The Voigt Regularization for Inviscid Hydrodynamic Models - In honor of the 60th birthday of Peter Constantin -

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Outline

α-Models of Turbulence

Viscous Camassa-Holm Equations (NS-α, LANS-α)

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} - \displaystyle\sum_{j=1}^{3} v_j \nabla u_j = -\nabla\pi + \nu\triangle\mathbf{v} \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{v} = \mathbf{u} - \alpha^2\triangle\mathbf{u} \end{cases}$$

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- Leray-α Model (Cheskidov, Holm, Olson, Titi, 2005)
- Clark-α Model (Clark, Ferziger, Reynolds, 1979; C. Cao, Holm, Titi, 2005)
- Simplified Bardina Model (Layton, Lewandowski 2006; Y. Cao, Lunasin, Titi, 2006)
- Modified Leray-α Model (Ilyin, Lunasin, Titi, 2006)

The Simplified Bardina Model

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$$\begin{cases} \partial_t(\mathbf{u} - \alpha^2 \triangle \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \triangle(\mathbf{u} - \alpha^2 \triangle \mathbf{u}) + \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0. \end{cases}$$

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- Applications to image inpainting. (Ebrahimi, Holst, Lunasin, 2009)

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- Sabra Shell model of Turbulence (Constantin, Levant, Titi, 2007)
- Computational Study with Sabra Shell Model: Structure functions of the Navier-Stokes-Voigt regularization are investigated in comparison to the those of the Navier-Stokes in the context of Sabra Shell Model. (Levant, Ramos, Titi, 2009)

Navier-Stokes-Voigt: Sabra Shell Model

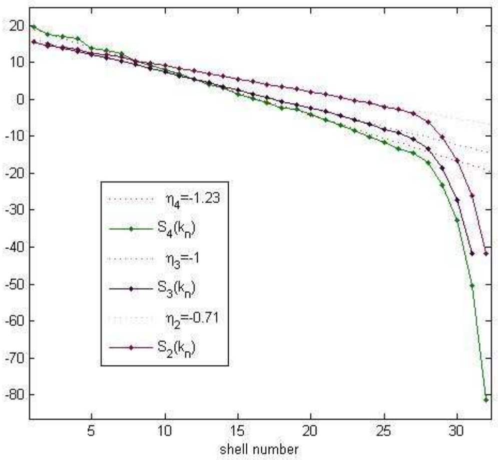
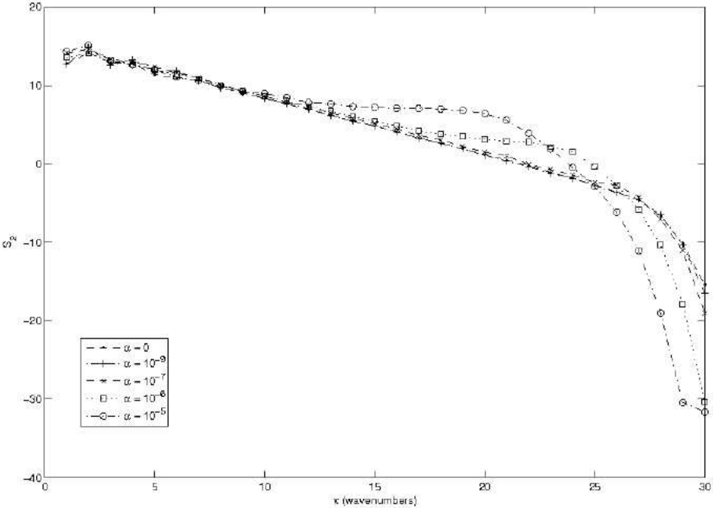
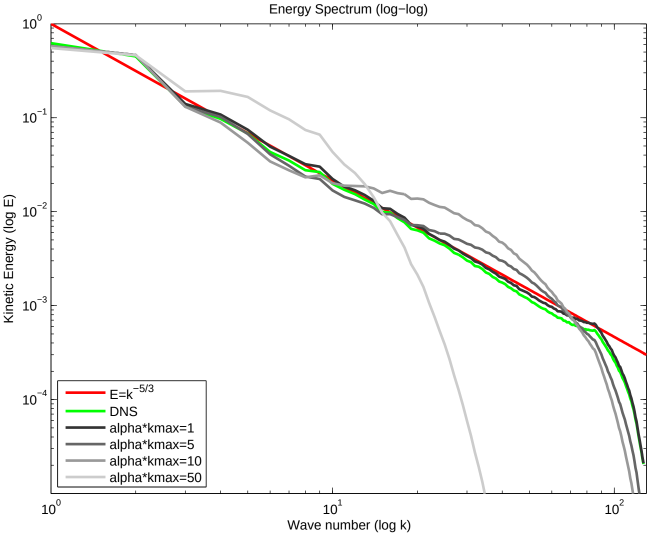


Image Credit: Levant, Ramos, Titi, Comm. Math. Sci., 2009.

Navier-Stokes-Voigt: Sabra Shell Model



Navier-Stokes-Voigt: 3D DNS Study



Joint with: Petersen, Wingate, Titi

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$$(\text{Voigt}) \begin{cases} -\alpha^2 u_{xxt} + u_t = \nu u_{xx} \text{ on } (0,2\pi)\times(0,T) \\ u(0,x) = u_0(x) \text{ on } (0,2\pi) \end{cases}$$

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$$\sum_{k\in\mathbb{Z}} (\alpha^2 k^2 + 1)\hat{u}_t^k e^{ikx} = \nu \sum_{k\in\mathbb{Z}} (-k^2)\hat{u}^k e^{ikx}$$

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- Coefficients do not decay exponentially as $t \to \infty$.
- As $k \to \infty$, we have the time scale α^2/ν.

The Euler-Voigt Model

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Energy Balance

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$$\begin{cases} -\alpha^2 \triangle \partial_t \mathbf{u} + \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0 \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0 \end{cases}$$

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Modified Energy Equality (Cao, Lunasin, Titi, 2006)

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Analytical Results: Regularity

$$(1) \qquad \begin{cases} -\alpha^2\partial_t\triangle\mathbf{u} + \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\triangle\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \end{cases}$$

Theorem (Global Existence and Uniqueness)(Y. Cao, Lunasin, Titi, 2006)

Let $\mathbf{u}_0 \in H^1$, $\nu \geq 0$. Then system (1) *has a unique solution in $C^1((-\infty,\infty), H^1)$ under either periodic or (if $\nu > 0$) homogeneous Dirichlet (no-slip) boundary conditions.*

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Theorem (H^s Regularity and Analyticity)(Larios, Titi, 2010)

Let $\mathbf{u}_0 \in H^s$, $s \geq 0$, $\nu \geq 0$. *Then system* (1) *has a unique solution in* $C^1((-\infty,\infty), V\cap H^s)$*, under periodic boundary conditions. Furthermore, if* $\mathbf{u}_0 \in V \cap C^\omega$*, then* $\mathbf{u} \in C^1((-\infty,\infty), V\cap C^\omega)$.

Analytical Results: Convergence

- Given initial data $\mathbf{u}_0 \in H^s$, $s \geq 3$.
- Let $\mathbf{u}$ be a solution to the Euler equations with initial data $\mathbf{u}_0$.
- Let $\mathbf{u}^\alpha$ be a solution of the Euler-Voigt equations with initial data $\mathbf{u}_0$.

Theorem (Convergence)(Larios, Titi, 2010)

Suppose $\mathbf{u} \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ *for* $s \geq 3$. *Then* $\mathbf{u}^\alpha \to \mathbf{u}$ *in* $L^\infty([0,T], L^2)$.

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Specifically,

$$\|\mathbf{u}(t) - \mathbf{u}^\alpha(t)\|_{L^2}^2 + \alpha^2 \|\nabla(\mathbf{u}(t) - \mathbf{u}^\alpha(t))\|_{L^2}^2 \leq C\alpha^2.$$

Analytical Results: Blow-Up Criterion

Consider the α-energy equality on $[0, T]$, an interval of existence and uniqueness for the 3D Euler equations. For $t \in [0, T]$,

$$\|\mathbf{u}^\alpha(t)\|_{L^2}^2 + \alpha^2 \|\nabla \mathbf{u}^\alpha(t)\|_{L^2}^2 = \|\mathbf{u}_0\|_{L^2}^2 + \alpha^2 \|\nabla \mathbf{u}_0\|_{L^2}^2$$

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Theorem (Blow-up Criterion)(Larios, Titi, 2010)

Suppose there exists a finite time $T_* > 0$ *such that*

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Then the Euler equations with initial data $\mathbf{u}_0$ *develop a singularity in the interval* $[0, T_*]$.

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Remark

Unlike the Beale-Kato-Majda criterion, in which one tracks a quantity arising from an equation which is not known to be well-posed, here we track a quantity which arises from a well-posed equation.

Surface Quasi-Geostrophic Equations

$$
\begin{aligned}
-\alpha^2\partial_t\triangle\theta + \partial_t\theta + (\mathbf{v}\cdot\nabla)\theta &= 0 \\
\mathbf{v} &= \nabla^{\perp}(-\triangle)^{-1/2}\theta \\
\theta(\mathbf{x},0) &= \theta_0(\mathbf{x})
\end{aligned}
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Khouider, Titi, 2008

- Global Regularity
- Convergence
- Blow-up criterion

Analytical Results

The 3D Magnetohydrodynamic-Voigt Model (Inviscid, Irresistive)

(2)
$$
\begin{cases}
\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \frac{1}{2}\nabla|\mathcal{B}|^2 = (\mathcal{B}\cdot\nabla)\mathcal{B}, \\
\partial_t \mathcal{B} + (\mathbf{u}\cdot\nabla)\mathcal{B} + \nabla q = (\mathcal{B}\cdot\nabla)\mathbf{u}, \\
\nabla\cdot\mathcal{B} = \nabla\cdot\mathbf{u} = 0, \\
\mathcal{B}(0) = \mathcal{B}_0,\ \mathbf{u}(0) = \mathbf{u}_0.
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Theorem (Global Regularity)(Larios, Titi, 2010)

Let $\mathbf{u}_0, \mathcal{B}_0 \in H^s$, *for* $s \geq 1$. *Then* (2) *has a unique solution* $(\mathbf{u}, \mathcal{B}) \in C^1((-\infty,\infty), H^s)$.

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Independently studied by Catania, Secchi, 2010, with strong initial data.

The Boussinesq Equations

Momentum Equation

$$\underbrace{\frac{\partial}{\partial t}\mathbf{u}}_{\textit{Acceleration}} + \underbrace{(\mathbf{u}\cdot\nabla)\mathbf{u}}_{\textit{Advection}} = \underbrace{-\nabla p}_{\substack{\textit{Pressure}\\\textit{Gradient}}} + \underbrace{\nu\triangle\mathbf{u}}_{\substack{\textit{Viscous}\\\textit{Diffusion}}} + \underbrace{\mathbf{k}\theta}_{\substack{\textit{Buoyancy}\\\textit{Force}}}$$

Continuity Equation

$$\nabla\cdot\mathbf{u} = 0$$

Transport Equation

$$\frac{\partial}{\partial t}\theta + \underbrace{(\mathbf{u}\cdot\nabla)\theta}_{\substack{\textit{Transport}\\\textit{by Velocity}}} = \underbrace{\kappa\triangle\theta}_{\substack{\textit{e.g., Thermal}\\\textit{Diffusion}}}$$

$\mathbf{u}$:= Velocity (vector field)

ν := Kinematic Viscosity

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The Boussinesq-Voigt Equations (2D or 3D)

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Let $\mathbf{u}_0 \in H^2$, $\theta_0 \in H^2$. Then there exists a unique global solution to (4).

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Theorem (Modified Brézis-Gallouet Inequality)(Larios, Lunasin, Titi, 2011)

For every $\epsilon > 0$, sufficiently small, and $\mathbf{w} \in H^2(\mathbb{T}^2)$,

$$\|\mathbf{w}\|_{L^\infty} \leq C\left(\|\nabla\mathbf{w}\|_{L^2}\epsilon^{-1/4} + \|\triangle\mathbf{w}\|_{L^2}e^{-1/\epsilon^{1/4}}\right),$$

where C is independent of ϵ.

Convergence as $\alpha \to 0$

Theorem (Larios, Lunasin, Titi, 2011)

Given initial data $\mathbf{u}_0, \theta_0 \in H^3$, *choose an arbitrary* $T \in (0, T_{max})$, *where* T_{max} *is the maximal time for which a solution to the Boussinesq-Voigt equations exists and is unique. Then* $\mathbf{u}^\alpha \to \mathbf{u}$ *in* $L^2([0,T], H^1)$ *and* $\theta^\alpha \to \theta$ *in* $L^2([0,T], L^2)$.

Blow-up Criterion

Energy Balance Equation

$$\alpha^2\|\mathbf{u}^\alpha(t)\|^2 + |\mathbf{u}^\alpha(t)|^2 = \alpha^2\|\mathbf{u}_0\|^2 + |\mathbf{u}_0|^2 + 2\int_0^t (\theta^\alpha(s)\mathbf{k}, \mathbf{u}^\alpha(s))\, ds.$$

Theorem (Larios, Lunasin, Titi, 2011)

Given initial data $\mathbf{u}_0, \theta_0 \in H^3$*, suppose that for some* $T_* < \infty$*, we have*

$$\sup_{t\in[0,T_*)} \limsup_{\alpha\to 0} \alpha^2\|\mathbf{u}^\alpha(t)\|^2 > 0.$$

Then the solutions to the 2D Boussinesq become singular in the time interval $[0, T_*)$.

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- Similar results for SQG, MHD, and Boussinesq equations.

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- Extend the Voigt-regularization to other fluid models.

Happy Birthday Professor Constantin!