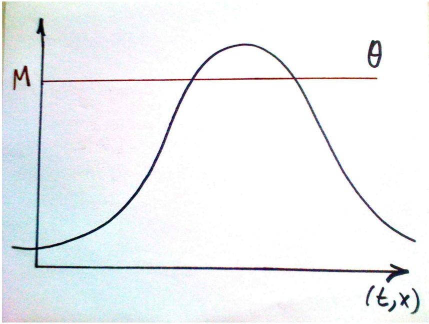


Figure: L^∞ bounds

L^∞ bounds

We want to show that $\theta < M$ (M depending on t_0 and U_0)



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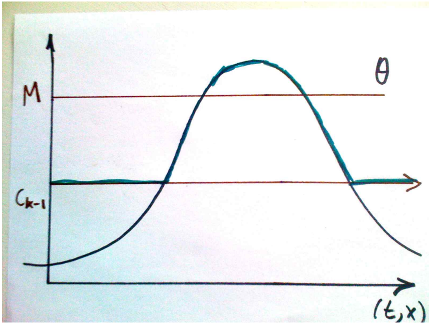
L^∞ bounds

We construct a sequence of C_k converging to M.



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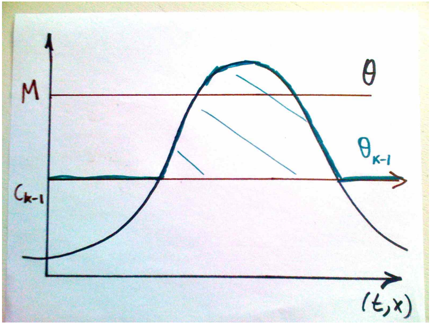
L^∞ bounds

We consider U_k "Energy/dissipation of energy" at the level set k.



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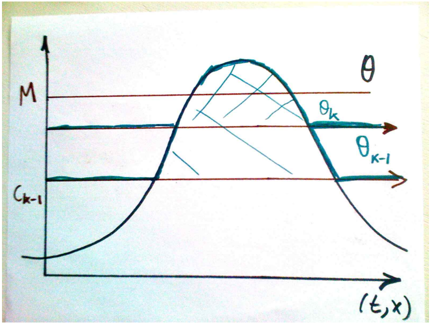
L^∞ bounds

$$U_k \leq \frac{C^k}{M t_0^N} U_{k-1}^{\beta}, \qquad \beta > 1.$$

(Sobolev, Chebychev inequality)



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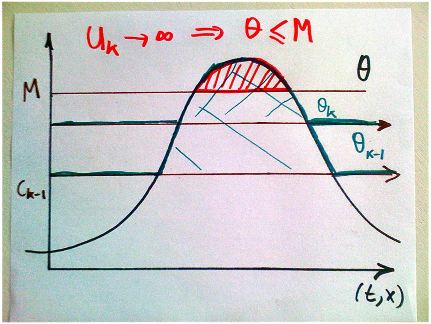


Figure: L^∞ bounds

From L^∞ to C^α

From Riesz transform we have

$$u \in L^\infty_t(BMO_x).$$

We consider the LINEAR equation:

$$\partial_t\theta + v\cdot\nabla\theta = \Delta^{1/2}\theta,$$
$$\mathrm{div} v = 0, \qquad v \in L^\infty_t(BMO_x).$$

- Competition between random walks and deterministic transport.
- $\Delta^{1/2}$: Fractional Laplacian associated to random walks with jumps (Levy processes).
- Critical case: Half Laplacian.

Main result

Theorem

(Caffarelli,V.) Let $\theta_0 \in L^2(\mathbb{R}^N)$. Then for every $t_0 > 0$, θ lies in $C^\alpha((t_0, \infty) \times \mathbb{R}^N)$ for a $\alpha > 0$.

Remark: Kiselev Nazarov gave an other proof based on duality arguments.

- use a lemma based on the L^∞ norm
- applied recursively on blow-ups of θ:

$$\tilde{\theta}(t, x) = \lambda\theta(\varepsilon t, \varepsilon x).$$

Local energy inequality

- C^α regularity is a LOCAL property.
- We need a local version (in x and t) of the energy equality on the truncation θ_k.
- We may use the extension of θ with a new variable $z > 0$ to keep memory of the non locality of the diffusion term.

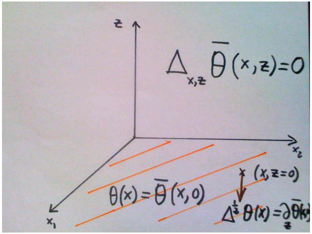


Figure: 2D projection

New energy estimate

$$\sup_{-1\leq t\leq 0}\left(\int_{B_1}\theta_k^2\,dx\right)+\int_{-1}^0\int_{B_1}\int_0^1|\nabla\overline{\theta}_k|^2\,dz\,dx\,dt$$
$$\leq C_p\left(\int_{-2}^0\int_{B_2}\theta_k^p\,dx\,dt+\int_{-2}^0\int_{B_2}\int_0^2\overline{\theta}_k^2\,dx\,dt\right).$$

- We have localized energy estimate in t, x, z
- But it gives only some partial knowledge on the boundary $z=0$. It is degenerated !!

The Equation

We consider the following nonlinear integral operator:

$$\partial_t \theta(t,x) - \int_{\mathbb{R}^N} \phi'(\theta(t,y) - \theta(t,x)) K(y-x)\,dy = 0,$$

Where ϕ is strictly convex and K is comparable with a fractional Laplacian:

$$K(-x) = K(x),$$
$$\frac{\Lambda^{-1}}{|x|^{N+s}} \leq K(x) \leq \frac{\Lambda}{|x|^{N+s}}.$$

$$\Lambda^{-1} \leq \phi''(x) \leq \Lambda.$$

This model appears in the phase transition literature and on issues of image processing (Giacomin Lebowitz Presutti, Gilboa Osher).

Variational problem

This is the Euler-Lagrange equation for the variational integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(\theta(y)-\theta(x))K(y-x)dydx.$$

The result

Theorem

(Caffarelli, Chan, V.) For any initial datum $\theta_0 \in H^1(\mathbb{R}^N)$, there exists a global classical solution to the problem with $\theta(0,\cdot) = \theta_0$ in the $L^2(\mathbb{R}^N)$ sense. Moreover $\nabla_x \theta \in C^\alpha((t_0, \infty) \times \mathbb{R}^N)$ for any $t_0 > 0$.

The solution is $C^{1,\alpha}$ in x.

Linear problem

we consider $w = D_e\theta$. It is solution to

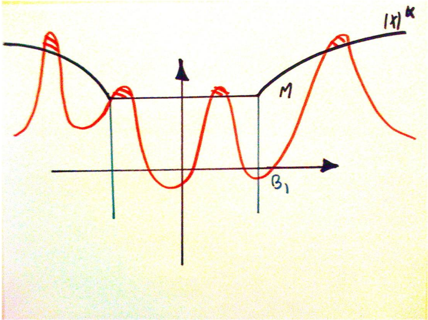
$$\partial_t w - \int_{\mathbb{R}^N} \{w(y) - w(x)\} K(t,x,y)\,dz = 0.$$

with

$$K(t,x,y) = \phi''(\theta(y) - \theta(x)) K(y-x).$$

- K is uniformly comparable to $\Delta^{s/2}$.
- But we CANNOT localize via an extension operator anymore.
- We have to stay GLOBAL.

Nonlinear case



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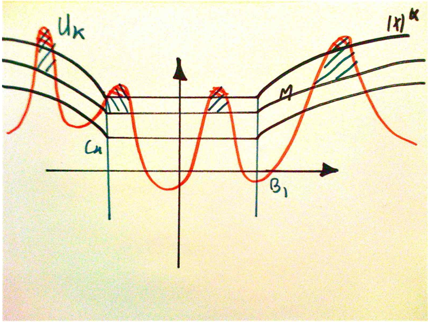

Figure: Nonlinear case

citations

- elliptic case: proved by Kassman, Kassman and Bass.
- Regularity theory for related models:
  - Very nonlinear integral operators: Caffarelli silvestre
  - Hamilton Jacobi with Fractional Laplacian: Barles, Imbert, Monneau

Thank you

THANK YOU

Thank you

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Bon anniversaire Peter !