New Global Solutions to the Lagrangian Averaged Navier-Stokes

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Incompressible Fluids, Turbulence, and Mixing
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Outline

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1. Introduction/History

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2. Approaches to Global Existence

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3. Statement of the theorem

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3. Statement of the theorem
4. Proof of the new global existence result

LANS equation

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- Incompressible Navier-Stokes Equation:

$$\partial_t u - \nu \triangle u + \text{div } (u \otimes u) = \nabla p$$
$$\text{div } u = 0, \quad u(0) = u_0.$$

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- Incompressible, isotropic Lagrangian Averaged Navier-Stokes Equation (more or less):

$$\partial_t u - \nu \triangle u + \text{div } (u \otimes u + \alpha^2 (1 - \alpha^2 \triangle)^{-1} (\nabla u \cdot \nabla u)) = \nabla p$$
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- Euler Equation Lagrangian:

$$L(F) = \int_I (u(t,x), u(t,x))_{L^2} dt$$

- F minimizes L if u satisfies the Euler equation:

$$\begin{aligned} \partial_t u + \text{div } (u \otimes u) &= 0, \\ \text{div } u &= 0. \end{aligned}$$

Lagrangian Averaging: Derivation of the LAE equations

- Averaged Lagrangian:

$$\tilde{L}(F) = \frac{1}{2}\int_0^T \int [(u,u)_{L^2} - \alpha^2(\triangle u, u)_{L^2}]dxdt.$$

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- F minimizes $\tilde{L}$ if u satisfies the Lagrangian Averaged Euler equation:

$$\partial_t u + \text{div}\ (u \otimes u + \alpha^2(1-\alpha^2\triangle)^{-1}(\nabla u \cdot \nabla u)) = \nabla p$$
$$\text{div}\ u = 0$$

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- Global Existence of LANS equation:
  - Marsden and Shkoller 2001 for $u_0 \in H^{3,2}(\mathbb{R}^3)$,
  - In previous work, we proved global existence for initial data $u_0 \in H^{3/4}(\mathbb{R}^3)$ and for $u_0 \in B_{2,q}^{(n/4)^+}(\mathbb{R}^n)$.

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- For Besov spaces, apply the Littlewood-Paley operator $\triangle_j$ to the equation, take the L^2 product with $\triangle_j u$, and finally sum over j.
- In general, requires the solution to be in an L^2-like space

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  3. Choose $W = [L^2, \dot{B}^{2/p-1}_{p,q}]_{\theta,q}$
- Technique gives unique global solution for arbitrary data in W.

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If X is a critical space, then $X \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$.

- Gallagher/Planchon result gives global solutions in $\dot{B}^{2/p-1}_{p,\infty}(\mathbb{R}^2)$ for large p.

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Theorem 1 (P.)

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- Replacement for the $\dot{B}^{2/p}_{p,q}(\mathbb{R}^2)$ result:

Theorem 2 (P.)

Let $v_0 \in B^{n/p}_{p,q}(\mathbb{R}^n)$ be divergence free. Then there exists a unique local solution v to the LANS equation with $v(0) = v_0$. If v_0 has a sufficiently small norm, v can be taken to be a global solution.

New Global Existence Result

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Theorem 3 (P.)

Let $U = B^{n/2}_{2,q}(\mathbb{R}^n)$ *and* $V = B^{n/p}_{p,q}(\mathbb{R}^n)$ *with* $3 \leq n < 6$. *Choose* $0 < \theta < 1$, *and define* s *and* $\tilde{p}$ *by*

$$s = \frac{n(1-\theta)}{2} + \frac{n\theta}{p}$$
$$\frac{1}{\tilde{p}} = \frac{1-\theta}{2} + \frac{\theta}{p}.$$

Then for any $w_0 \in B^s_{\tilde{p},q}(\mathbb{R}^n)$ *there exists a unique global solution to the LANS equation* $w \in BC([0,\infty) : B^s_{\tilde{p},q}(\mathbb{R}^n))$ *with* $w(0) = w_0$.

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4. Unify results

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- Let v be known global solution with $v(0) = v_0$. Then

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\begin{aligned}
&\partial_t(u+v) - \triangle(u+v) + \operatorname{div}\,((u+v)\otimes(u+v)) \\
&+ \operatorname{div}\,(1-\alpha^2\triangle)^{-1}((\nabla(u+v)+\nabla(u+v)^T) \\
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- Simplifies to (essentially)

$$\begin{aligned}\partial_t u - \triangle u + \operatorname{div}\,(u\otimes u + u\otimes v + v\otimes u)\\+\operatorname{div}\,(1-\alpha^2\triangle)^{-1}(\nabla u\nabla u + \nabla u\nabla v) = \nabla p,\end{aligned} \tag{1}$$

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- Called the modified LANS (mLANS) equation.

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Theorem 4 (P.)

Let $u_0 \in B^{n/2}_{2,q}(\mathbb{R}^n)$ and let v as above be given. Then there exists a local solution u to the mLANS equation such that

$$u \in BC([0,T) : B^{n/2}_{2,q}(\mathbb{R}^n)) \cap \dot{C}^T_{a;s_2,2,q},$$

where $a = (s_2 - n/2)/2$, $0 < s_2 - s_1 < 1$, and T depends only on $\|u_0\|_{n/2,2,q}$.

Proof of the Local Solution

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- By Dumahel's principle, LANS becomes

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$$\|uv\|_{B^s_{p,q}} \leq \|u\|_{B^{s_1}_{p_1,q}} \|v\|_{B^{s_2}_{p_2,q}},$$

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- Avoids taking a high-regularity norm of v.

Step 3: An *a priori* estimate

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Theorem 5 (P.)

Let u be a solution to the mLANS equation with small-data global LANS solution v. Then

$$\|u\|_{B^r_{2,q}} \leq \|u(0)\|_{B^r_{2,q}} \exp\Big(\int_0^t (\|u(s)\|_{B^{1+n/2}_{2,q}} + \|v(s)\|_{B^{1+n/p}_{p,q}})\Big) ds,$$

where $2 < r < 3$.

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- Global existence then follows from standard extension methods.

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- Re-state the mLANS as:

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\begin{aligned}
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  where

$$\begin{aligned}V &= \nabla_u(1-\alpha^2\triangle)u - \alpha^2(\nabla u)^T\cdot\triangle u\\ R_1 &= \nabla_u(1-\alpha^2\triangle)v\\ R_2 &= \nabla_v(1-\alpha^2\triangle)u\\ R_3 &= -\alpha^2(\nabla u)^T\cdot\triangle v\\ R_4 &= -\alpha^2(\nabla v)^T\cdot\triangle u.\end{aligned}$$

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where

$$\begin{aligned}
I_1 &= C\|\triangle_j u\|_2\|\triangle_j(\nabla_u(1-\alpha^2\triangle)v)\|_2,\\
I_2 &= C\|\triangle_j u\|_2\|\triangle_j(\nabla_v(1-\alpha^2\triangle)u)\|_2,\\
J_1 &= C\|\triangle_j u\|_2\|\triangle_j\alpha^2(\nabla u)^T\cdot\triangle v\|_2,\\
J_2 &= C\|\triangle_j u\|_2\|\triangle_j\alpha^2(\nabla v)^T\cdot\triangle u\|_2.
\end{aligned}$$

Proof of Theorem 5

- Apply $\triangle_j$ to both sides, take L^2 inner product with $2\triangle_j u$:

$$\frac{1}{2}\frac{d}{dt}(\|\triangle_j u\|_{L^2}^2 + \alpha^2\|A^{1/2}\triangle_j u\|_{L^2}^2) \leq W + I_1 + I_2 + J_1 + J_2,$$

where

$$\begin{aligned} I_1 &= C\|\triangle_j u\|_2\|\triangle_j(\nabla_u(1-\alpha^2\triangle)v)\|_2, \\ I_2 &= C\|\triangle_j u\|_2\|\triangle_j(\nabla_v(1-\alpha^2\triangle)u)\|_2, \\ J_1 &= C\|\triangle_j u\|_2\|\triangle_j\alpha^2(\nabla u)^T\cdot\triangle v\|_2, \\ J_2 &= C\|\triangle_j u\|_2\|\triangle_j\alpha^2(\nabla v)^T\cdot\triangle u\|_2. \end{aligned}$$

- W similar, only contains u terms.

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$$K_3 = \sum_{k\geq j-3} \triangle_j\left(\triangle_k u \sum_{l=-1}^{l=1} \nabla\triangle_{k+l} v\right)$$
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$$\tilde{M}_2 = \|\triangle_k v\|_{L^p} \sum_{m<k-2} 2^{3m}\|\triangle_m u\|_{L^2}$$

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$$\tilde{N} = (\sum_{l=-1}^{l=1} 2^{(3+n/p)(k+l)}\|\triangle_{k+l} v\|_{L^p}),$$

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is finite, which requires $r < 3$.

Constructing the new global solution to the LANS equation

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- So by uniqueness, $w = u + v$ is a global solution.