Recent progress on Boussinesq systems

Jean-Claude Saut
University Paris-Sud

Incompressible fluids, turbulence and mixing
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- A brief review on Boussinesq systems for surface water waves.
- To survey recent results on the well-posedness of Boussinesq systems, mainly :
- The strongly dispersive Boussinesq system (with F. Linares and D. Pilod).
- Long time existence $(1/\epsilon)$ for most of Boussinesq systems (with Li Xu).
- Emphasize the following fact : most of nonlinear dispersive equations or systems are not derived from first principles but through some asymptotic expansion and are not supposed to be "good" models for all time. So the classical dichotomy *local well-posedness* versus *finite time blow-up* should be probably be replaced by questions on *long time* existence (with respect to some parameters). Then using only methods of harmonic analysis (even the more fancy ones) seems to be useless...

- *Boussinesq* regime for surface water waves (with flat bottom) : $h =$ mean depth of the fluid layer, $a =$ typical amplitude of the wave and $\lambda =$ a typical (horizontal) wave length, one assumes that

$$a/h \sim (h/\lambda)^2 = \epsilon \ll 1.$$

- One gets (for instance by expanding the Dirichlet to Neumann operator with respect to ϵ) the (a, b, c, d) family of systems (Bona-Chen-S. 2002) for the amplitude ζ of the wave and an approximation $\mathbf{v}$ of the horizontal velocity taken at some height in the infinite strip domain :

$$\begin{cases} \partial_t\zeta + \nabla\cdot\mathbf{v} + \epsilon\nabla\cdot(\zeta\mathbf{v}) + \epsilon\big(a\nabla\cdot\Delta\mathbf{v} - b\Delta\partial_t\zeta\big) = 0, \quad (O(\epsilon^2)) \\ \partial_t\mathbf{v} + \nabla\zeta + \frac{\epsilon}{2}\nabla(|\mathbf{v}|^2) + \epsilon\big(c\nabla\Delta\zeta - d\Delta\partial_t\mathbf{v}\big) = 0, \quad (O(\epsilon^2)) \end{cases} \tag{1}$$

with the initial data

$$(\zeta, \mathbf{v})^T|_{t=0} = (\zeta_0, \mathbf{v}_0)^T \tag{2}$$

a, b, c, d are modelling parameters which satisfy the constraint $a + b + c + d = \frac{1}{3}$. Those three degrees of freedom arise from the height at which the horizontal velocity is taken and from a double use of the *BBM trick*.

Advantages of the (a, b, c, d) systems :

- Choose (a, b, c, d) for better mathematical/numerical properties.
- Choose (a, b, c, d) to fit better the water waves dispersion relation.

- Dispersion matrix :

$$D = \epsilon \begin{pmatrix} -b\Delta\partial_t & a\Delta\nabla \\ c\Delta\nabla\cdot & -d\Delta\partial_t \end{pmatrix}.$$

- The corresponding non zero eigenvalues are

$$\lambda_{\pm}(\xi) = \pm i|\xi| \left(\frac{(1-\epsilon a|\xi|^2)(1-\epsilon c|\xi|^2)}{(1+\epsilon d|\xi|^2)(1+\epsilon b|\xi|^2)} \right)^{\frac{1}{2}}.$$

- The linearized system around the null solution is well-posed provided that

$$a \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0, \tag{3}$$

$$\text{or} \quad a = c > 0, \quad b \geq 0, \quad d \geq 0. \tag{4}$$

Note that the eigenvalues $\lambda(\xi)$ can have order $3, 2, 1, 0, -1$.

- Order 3 : "KdV" type.
- Order 2 : "Schrödinger" type.

- When $b = d$ the Boussinesq systems are Hamiltonian with Hamiltonian (recall that generically $a, c \leq 0$) :

$$H(\zeta, \mathbf{v}) = \frac{1}{2}\int_{\mathbb{R}^d}[-c|\nabla\zeta|^2 - a|\nabla\mathbf{v}|^2 + \zeta^2 + |\mathbf{v}|^2(1+\zeta)].$$

- Useless in 2D.

When surface tension is taken into account, one obtains a similar class of Boussinesq systems (Daripa-Das 2003) with a changed into $a-\tau$ where $\tau \geq 0$ is the Bond number which measures surface tension effects. The constraint on the parameters a, b, c, d is the same and condition (19) reads now

$$\begin{cases} a-\tau \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0, \\ \text{or} \\ a-\tau = c > 0, \quad b \geq 0, \quad d \geq 0. \end{cases} \tag{5}$$

The complete *rigorous* justification of the Boussinesq systems involves five main steps.

- Formal derivation of the models (Bona-Chen-S.2002)
- The proof of the *consistency* of the asymptotic systems with the full Euler system with free surface. This has been carried out in BCS (2002) and Bona-Colin-Lannes (2005).
- The proof of the well-posedness of the Euler system on the correct time scales $1/\epsilon$. This difficult step has been achieved in Alvarez-Samaniego, Lannes (2008) in absence of surface tension (see M. Ming-P. Zhang-Z. Zhang 2011 for gravity-capillary waves in the weakly transverse regime).
- Establishing long time, (that is on time scales of order at least $1/\epsilon$), existence of solutions to the Boussinesq systems satisfying uniform bounds with respect to ϵ.
- Assuming the previous steps, proving the optimal error estimates $O(\epsilon^2 t)$ (see Bona-Colin-Lannes 2005).

Previous results in one dimension

- All linearly well-posed systems are (nonlinearly) locally-well-posed (Bona-Chen-S. 2004).
- Global well-posedness (under a non cavitation assumption and for small enough Hamiltonian) for the very specific systems which are Hamiltonian, *eg* when $b = d > 0$, and $c \leq 0$, $a < 0$. Does not extend in $2D$ since then the Hamiltonian does not control any Sobolev norm.
- When $a = c = d = 0$ and $b = \frac{1}{3}$, (a version of the original Boussinesq system), global well-posedness under a non-cavitation condition for initial data which are compactly supported perturbations of constant states (Schonbek 1981, Amick 1984). Key observation : this Boussinesq system van be viewed as a perturbation of the Saint-Venant quasilinear hyperbolic system and use its properties (entropy,..). Does not seem to work in $2D$.

Previous results in 2D

- Various local well-posedness results (Dougalis-Mitsotakis-S. 2007, Cung The Anh 2010) for the *weakly dispersive systems*.
- It turns out that those results, which use in some way the dispersive properties of the systems, yield only existence on time intervals of lengths $O(1/\sqrt{\epsilon})$.

We first focus now on the *more dispersive* Boussinesq system, corresponding to $b = d = 0$, $a = c = 1/6$. Though not the more relevant for modeling purposes due in particular to its computational difficulties and its "bad" behavior to short waves, it is mathematically challenging (this is one of the few physically relevant 2D versions of the KDV equation...) and show the limitations of the purely *dispersive* method to get then long time well-posedness results.

- Joint with F. Linares and D. Pilod (2011).

$$\begin{cases} \partial_t\zeta + \operatorname{div}\mathbf{v} + \epsilon\operatorname{div}(\zeta\mathbf{v}) + \epsilon\operatorname{div}\Delta\mathbf{v} = 0 \\ \partial_t\mathbf{v} + \nabla\zeta + \epsilon\frac{1}{2}\nabla(|\mathbf{v}|^2) + \epsilon\nabla\Delta\zeta = 0 \end{cases}, \quad (x_1, x_2) \in \mathbb{R}^2,\ t \in \mathbb{R}, \tag{6}$$

Theorem

Let $s > \frac{3}{2}$ and $0 < \epsilon \leq 1$ be fixed. Then for any $(\zeta_0, \mathbf{v}_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)^2$ with curl $\mathbf{v}_0 = 0$*, there exist a positive time $T = T(\|(\zeta_0, \mathbf{v}_0)\|_{H^s \times (H^s)^2})$, a space $Y^s_{T_\epsilon}$ such that*

$$Y^s_{T_\epsilon} \hookrightarrow C\big([0, T_\epsilon]\ ;\ H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)^2\big), \tag{7}$$

and a unique solution $(\zeta, \mathbf{v})$ to (6) in $Y^s_{T_\epsilon}$ satisfying $(\zeta, \mathbf{v})_{|_{t=0}} = (\eta_0, \mathbf{v}_0)$, where $T_\epsilon = T\epsilon^{-\frac{1}{2}}$, uniformly bounded on $[0, T_\epsilon]$.
Moreover, for any $T' \in (0, T_\epsilon)$, there exists a neighborhood Ω^s of $(\zeta_0, \mathbf{v}_0)$ in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)^2$ such that the flow map associated to (6) is smooth from Ω^s into $Y^s_{T'}$.

After diagonalization of the linear part, the system reduces to (a component has been eliminated by the curl-free assumption) :

$$\begin{cases} \partial_t w_1 + i(1+\epsilon\Delta)\sqrt{-\Delta}w_1 + (I+II)(w_1,w_2) = 0 \\ \partial_t w_2 - i(1+\epsilon\Delta)\sqrt{-\Delta}w_2 + (-I+II)(w_1,w_2) = 0 \end{cases}, \tag{8}$$

where

$$\begin{aligned} I = I(w_1,w_2) \; =& (w_1-w_2)\sqrt{-\Delta}(w_1+w_2) + \partial_{x_1}(w_1-w_2)R_1(w_1+w_2) \\ &+ \partial_{x_2}(w_1-w_2)R_2(w_1+w_2), \end{aligned} \tag{9}$$

$$II = II(w_1,w_2) := i\sqrt{-\Delta}\left((R_1(w_1+w_2))^2 + (R_2(w_1+w_2))^2\right). \tag{10}$$

and R_1, R_2 are the Riesz transforms.

The proof consists in implementing a fixed point argument on the Duhamel formulation by using various dispersive estimates on the linear part : Strichartz (with smoothing), Kato type smoothing, maximal function estimate (see below).

- It seems that this "KdV-KdV" system cannot (apparently) be solved by "elementary" energy methods.

Linear system :

$$\begin{cases} u_t \pm L_\epsilon u = 0, \\ u(.,0) = u_0, \end{cases} \tag{11}$$

where $L_\epsilon = i(I + \epsilon\Delta)\sqrt{-\Delta} = -i\varphi_\epsilon(D)$, *i.e.* $\varphi_\epsilon(\xi) = \epsilon|\xi|^3 - |\xi|$. In the sequel, we will identify $\varphi_\epsilon(\xi) = \varphi_\epsilon(|\xi|)$. Moreover, we will denote by $U_\epsilon^\pm(t)u_0$ the solution of (11), *i.e.*

$$U_\epsilon^\pm(t)u_0 = \left(e^{\pm it\varphi_\epsilon}\widehat{u_0}\right)^\vee. \tag{12}$$

Let $\left\{ Q_\alpha \right\}_{\alpha \in \mathbb{Z}^2}$ denote a family of nonoverlapping cubes of unit size such that $\mathbb{R}^2 = \bigcup_{\alpha \in \mathbb{Z}^2} Q_\alpha$.

Theorem

Let $T > 0$ and $\epsilon > 0$. Then, it holds that

$$\sup_{\alpha} \left(\int_{Q_\alpha} \int_0^T \left| P_{>\epsilon^{-\frac{1}{2}}} D_x^1 U_\epsilon^{\pm}(t) u_0(x) \right|^2 dt dx \right)^{\frac{1}{2}} \lesssim \epsilon^{-\frac{1}{2}} \| u_0 \|_{L^2_x}, \quad (13)$$

where the implicit constant does not depend on ϵ and T.

- Maximal function estimate in the n-dimensional case.

Unitary group $U_\epsilon^{\pm}(t) = e^{\pm it\varphi_\epsilon(D)}$, where $\varphi_\epsilon(\xi) = \varphi_\epsilon(|\xi|) = \epsilon|\xi|^3 - |\xi|$ and $\xi \in \mathbb{R}^n$, for $n \geq 2$. Let $\{Q_\alpha\}_{\alpha\in\mathbb{Z}^n}$ denote the mesh of dyadic cubes of unit size.

Theorem

With the above notation, for any $s > \frac{3n}{4}$, $\epsilon > 0$ and $T > 0$ satisfying $\epsilon T \leq 1$, it holds that

$$\Big(\sum_{\alpha\in\mathbb{Z}^n} \sup_{|t|\leq T} \sup_{x\in Q_\alpha} \big|U_\epsilon^{\pm}(t)u_0(x)\big|^2\Big)^{\frac{1}{2}} \lesssim (1 + T^{\frac{n}{4}-\frac{1}{4}})\|u_0\|_{H^s}, \qquad (14)$$

where the implicit constant does not depend on ϵ and T.

- Strichartz estimates

Theorem
Let $\epsilon > 0$, $T > 0$ and $0 \le \theta \le \frac{1}{2}$. Then, it holds that

$$\|D_x^\theta U_\epsilon^\pm u_0\|_{L_T^3 L_x^\infty} \lesssim \epsilon^{-\frac{1}{3}-\frac{\theta}{2}} \|u_0\|_{L^2}, \tag{15}$$

where the implicit constant is independent of ϵ and T.

- Remarks on the *Schrödinger* case (Linares-Pilod-S 2011).

When $a < 0$, $c < 0$, b or $d = 0$, the eigenvalues are of order 2 and, after diagonalization, the linear part of the corresponding Boussinesq systems looks like, for large frequencies, to an uncoupled system of two linear Schrödinger equations.
In this case one can obtain order $1/\sqrt{\epsilon}$ well-posedness results by relatively elementary energy methods.
There are two cases.

- First case.

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{v} + \epsilon[\nabla \cdot (\eta \mathbf{v}) + a\nabla \cdot \Delta \mathbf{v}] = 0 \\ \mathbf{v}_t + \nabla \eta + \epsilon[\frac{1}{2}\nabla |\mathbf{v}|^2 + c\nabla\Delta\eta - d\Delta \mathbf{v}_t] = 0, \end{cases} \tag{16}$$

where $a < 0$, $c < 0$, $d > 0$.

Theorem

Let $s \geq 3/2$ and $0 < \epsilon \leq 1$ be given.
(i) Assume that $a = c$. Then, for every $(\eta_0, \mathbf{v}_0) \in H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)^2$, there exist $T = T(||\eta_0||_{H^s}, ||\mathbf{v}_0||_{H^{s+1}})$ and a unique solution

$$(\eta, \mathbf{v}) \in C([0, T_\epsilon]; H^s(\mathbb{R}^2)) \times C([0, T_\epsilon]; H^{s+1}(\mathbb{R}^2)^2),$$

where $T_\epsilon = T\epsilon^{-1/2}$, of (16) with initial data $(\eta_0, \mathbf{v}_0)$, which is uniformly bounded on $[0, T_\epsilon]$.
(ii) Assume that $a \neq c$. Then, for every $(\eta_0, \mathbf{v}_0) \in H^{s+1}(\mathbb{R}^2) \times H^{s+2}(\mathbb{R}^2)^2$, there exist $T = T(||\eta_0||_{H^{s+1}}, ||\mathbf{v}_0||_{H^{s+2}})$ and a unique solution

$$(\eta, \mathbf{v}) \in C([0, T_\epsilon]; H^{s+1}(\mathbb{R}^2)) \times C([0, T_\epsilon]; H^{s+2}(\mathbb{R}^2)^2),$$

- Second case.

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{v} + \epsilon[\nabla \cdot (\eta \mathbf{v}) + a\nabla \cdot \Delta \mathbf{v} - b\Delta \eta_t] = 0 \\ \mathbf{v}_t + \nabla \eta + \epsilon[\frac{1}{2}\nabla |\mathbf{v}|^2 + c\nabla \Delta \eta] = 0. \end{cases} \tag{17}$$

We will here restrict the velocity $\mathbf{v}$ to irrotational motions, a situation which is relevant since the Boussinesq systems are derived for potential flows. Note also that the curlfree condition is preserved by the evolution of (17). When $\mathbf{v}$ is curlfree, the term $\frac{1}{2}\nabla|\mathbf{v}|^2$ in the second equation of (17) writes as two transport equations namely $(\mathbf{v} \cdot \nabla v_1, \mathbf{v} \cdot \nabla v_2)^T$ where $\mathbf{v} = (v_1, v_2)^T$. This will permit to perform energy estimates on $\mathbf{v}$.

Theorem

Let $s > 2$ and $0 < \epsilon \leq 1$ be given.
(i) Assume that $a = c$. Then, for every $(\eta_0, \mathbf{v}_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)^2$ *with* $\operatorname{curl} \mathbf{v} = 0$ *there exist* $T = T(||\eta_0||_{H^{s+1}}, ||\mathbf{v}_0||_{H^s})$ *and a unique solution*

$$(\eta, \mathbf{v}) \in C([0, T_\epsilon]; H^{s+1}(\mathbb{R}^2)) \times C([0, T_\epsilon]; H^s(\mathbb{R}^2)^2),$$

where $T_\epsilon = T\epsilon^{-1/2}$, of (16) with initial data $(\eta_0, \mathbf{v}_0)$, which is uniformly bounded on $[0, T_\epsilon]$.
(ii) Assume that $a \neq c$. Then, for every $(\eta_0, \mathbf{v}_0) \in H^{s+2}(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)^2$ *with* $\operatorname{curl} \mathbf{v} = 0$ *there exist* $T = T(||\eta_0||_{H^{s+2}}, ||\mathbf{v}_0||_{H^{s+1}})$ *and a unique solution*

$$(\eta, \mathbf{v}) \in C([0, T_\epsilon]; H^{s+2}(\mathbb{R}^2)) \times C([0, T_\epsilon]; H^{s+1}(\mathbb{R}^2)^2),$$

- **Existence on long time.**

In Bona-Colin-Lannes (BCL 2008) a new class of *fully symmetric* systems was introduced. The idea was, when the dispersive part is skew-adjoint, that is when $a = c$, to perform the change of variables

$$\tilde{\mathbf{v}} = \mathbf{v}(1 + \frac{\epsilon}{2}\zeta)$$

to get (up to terms of order ϵ^2) systems having a *symmetric* nonlinear part. Those transformed systems can be viewed as skew-adjoint perturbations of *symmetric* first order hyperbolic systems (when $\mathbf{v}$ is curlfree up to $0(\epsilon)$) and existence on time scales of order $1/\epsilon$ follows classically. But of course the transformed systems are not members of the a, b, c, d class of Boussinesq systems and this does not solve the long time existence (though this justifies in some sense this class as a *good* one to approximate the full Euler system with free boundary, see BCL).

- We now aim to proving long time ($1/\epsilon$) existence for the original Boussinesq systems.

- A result via Nash-Moser (M.Ming-P.Zhang-JCS 2011, *A drop hammer to kill a fly...*).
  Let $T > 0$ and $s \in \mathbb{R}$. For any $\epsilon \in (0,1)$, we define the norm to the Banach space X^s_ϵ by

1) Generic case $(a, c < 0,\ b, d > 0)$:
$|U|_{X^s_\epsilon} = |(V,\eta)|_{X^s_\epsilon} \equiv |(1-\epsilon\Delta)^{\frac{3}{2}} V|_{H^s} + |(1-\epsilon\Delta)^{\frac{3}{2}}\eta|_{H^s}$,
2) Purely BBM case $(a = c = 0,\ b, d > 0)$:
$|U|_{X^s_\epsilon} = |(V,\eta)|_{X^s_\epsilon} \equiv |(1-\epsilon\Delta)^{\frac{1}{2}} V|_{H^s} + |(1-\epsilon\Delta)^{\frac{1}{2}}\eta|_{H^s}$.

Theorem

Let $m_0 > \frac{d}{2}$ be a large positive integer. Let $D > 3$, $P > P_{min} = 3 + \frac{D}{D-1}\left(\sqrt{3} + \sqrt{2(2+D)}\right)^2$, $s \geq m_0 + 1$. Then for any $(U_0^\epsilon)_{0<\epsilon<\epsilon_0}$ which is uniformly bounded in X_ϵ^{s+P} and satisfying

$$1 + \epsilon\eta_0^\epsilon > 0 \qquad \text{uniformly for} \quad \epsilon \in (0, \epsilon_0),$$

there exists $T > 0$ so that the Boussinesq systems for both generic case and purely BBM case has a unique family of solutions $(U^\epsilon)_{0<\epsilon<\epsilon_0}$ which is uniformly bounded in $C([0, T/\epsilon]; X_\epsilon^{s+D})$.

- Use a Nash-Moser theorem to get well-posedness on time scales of order $1/\epsilon$. (see Alvarez-Samaniego, Lannes 2008 for a Nash-Moser approach to Green-Naghdi systems which are much more nonlinear...).
- Works for the "generic" case and the BBM/BBM case (and probably for other as well).
- Apparently the first existence result on time scales $1/\epsilon$ (except the few aforementionned global 1D situations).
- Shortcoming : the method is not a natural one ; loss of derivatives ; high regularity needed.

- A more natural and friendly approach (Li Xu, JCS, J.Math.Pures Appl. 2011).

Definition

For any $s \in \mathbb{R}$, $k \in \mathbb{N}$, $\epsilon \in (0,1)$, the Banach space $X^s_{\epsilon^k}(\mathbb{R}^n)$ is defined as $H^{s+k}(\mathbb{R}^n)$ equipped with the norm :

$$|u|^2_{X^s_{\epsilon^k}} = |u|^2_{H^s} + \epsilon^k |u|^2_{H^{s+k}}. \tag{18}$$

k, and later k' are positive numbers which depend on the sign of (a,b,c,d).
For instance $(k,k') = (3,3)$ when $a, c < 0$, $b, d > 0$.

Recall that the (a,b,c,d) systems are linearly well-posed when

$$a \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0, \tag{19}$$

$$\text{or} \quad a = c > 0, \quad b \geq 0, \quad d \geq 0. \tag{20}$$

Theorem

Let $t_0 > \frac{n}{2}$, $s \geq t_0 + 2$ *if* $b \neq d$, $s \geq t_0 + 4$ *if* $b = d = 0$. *Let* a, b, c, d *satisfy the condition* (19). *Assume that* $\zeta_0 \in X^s_{\epsilon^k}(\mathbb{R}^n), \mathbf{v}_0 \in X^s_{\epsilon^{k'}}(\mathbb{R}^n)$ *satisfy the (non-cavitation) condition*

$$1 - \epsilon\zeta_0 \geq H > 0, \quad H \in (0,1), \tag{21}$$

Then there exists a constant $\tilde{c}_0$ *such that for any* $\epsilon \leq \epsilon_0 = \frac{1-H}{\tilde{c}_0(|\zeta_0|_{X^s_{\epsilon^k}} + |\mathbf{v}_0|_{X^s_{\epsilon^{k'}}})}$, *there exists* $T > 0$ *independent of* ϵ, *such that* (1)-(24) *has a unique solution* $(\zeta, \mathbf{v})^T$ *with* $\zeta \in C([0, T/\epsilon]; X^s_{\epsilon^k}(\mathbb{R}^n))$ *and* $\mathbf{v} \in C([0, T/\epsilon]; X^s_{\epsilon^{k'}}(\mathbb{R}^n))$. *Moreover,*

$$\max_{t \in [0, T/\epsilon]} (|\zeta|_{X^s_{\epsilon^k}} + |\mathbf{v}|_{X^s_{\epsilon^{k'}}}) \leq \tilde{c}(|\zeta_0|_{X^s_{\epsilon^k}} + |\mathbf{v}_0|_{X^s_{\epsilon^{k'}}}). \tag{22}$$

Here $\tilde{c} = C(H^{-1})$ *is a nondecreasing function of its argument.*

The idea of the proof is to perform a suitable symmetrization of a linearized system and then to implement a energy method on an approximate system. Our method is of "hyperbolic" spirit.

Setting $\mathbf{V} = (\zeta, \mathbf{v})^T$, $\mathbf{U} = (\eta, \mathbf{u})^T = \epsilon \mathbf{V}$, we rewrite (1) as

$$\begin{cases} (1 - b\epsilon\Delta)\partial_t \eta + \nabla \cdot \mathbf{u} + \nabla \cdot (\eta \mathbf{u}) + a\epsilon \nabla \cdot \Delta \mathbf{u} = 0, \\ (1 - d\epsilon\Delta)\partial_t \mathbf{u} + \nabla \eta + \frac{1}{2}\nabla(|\mathbf{u}|^2) + c\epsilon \nabla \Delta \eta = 0. \end{cases} \tag{23}$$

with the initial data

$$(\eta, \mathbf{u})^T|_{t=0} = (\epsilon\zeta_0, \epsilon\mathbf{v}_0)^T \tag{24}$$

If $b > 0, d \geq 0$ or $b = d = 0$, let $g(D) = (1 - b\epsilon\Delta)(1 - d\epsilon\Delta)^{-1}$, then (23) is equivalent after applying $g(D)$ to the second equation to the condensed system :

$$(1 - b\epsilon\Delta)\partial_t \mathbf{U} + M(\mathbf{U}, D)\mathbf{U} = 0, \tag{25}$$

where

$$M(\mathbf{U}, D) = \begin{pmatrix} \mathbf{u} \cdot \nabla & (1 + \eta + a\epsilon\Delta)\partial_{x_1} & (1 + \eta + a\epsilon\Delta)\partial_{x_2} \\ g(D)(1 + c\epsilon\Delta)\partial_{x_1} & g(D)(u_1\partial_{x_1}) & g(D)(u_2\partial_{x_1}) \\ g(D)(1 + c\epsilon\Delta)\partial_{x_2} & g(D)(u_1\partial_{x_2}) & g(D)(u_2\partial_{x_2}) \end{pmatrix}. \tag{26}$$

In order to solve the system (1)-(24), we consider the following linear system in $\mathbf{U}$

$$(1 - b\epsilon\Delta)\partial_t \mathbf{U} + M(\underline{\mathbf{U}}, D)\mathbf{U} = \mathbf{F}, \tag{27}$$

together with the initial data

$$\mathbf{U}|_{t=0} = \mathbf{U}_0, \tag{28}$$

The key point to solve the linear system (27)-(28) is to search a symmetrizer $S_{\underline{\mathbf{U}}}(D)$ of $M(\underline{\mathbf{U}}, D)$ such that the principal part of $iS_{\underline{\mathbf{U}}}(\xi)M(\underline{\mathbf{U}}, \xi)$ is self-adjoint and $S_{\underline{\mathbf{U}}}(\xi)$ is positive and self-adjoint under a smallness assumption on $\underline{\mathbf{U}}$. It is not difficult to find that :

(i) if $b = d$, $g(D) = 1$, $S_{\underline{\mathbf{U}}}(D)$ is defined by

$$\begin{pmatrix} 1 + c\epsilon\Delta & \underline{u}_1 & \underline{u}_2 \\ \underline{u}_1 & 1 + \underline{\eta} + a\epsilon\Delta & 0 \\ \underline{u}_2 & 0 & 1 + \underline{\eta} + a\epsilon\Delta \end{pmatrix}; \qquad (29)$$

(ii) if $b \neq d$, $S_{\underline{\mathrm{U}}}(D)$ is defined by

$$\begin{pmatrix} (1+c\epsilon\Delta)^2 g(D) & g(D)(\underline{u}_1(1+c\epsilon\Delta)) & g(D)(\underline{u}_2(1+c\epsilon\Delta)) \\ g(D)(\underline{u}_1(1+c\epsilon\Delta)) & (1+\underline{\eta}+a\epsilon\Delta)(1+c\epsilon\Delta) & 0 \\ g(D)(\underline{u}_2(1+c\epsilon\Delta)) & 0 & (1+\underline{\eta}+a\epsilon\Delta)(1+c\epsilon\Delta) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{u}_1\underline{u}_1 & \underline{u}_1\underline{u}_2 \\ 0 & \underline{u}_1\underline{u}_2 & \underline{u}_2\underline{u}_2 \end{pmatrix} (g(D)-1). \tag{30}$$

Then we define the energy functional associated with (27) as

$$E_s(\mathbf{U}) = \left((1-b\epsilon\Delta)\Lambda^s\mathbf{U}, S_{\underline{\mathbf{U}}}(D)\Lambda^s\mathbf{U}\right)_2. \tag{31}$$

One can show that $E_s(\mathbf{U})$ defined in (31) is trully a energy functional equivalent to some $X^s_{\epsilon^k}(\mathbb{R}^2)$ norm provided a smallness condition is imposed on $\underline{\mathbf{U}}$, which is satisfied (for ϵ small enough) if

$$1+\underline{\eta} \geq H > 0, \quad |\underline{\mathbf{U}}|_\infty \leq \kappa(H,a,b,c,d), \quad |\underline{\mathbf{U}}|_{H^s} \leq 1, \quad \text{for} \quad t \in [0,T']. \tag{32}$$

For the nonlinear system, if ϵ is small enough, this smallness condition holds for its solution $(\zeta, \mathbf{v})^T$, i.e., $(\eta, \mathbf{u})^T \equiv \epsilon(\zeta, \mathbf{v})^T$ satisfies (32).

- The main (painful) ask is to derive a priori estimates on the linearized system, in the various cases.

- Construction of the nonlinear solution by an iterative scheme :

We construct the approximate solutions $\{\mathbf{V}^n\}_{n\geq 0} = \{(\eta^n, \mathbf{v}^n)^T\}_{n\geq 0}$ with $\mathbf{U}^{n+1} = \epsilon \mathbf{V}^n$ solution to the linear system

$$(1 - b\epsilon\Delta)\partial_t \mathbf{U}^{n+1} + M(\mathbf{U}^n, D)\mathbf{U}^{n+1} = 0, \quad \mathbf{U}^{n+1}|_{t=0} = \epsilon \mathbf{V}_0 \equiv \mathbf{U}_0, \tag{33}$$

and with $\mathbf{U}^0 = \mathbf{U}_0$. Given $\mathbf{U}^n$ satisfying the above assumption, the linear system (33) is unique solvable.

- Our proof does not seem to work for the *strongly dispersive case* $b = d = 0$, $a = c = 1/6$, at least in the 2D case.
- The proofs using dispersion (that is high frequencies) do not take into account the algebra of the nonlinear terms. They allows initial data in relatively big Sobolev spaces but seem to give only existence times of order $O(1/\sqrt{\epsilon})$.
- The existence proofs on existence times of order $1/\epsilon$ are of "hyperbolic" nature.They do not take into account the dispersive effects (treated as perturbations).
- Is it possible to go till $O(1/\epsilon^2)$, or to get global existence. Plausible in one D (the Boussinesq systems should evolves into an uncoupled system of KdV equations). Not so clear in 2D... One should there use dispersion.

  Use of a normal form technique (*à la Germain-Masmoudi-Shatah*) ?

- For (possible) use in control theory, one should get similar results on domains such as half-planes, strips,...with the *ad hoc* boundary conditions.

HAPPY BIRTHDAY PETER !