DYNAMIC TRANSITIONS OF SURFACE TENSION DRIVEN CONVECTION

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In honor of Peter Constantin's 60th birthday,
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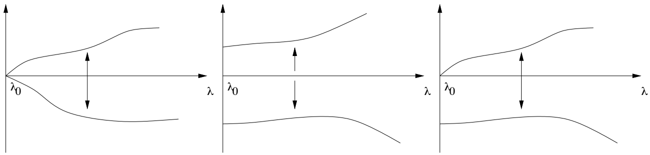
Brief Summary of Dynamic Transition Theory and Pattern Formation

- Consider a system that is in an equilibrium state which loses its stability as a control parameter exceeds a critical threshold.
  - What are the spatial and or spatio-temporal structures of the emergent states?
  - What is the stability/robustness of these new states?
- **Philosophy of Dynamic Transition Theory:** to search for the complete set of transition states, often represented by a local attractor.
- **Three Types of Transitions:**



(a) Type-I (b) Type-II (c) Type-III

THE PROBLEM

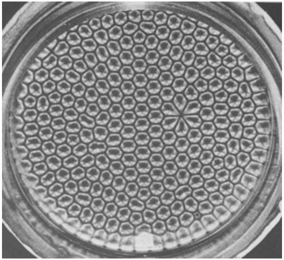


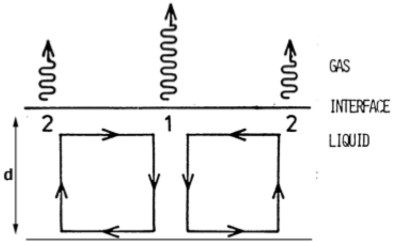
FIGURE: Kochmieder, 1974.

Rayleigh (1916) was the first to explain Bénard (1900)'s experiments using linear stability. But there were several discrepancies between the experiments and his results: the values of the critical temperature gradient and the wavelength of the pattern did not match. Block (1956) and Pearson (1958) proposed that the reason was the presence of free surface. On the free surface, there is surface tension:

$$\sigma = \sigma_0(1 - \gamma_T(T - T_0))$$

Surface tension gradient gives rise to shear stress on the interface which induces bulk motion when the fluid is viscous known as Marangoni (1871) effect.

The Marangoni Effect



Suppose a volume of liquid moves upward to point 2 due to infinitesimal perturbation. Since its temperature is higher, the surface tension is slightly decreased. Hence a flow develops from point 2 to point 1.

- Destabilizing Factors: Buoyancy, surface tension.
- Stabilizing Factors: Diffusion of heat and momentum

Mathematical Setting

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \mathrm{Pr}\left(-\nabla p + \Delta \mathbf{u} + \mathrm{Ra}\, \theta\, \vec{k}\right), \\
&\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla)\, \theta = w + \Delta \theta, \\
&\nabla \cdot \mathbf{u} = 0, \\
&\mathbf{u}\,(0) = \mathbf{u}_0, \quad \theta\,(0) = \theta_0.
\end{aligned} \tag{1}
$$

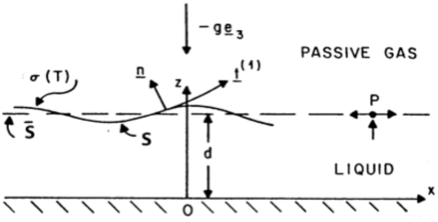
$u = (u, v, w)$ is the velocity field, θ is the temperature, $\vec{k} = (0, 0, 1)$. Unknowns represent deviations around the following steady state:

$$
\overline{u} = 0, \qquad \overline{T} = T_0 + (T_1 - T_0)\, \frac{x_3}{h}
$$

$\mathrm{Pr} = \nu/\kappa$	the Prandtl number,
$\mathrm{Ra} = \dfrac{a\, g\, (T_0 - T_1)}{\kappa\nu} h^3$	the Rayleigh number,

Here g is the gravitational acceleration, ν is the kinematic viscosity, κ is the thermal diffusivity, T_0 is the reference temperature at $x_3 = 0$.

Top Boundary Conditions



Let S be parametrized $z = d + \eta(x, y, t)$. The tangents to the interface are $\mathbf{t_1}$, $\mathbf{t_2}$ and the normal is $\mathbf{n}$.

- The kinematic condition:

$$w = u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial t}$$

- The normal stress balance

$$\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = 2H\sigma$$

- The tangential stress balance

$$\mathbf{t_i} \cdot \mathbf{T} \cdot \mathbf{n} = \mathbf{t_i} \cdot \nabla\sigma$$

$\mathbf{T}$ is the stress tensor, $2H$ is the mean curvature of the interface.

The non-dimensional domain is $\Omega = (0, L_1) \times (0, L_2) \times (0, 1)$.
Flat Top Surface $(\eta = 0)$:

$$\frac{\partial (u, v)}{\partial z} + \lambda \nabla_H \theta = w = \frac{\partial \theta}{\partial z} + \text{Bi}\, \theta = 0, \quad \text{at } z = 1$$

$\text{Bi} \geq 0$, the Biot number

$$\lambda = \frac{\xi_0 \gamma_T (T_0 - T_1) h^2}{\rho_0 \nu \kappa} > 0,$$ the Marangoni number

- **Lateral Sides**: "slippery" boundary conditions for the velocity, thermally insulating.
- **Bottom Surface**: $u = v = w = \theta = 0 \qquad \text{at } z = 0.$

Eigenvalue problem

Unlike pure RB Problem, linear operator is not self adjoint.

$$\begin{aligned}
&\mathrm{Pr}\,(-\nabla p + \Delta \mathbf{u}) = \beta \mathbf{u}, \\
&w + \Delta\theta = \beta\theta, \\
&\nabla\cdot\mathbf{u} = 0,
\end{aligned} \tag{2}$$

Eigenfunctions

$$\begin{aligned}
&w_I = W_I\,(z)\cos L_1^{-1} i_x \pi x \cos L_2^{-1} i_y \pi y, \\
&\theta_I = \Theta_I\,(z)\cos L_1^{-1} i_x \pi x \cos L_2^{-1} i_y \pi y, \qquad I = (i_x, i_y) \in \mathbb{Z}\times\mathbb{Z}.
\end{aligned}$$

Let

$$\alpha_I^2 = (L_1^{-1} i_x)^2\pi^2 + (L_2^{-1} i_y)\pi^2, \quad D = \frac{d}{dz}. \tag{3}$$

$$U_I(z) = -\frac{L_1^{-1} i_x \pi}{\alpha_I^2} DW_I(z), \qquad V_I(z) = -\frac{L_2^{-1} i_y \pi}{\alpha_I^2} DW_I(z), \tag{4}$$

$$\begin{aligned}
&\mathrm{Pr}^{-}1\beta(\lambda)(D^2-\alpha^2)W = (D^2-\alpha^2)^2 W + \mathrm{Ra}\,\alpha^2\Theta, \\
&\beta(\lambda)\Theta - W = (D^2-\alpha^2)\Theta
\end{aligned}$$

Critical Marangoni Number

Set $Ra = 0$ afterwards. Solving for $\beta = 0$:

$$\lambda_c = \min_{\alpha} \frac{8\alpha\,(\alpha\cosh\alpha + \mathrm{Bi}\sinh\alpha)\,(\alpha - \cosh\alpha\sinh\alpha)}{\alpha^3\cosh\alpha - \sinh^3\alpha}. \qquad (5)$$

$$\alpha_I^2 = i_x^2\pi^2 L_1^{-2} + i_y^2\pi^2 L_2^{-2}, \quad I = (i_x, i_y)$$

Pearson (1958): In the horizontally infinite case $\lambda_c \approx 79.6$ when $Bi = 0$ with critical wave number $\alpha = 1.99$.

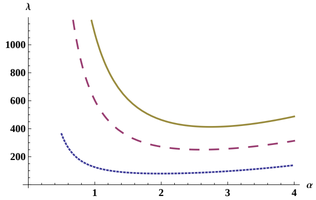
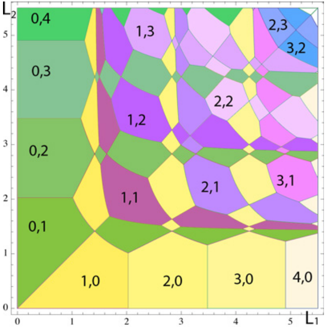


FIGURE: Marginal stability curves at $Bi = 0$ (dotted), $Bi = 5$ (dashed), $Bi = 10$(continuous).

$\lambda_c \to \infty$ as $Bi \to \infty$ (No instability)



THEOREM

There is a set of critical indices $\mathcal{C} = \{I = (i_x, i_y) \in \mathbb{Z}^2_{\geq 0} \mid \alpha_I \text{ minimizes } (5)\}$ *finite and non-empty such that*

$$\beta_{(J,1)}(\lambda) = \begin{cases} < 0 & \lambda < \lambda_c \\ = 0 & \lambda = \lambda_c \\ > 0 & \lambda > \lambda_c, \end{cases} \qquad \forall J \in \mathcal{C} \tag{6}$$

$$Re\beta_{(J,k)}(\lambda_c) < 0 \qquad \forall J \notin \mathcal{C}. \tag{7}$$

Vrentas (2004) proved that the $\beta_{(J,1)}$, $J \in \mathcal{C}$ are always real!

Rolls, Rectangles and Hexagons

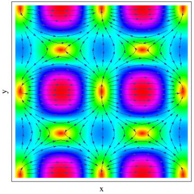


Figure: Rectangles have wave indices I=(m,n), $m \neq 0$, $n \neq 0$

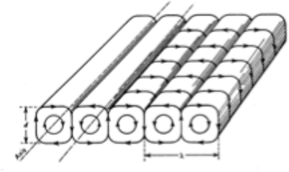


Figure: Rolls have wave indices I=(0,n), $n \neq 0$.

When $\frac{L_1}{L_2} = \frac{m}{n\sqrt{3}}$ with m, n two positive integers, the vector field $x_I\phi_I + x_J\phi_J$ defines a hexagonal pattern when $x_I = \pm 2x_J$.

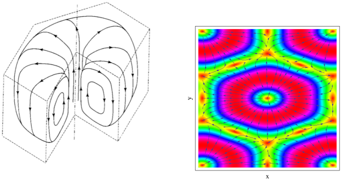


Figure: Hexagon structure

Some of the previous works on the pattern formation:
- Scanlon and Segel (1967)...the infinite vertical layer
- Rosenblat, Davis, Homsy (1982)...competition of rolls
- Cloot, Lebon (1984)
- Dauby, Lebon, Colinet, Legros (1993)...Partial results about hexagons
- Dijkstra (1995)...Numerical study in rigid containers

There are numerous other experimental, theoretical and numerical results.

Statement of The Main Theorems

$$b_2 = \sum_{l \geq 1 \text{ and } S=(0,0),(0,4i_y)} G_s(\phi_{(0,2i_y,1)}, \phi_{(S,l)}, \phi^*_{(0,2i_y,1)})\Phi^l_S.$$

where $G_s(\phi_1, \phi_2, \phi_3) = G(\phi_1, \phi_2, \phi_3) + G(\phi_2, \phi_1, \phi_3)$ and G is the tri-linear operator. Φ^l_S are coefficients which are computable.

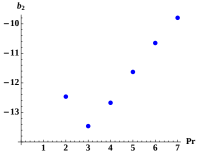


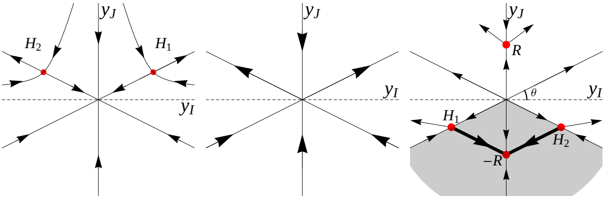
Figure: The value of b_2 for $L_2 = 3.02$ and $L_1 = \frac{2L_2}{\sqrt{3}}$ and $Bi = 0$.

Theorem

Assume $\mathcal{C} = \{I = (i_x, i_y), J = (0, 2i_y)\}$ such that $\frac{L_1}{L_2} = \frac{i_x}{i_y\sqrt{3}}$, where i_x and i_y are two positive integers. Then the system goes a dynamic transition at $\lambda = \lambda_c$. The transition is Type-II if $b_2 > 0$ and Type-III if $b_2 < 0$.

Theorem ($b_2 < 0$ case)

The system undergoes a Type-III transition at $\lambda = \lambda_c$:



(a) $\lambda < \lambda_c$ (b) $\lambda = \lambda_c$ (c) $\lambda > \lambda_c$

$$\pm R^\lambda = (y_I, y_J) = \pm(0, \sqrt{\beta(\lambda)/-b_2}\,) + O(\beta(\lambda))$$ *rolls*

$$H_i^\lambda = \frac{\beta(\lambda)}{a_1}(2(-1)^i, -1) + O(\beta(\lambda)^2),\ i = 1, 2$$ *hexagons*

THEOREM ($b_2 < 0$ CASE)

I) *There is a neighborhood $\mathcal{U}$ of $\phi = 0$ in the phase space $H \subset L^2(\Omega)^4$ such that for any $\lambda_c < \lambda < \lambda_c + \epsilon$ with some $\epsilon > 0$,*

$$\overline{\mathcal{U}} = \overline{\mathcal{U}_1^\lambda} \cup \overline{\mathcal{U}_2^\lambda}, \qquad \mathcal{U}_1^\lambda \cap \mathcal{U}_2^\lambda = \emptyset$$

$$\lim_{\lambda \to \lambda_c} \limsup_{t \to \infty} ||S_\lambda(t, \varphi)||_H = 0 \qquad \forall \varphi \in \mathcal{U}_1^\lambda,$$

$$\limsup_{t \to \infty} ||S_\lambda(t, \varphi)||_H \geq \delta > 0 \qquad \forall \varphi \in \mathcal{U}_2^\lambda,$$

for some $\delta > 0$. Moreover $\mathcal{P}(U_1^\lambda)$ and $\mathcal{P}(U_2^\lambda)$ are sectorial regions given by:

$$\mathcal{P}(\mathcal{U}_1^\lambda) = \mathcal{U} \cap \{x \in \mathbb{R}^2 \mid \pi + \theta < arg(x) < 2\pi - \theta\},$$

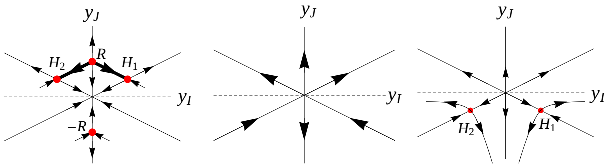
$$\overline{\mathcal{P}(\mathcal{U}_2^\lambda)} = \mathcal{U} \cap \{x \in \mathbb{R}^2 \mid -\theta < arg(x) < \pi + \theta\},$$

$\theta = \arctan 1/2$ and $\mathcal{P}$ is the projection onto the center-unstable space.

II) *The system bifurcates to an attractor Σ_λ which consists of three steady states $H_1^\lambda, H_2^\lambda, -R^\lambda$ and heteroclinic orbits connecting $-R^\lambda$ to H_1^λ and $-R^\lambda$ to H_2^λ and has basin of attraction $\mathcal{U}_1^\lambda$.*

THEOREM ($b_2 > 0$ CASE)

The system undergoes a Type-II transition at $\lambda = \lambda_c$:



(d) $\lambda < \lambda_c$ (e) $\lambda = \lambda_c$ (f) $\lambda > \lambda_c$

FIGURE: Drastic transition.

Sketch of the Proof

- Consider the reduction of the system on to the center manifold:

$$\phi = y_I \phi_I + y_J \phi_J + \Phi(y_I, y_J)$$

- Solve the adjoint linear system.
- Determine the resonant modes $\mathcal{S}$. For $I = (I_x, I_y)$ and $J = (0, 2I_y)$:

$$\mathcal{S} = \{(0,0), (2I_x, 0), J, 2I, 2J, I + J, I\}$$

- Approximate the center manifold $\Phi(y, \lambda)$ using:

$$-\mathcal{L}_\lambda \Phi(y, \lambda) = y_I y_J P_2 G(\psi_I, \psi_J) + o(2), \tag{8}$$

where $\mathcal{L}_\lambda = L_\lambda \mid_{E_2}$, E_2 is the subspace spanned by the stable modes, P_2 is the projection onto E_2 and G is the nonlinear operator.

$$\begin{aligned} \frac{dy_I}{dt} &= \beta(\lambda) y_I + a_1 y_J y_I + y_I(a_2 y_I^2 + a_3 y_J^2) + o(3), \\ \frac{dy_J}{dt} &= \beta(\lambda) y_J + b_1 y_I^2 + y_J(b_2 y_J^2 + b_3 y_I^2) + o(3). \end{aligned} \tag{9}$$

$$a_1 = 4b_1, \qquad a_3 = 2b_3, \qquad 4a_2 = a_3 + b_2.$$

OPEN QUESTIONS

- The case of deformable surface ($\eta \neq 0$) is still open. Needs a combination of the dynamic transition theory with numerical methods.
- At the onset of convection, α_c changes with increasing Marangoni number. Why???
- Can we resolve the uncertainty in the case of Type-II regions?

SUMMARY OF RESULTS

- It is known that **without** surface tension, we have always Type-I transition (Ma–Wang, 2004). **With** surface-tension driven convection, the transition is Type-II or III. This is the first time we encountered a Type-III transition in a physically realistic model.
- The hexagonal and other patterns are metastable, leading to uncertainty in pattern selection.
- The reduced equations are universal in the sense that they are ubiquitous in the pattern selection of hexagons. We recently encountered the same reduced equations in Cahn-Hilliard equation with a nonlocal term (joint with H. Liu, S. Wang, P. Zhang).
- The well-posedeness and the existence of the global attractor of the problem has also been studied.

Thank you!