Nonlocal maximum principles for active scalars

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Introduction and background

Active scalars:

$$\theta_t = (u \cdot \nabla)\theta - (-\Delta)^{\alpha}\theta, \;\; \theta(x,0) = \theta_0(x),$$

where the vector field u is determined from θ. We will usually consider the equation on $\mathbb{R}^d$ or $\mathbb{T}^d$, with $0 \le \alpha \le 1$.

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3. The surface quasi-geostrophic equation (SQG): $d = 2$, $0 \leq \alpha \leq 1$, $u = R^{\perp}\theta \equiv \nabla^{\perp}(-\Delta)^{-1/2}\theta$. Constantin-Majda-Tabak (1994). 4. The modified SQG: $d = 2$, $u = \nabla^{\perp}(-\Delta)^{-\gamma}\theta$, $1/2 < \gamma < 1$. Interpolates between 2D Euler and SQG equations.

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5. The Hilbert transform model: $d = 1$, $u = H\theta$, where $H\theta$ is the Hilbert transform of θ. Cordoba-Cordoba-Fontelos (2005)

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Complete answer in critical case: K-Nazarov-Volberg and Caffarelli-Vasseur.

Theorem (KNV)

Assume that the initial data θ_0 *is smooth and periodic. Then the critical SQG (and Burgers, and Hilbert) equation has a unique global solution which is smooth and real analytic in* x *for any* $t > 0$. *Moreover,*

$$\|\nabla\theta(x,t)\|_{L^\infty} \leq \|\nabla\theta_0\|_{L^\infty} \exp\exp(C\|\theta_0\|_{L^\infty}).$$

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Idea of the proof.

Definition

$\omega(\xi)$ is a modulus of continuity if ω is $(0,\infty) \mapsto (0,\infty)$, increasing, concave, piecewise C^2. $f(x)$ obeys ω if $|f(x) - f(y)| < \omega(|x-y|)$ for all x, y.

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We find ω that is preserved by critical SQG evolution: if θ_0 obeys it, so does $\theta(x,t)$ for every $t > 0$.

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Properties: $\omega(0) = 0$, $\omega'(0) = 1$, $\omega''(0) = -\infty$, $\omega(\xi) \sim \log\log\xi$ for large ξ.

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K-Nazarov 2008: a third approach.

Theorem

Assume that $\theta(x,t)$, $u(x,t)$ *are* $C^\infty(\mathbb{T}^d)$ *for all* $t \in [0,T]$, *and that*

$$\theta_t = (u \cdot \nabla)\theta - (-\Delta)^{1/2}\theta \tag{1}$$

holds for any $t \geq 0$. *Assume that the velocity* u *is divergence free and satisfies a uniform bound* $\|u(\cdot,t)\|_{\mathrm{BMO}} \leq B$ *for* $t \in [0,T]$. *Then there exists* $\beta = \beta(B,d) > 0$ *such that*

$$\|\theta(x,t)\|_{C^\beta(\mathbb{T}^d)} \leq C(\theta(x,0))$$

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The method is based on dualizing the equation (1) with appropriate class of test functions, and then studying the evolution of these test functions.
2011: Constantin-Vicol, nonlinear maximum principle.

The supercritical case

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(1) Conditional regularity. Constantin, Wu 2008: if a weak solution $\theta(x,t)$ of the supercritical SQG satisfies $\theta(x,t) \in C([t_0,t], C^\beta)$, $\beta > 1-2\alpha$, then $\theta(x,t) \in C((t_0,t], C^\infty)$.

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(2) Finite time regularization. Silvestre 2009: finite time regularization of slightly supercritical SQG equation: if $\alpha = \frac{1}{2} - \epsilon$, then there exists $T = T(\alpha, \theta_0)$ such that the solution is smooth for $t > T$. A similar result for the Burgers equation by Chan, Czubak and Silvestre (2010).

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(3) Global regularity for slightly supercritical SQG. Dabkowski-K-Vicol 2011.

The very slightly supercritical SQG

Let $m(\xi)$ be a smooth, radial, positive, non-decreasing function on $\mathbb{R}^2$ satisfying

$$\lim_{\xi\to\infty} \frac{m(\xi)}{\log\log|\xi|} = 0, \quad |\xi|^k |\partial_\xi^k m(\xi)| \leq Cm(\xi) \tag{2}$$

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$$\partial_t \theta = (u \cdot \nabla)\theta - (-\Delta)^{1/2}\theta, \;\; \theta(x, 0) = \theta_0(x), \tag{3}$$

where $u = \nabla^{\perp} \Lambda^{-1} m(\Lambda)$, $\Lambda = (-\Delta)^{1/2}$.

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The supercriticality is very slight. But it does destroy the scaling.

The very slightly supercritical SQG: origins of the loglog

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Assume that θ_0 obeys $\omega : |\theta_0(x) - \theta_0(y)| < \omega(|x-y|)$.
If the solution $\theta(x,t)$ ever loses ω, there must exist t_1, x, y such that $\theta(x,t_1) - \theta(y,t_1) = \omega(|x-y|)$ while $\theta(x,t)$ obeys ω for $t < t_1$. Let $|x-y| \equiv \xi$, $e = (y-x)/|y-x|$.

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Consider

$$\partial_t \left(\theta(x,t) - \theta(y,t)\right)|_{t=t_1} = \text{flow term} + \text{dissipation term},$$

where the flow term is equal to

$$(u\cdot\nabla)\theta(x,t_1) - (u\cdot\nabla)\theta(y,t_1) \leq \Omega(\xi)\omega'(\xi),$$

where $\Omega(\xi)$ is such that $|(u(x)-u(y))\cdot e| \leq \Omega(|x-y|)$.
For the SQG, one can use

$$\Omega(\xi) = A\left(\int_0^\xi \frac{\omega(\eta)}{\eta}d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)}{\eta^2}d\eta\right)$$

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The diffusion term is equal to

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where

$$D(\xi)=\frac{1}{\pi}\left(\int_0^{\xi/2}\frac{\omega(\xi+2\eta)+\omega(\xi-2\eta)-2\omega(\xi)}{\eta^2}\,d\eta+\right.$$
$$\left.\int_{\xi/2}^{\infty}\frac{\omega(\xi+2\eta)-\omega(2\eta-\xi)-2\omega(\xi)}{\eta^2}\,d\eta\right).$$

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Therefore,

$$\partial_t\left(\theta(x,t)-\theta(y,t)\right)\big|_{t=t_1} \le \Omega(\xi)\omega'(\xi) + D(\xi).$$

If $\Omega(\xi)\omega'(\xi) + D(\xi) < 0$ for all $\xi > 0$, then ω is conserved by evolution.

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For large ξ, the balance that emerges is $\Omega(\xi)\omega'(\xi)$ vs $c\omega(\xi)/\xi$. Given that for large ξ, $\Omega(\xi) \lesssim \omega(\xi)\log\xi$, this dictates at most double logarithmic growth for ω.

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And indeed, one can find a modulus of continuity ω that is growing as double logarithm and is conserved by evolution.

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And indeed, one can find a modulus of continuity ω that is growing as double logarithm and is conserved by evolution.

It turns out that one can trade some of this growth for supercriticality.

When we introduce the modified $u = \nabla^{\perp}\Lambda^{-1}m(\Lambda)$, $\Omega(\xi)$ becomes larger:

$$\Omega(\xi) = A\left(\int_0^{\xi} \frac{\omega(\eta)m(\eta^{-1})}{\eta}d\eta + \xi\int_{\xi}^{\infty} \frac{\omega(\eta)m(\eta^{-1})}{\eta^2}d\eta\right).$$

The very slightly supercritical SQG: origins of the loglog

For large ξ, the balance that emerges is $\Omega(\xi)\omega'(\xi)$ vs $c\omega(\xi)/\xi$. Given that for large ξ, $\Omega(\xi) \lesssim \omega(\xi)\log\xi$, this dictates at most double logarithmic growth for ω.
And indeed, one can find a modulus of continuity ω that is growing as double logarithm and is conserved by evolution.
It turns out that one can trade some of this growth for supercriticality.

When we introduce the modified $u = \nabla^\perp \Lambda^{-1} m(\Lambda)$, $\Omega(\xi)$ becomes larger:

$$\Omega(\xi) = A\left(\int_0^\xi \frac{\omega(\eta)m(\eta^{-1})}{\eta}d\eta + \xi\int_\xi^\infty \frac{\omega(\eta)m(\eta^{-1})}{\eta^2}d\eta\right).$$

We can no longer construct a single modulus ω and then use scaling.
Instead, we need to construct a family of moduli ω_B conserved by evolution such that every smooth initial data will obey ω_B for some B.

The very slightly supercritical SQG

The family ω_B can be defined as follows. Let κ be a sufficiently small parameter. Define $\delta(B)$ by

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$$\omega_B'(\xi) = \begin{cases} B - \frac{B^2}{8\kappa}\xi m(\xi^{-1})\left(4 + \log\frac{\delta(B)}{\xi}\right), & 0 < \xi \le \delta(B) \\ \frac{\gamma}{\xi(4+\log(\xi/\delta(B)))m(\delta(B)^{-1})} & \xi > \delta(B). \end{cases}$$

One can check that if κ, γ are sufficiently small, then ω_B is conserved by evolution.

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Given initial data θ_0, we need $\omega_B\left(2\|\theta_0\|_{L^\infty}/\|\nabla\theta_0\|_{L^\infty}\right) \geq 2\|\theta_0\|_{L^\infty}$ for θ_0 to obey ω_B.

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But for any fixed a,

$$\int_{\delta(B)}^{a} \frac{\gamma}{\xi(4+\ln(\xi/\delta(B)))m(\delta(B)^{-1})} d\xi = \frac{\gamma}{m(\delta(B)^{-1})}\ln(1+\ln(a/\delta(B))) \to \infty.$$

So any θ_0 obeys ω_B with B large enough.

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For example, finite time regularization is well known and straightforward for Navier-Stokes equations in 3D.

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Consider a regularized active scalar

$$\partial_t \theta = (u \cdot \nabla)\theta - (-\Delta)^\alpha \theta + \epsilon \Delta \theta, \;\; \theta(x,0) = \theta_0(x), \;\; \epsilon > 0.$$

A viscosity solution of active scalar is a weak solution which is a limit of a sequence of regularized solutions as $\epsilon \to 0$.

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A viscosity solution of active scalar is a weak solution which is a limit of a sequence of regularized solutions as $\epsilon \to 0$.
I will focus on the Burgers ($u = \theta$, set on $\mathbb{T}^1$) and SQG ($u = \nabla^\perp (-\Delta)^{-1/2}\theta$, set on $\mathbb{T}^2$) cases.

The main application: statement

Theorem

Assume $0 < \alpha < 1/2$ *and* θ_0 *is periodic and smooth. Let* $\theta(x, t)$ *be viscosity solution of the Burgers or SQG equation. Then there exist* $0 < T_1(\alpha, \theta_0) \leq T_2(\alpha, \theta_0) < \infty$ *such that* $\theta(x, t)$ *is smooth for* $0 < t < T_1$ *and* $t > T_2$.

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The proofs for all cases are quite similar, except in the SQG case there is an additional, non-trivial difficulty to resolve.

The general criterion

Theorem

Let $\theta(x,t)$ *be a periodic smooth solution of a regularized active scalar equation. Suppose that* $\omega(\xi,t)$ *is continuous on* $(0,\infty)\times[0,T]$*, piecewise* C^1 *in time variable and that for each fixed* $t\geq 0$*,* $\omega(\cdot,t)$ *is a modulus of continuity. Let the initial data* $\theta_0(x)$ *obey* $\omega(\xi,0)$*. Then* $\theta(x,T)$ *obeys the modulus of continuity* $\omega(\xi,T)$ *provided that* $\omega(\xi,t)$ *satisfies*

$$\partial_t\omega(\xi,t) > \Omega(\xi,t)\partial_\xi\omega(\xi,t) + D_\alpha(\xi,t) + 2\epsilon\partial^2_{\xi\xi}\omega(\xi,t)$$

for all $\xi>0,\ T\geq t>0$.

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Here $\Omega(\xi,t)$ is determined as previously (but may now depend on time). The $D_\alpha(\xi,t)$ term is similar to the dissipative $D(\xi)$ term appearing in the critical case; η^2 in the denominator needs to be replaced with $\eta^{1+2\alpha}$.

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The supercritical Burgers: finite time regularization

Here again $\Omega(\xi) = \omega(\xi)$. Define a modulus of continuity

$$\omega(\xi) = \begin{cases} H(\xi/\delta)^{\beta}, & 0 \le \xi \le \delta \\ H, & \xi > \delta \end{cases} \tag{4}$$

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The first observation is

Proposition

Consider the supercritical ($0 < \alpha < 1/2$) Burgers (regularized) equation. Fix $1 > \beta > 1 - 2\alpha$. There exists a constant $C_1 = C_1(\alpha, \beta)$ such that if $H \le C_1 \delta^{1-2\alpha}$, then the equations preserve ω given by (4), independently of $\epsilon > 0$.

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All we need to show here is that $\omega(\xi)\omega'(\xi) + D_\alpha(\xi) < 0$ for all $\xi > 0$.

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All we need to show here is that $\omega(\xi)\omega'(\xi) + D_\alpha(\xi) < 0$ for all $\xi > 0$.

The Proposition is of course not sufficient for global regularity due to the restriction $H \le C_1 \delta^{1-2\alpha}$ (not every initial data will obey such modulus of continuity).

The supercritical Burgers: finite time regularization

Define the following derivative family of "moduli of continuity"

$$\omega(\xi, \xi_0) = \begin{cases} \beta H \delta^{-\beta} \xi_0^{\beta-1} \xi + (1-\beta) H \delta^{-\beta} \xi_0^{\beta}, & 0 < \xi < \xi_0 \\ H(\xi/\delta)^{\beta}, & \xi_0 \leq \xi \leq \delta \\ H, & \xi > \delta \end{cases}$$

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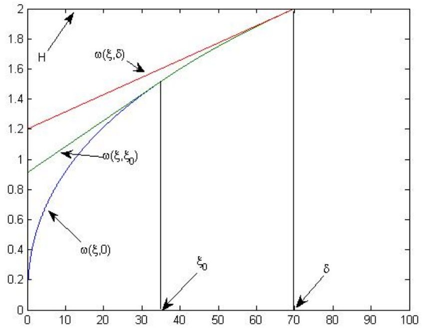

To the left is the sketch of $\omega(\xi,\xi_0)$, $\omega(\xi,\delta)$ and $\omega(\xi,0)$. The modulus $\omega(\xi,0) = H(\xi/\delta)^{\beta}$ is just Hölder on $0 < \xi < \delta$. The modulus $\omega(\xi,\delta)$ is piecewise linear and $\omega(0,\delta) = (1-\beta)H > 0$.

The supercritical Burgers: finite time regularization

Theorem

Let $0 < \alpha < 1/2$, $\beta > 1 - 2\alpha$, $\epsilon > 0$. *Assume that the initial data* $\theta_0(x)$ *for the Burgers (regularized) equation obeys* $\omega(\xi, \delta)$. *Then there exist positive constants* $C_{1,2} = C_{1,2}(\alpha, \beta)$ *such that if* $\xi_0(t)$ *is a solution of*

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and $H \leq C_1 \delta^{1-2\alpha}$, *then the solution* $\theta(x,t)$ *obeys* $\omega(\xi, \xi_0(t))$ *for all* t. *All constants and statements are independent of* $\epsilon > 0$.

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To prove this Theorem, we need to show that for $\omega(\xi, \xi_0(t))$ we have

$$\partial_t \omega > \omega \partial_\xi \omega + D_\alpha(\xi),$$

for all $\xi > 0$ and $0 \leq t \leq T$ - provided that C_1 and C_2 are sufficiently small.

The supercritical SQG case: the main issue

If $\omega(\xi)$ has a jump at $\xi = 0$, we only control L^∞ norm of θ. For the SQG equation, the components of u are Riesz transforms of θ, and we control them only in BMO.

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Idea: a better dissipation estimate to control potential singularities in u.

Suppose $x - y$ is directed along the first coordinate. The estimate

$$-(-\Delta)^\alpha \theta(x, t_1) + (-\Delta)^\alpha \theta(y, t_1) \leq D_\alpha(\xi, t)$$

is only sharp if $\theta(x_1, x_2, t_1) = \frac{1}{2}\omega(2x_1, t_1)$ if $x_1 > 0$ and odd in x_1.

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But observe that in this case $u(x, t_1)$ and $u(y, t_1)$ are perpendicular to $x - y$ and so nonlinear term vanishes!

Set $x = (\xi/2, 0)$ and $y = (-\xi/2, 0)$.

Lemma

Assume that x, y, ξ, and t_1 are in breakthrough scenario. Then $-(-\Delta)^\alpha\theta(x, t_1) + (-\Delta)^\alpha\theta(y, t_1) \le D_\alpha(\xi, t_1) + D_\alpha^\perp(\xi, t_1)$, *where*

$$D_\alpha^\perp(\xi, t_1) \le -C \int_{(\frac{1}{2}-c)\xi}^{(\frac{1}{2}+c)\xi} d\eta \int_{-c\xi}^{c\xi} \frac{\omega(2\eta, t_1) - \theta(\eta, \nu, t_1) + \theta(-\eta, \nu, t_1)}{\left(\left(\frac{\xi}{2} - \eta\right)^2 + \nu^2\right)^{1+\alpha}} \, d\nu.$$

Here $C, c > 0$ are fixed constants that may depend only on α.

Observe that we always have $D_\alpha^\perp(\xi, t_1) \le 0$.

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Observe that we always have $D_\alpha^\perp(\xi, t_1) \leq 0$.
On the other hand, recall that

$$u(x) - u(y) = C\left(P.V. \int \frac{(x-z)^\perp}{|x-z|^3}\theta(z)\, dz - P.V. \int \frac{(y-z)^\perp}{|y-z|^3}\theta(z)\, dz\right).$$

Let Q_x, Q_y be small squares with centers at x, y and side length $2c\xi$.

The supercritical SQG case: the key estimate

The integrals in z away from Q_x and Q_y respectively can be estimated by

$$A\left(\xi \int_{\xi}^{\infty} \frac{\omega(r)}{r^2}\, dr + \omega(\xi)\right)$$

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The potentially singular part is

$$\left|\int_{Q_x}\frac{(x-z)^{\perp}\cdot e}{|x-z|^3}\,\theta(z)\,dz-\int_{Q_y}\frac{(y-z)^{\perp}\cdot e}{|y-z|^3}\,\theta(z)\,dz\right|=$$

$$\left|\int_{(\frac{1}{2}-c)\xi}^{(\frac{1}{2}+c)\xi}d\eta\int_{-c\xi}^{c\xi}\frac{\nu}{\left(\left(\frac{\xi}{2}-\eta\right)^2+\nu^2\right)^{3/2}}(\theta(\eta,\nu)-\theta(-\eta,\nu))\,d\nu\right|$$

$$=\left|\int_{(\frac{1}{2}-c)\xi}^{(\frac{1}{2}+c)\xi}d\eta\int_{0}^{c\xi}\frac{\nu(\theta(\eta,\nu)-\theta(-\eta,\nu)-\theta(\eta,-\nu)+\theta(-\eta,-\nu))}{\left(\left(\frac{\xi}{2}-\eta\right)^2+\nu^2\right)^{3/2}}\,d\nu\right|$$

The supercritical SQG case: the key estimate

Recall that $D_\alpha^\perp(\xi)$ is equal to

$$-C \int_{(\frac{1}{2}-c)\xi}^{(\frac{1}{2}+c)\xi} d\eta \int_0^{c\xi} \frac{2\omega(2\eta) - \theta(\eta,\nu) + \theta(-\eta,\nu) - \theta(\eta,-\nu) + \theta(-\eta,-\nu)}{\left(\left(\frac{\xi}{2}-\eta\right)^2 + \nu^2\right)^{1+\alpha}} \, d\nu.$$

Now on the region of integration

$$0 < \frac{\nu}{\left(\left(\frac{\xi}{2}-\eta\right)^2 + \nu^2\right)^{3/2}} \leq \frac{C\xi^{2\alpha}}{\left(\left(\frac{\xi}{2}-\eta\right)^2 + \nu^2\right)^{1+\alpha}}$$

and

$$|\theta(\eta,\nu) - \theta(-\eta,\nu) - \theta(\eta,-\nu) + \theta(-\eta,-\nu)| \leq$$
$$2\omega(2\eta) - \theta(\eta,\nu) + \theta(-\eta,\nu) - \theta(\eta,-\nu) + \theta(-\eta,-\nu).$$

The supercritical SQG case: finite time regularization

Lemma

Suppose that $u = \nabla^{\perp}(-\Delta)^{-1/2}\theta$, ω *is a modulus of continuity, and* x, y, ξ *are as above (we are in a breakthrough scenario). Let* $e = \frac{x-y}{|x-y|}$. *Then we have*

$$|(u(x) - u(y)) \cdot e| \leq \Omega(\xi), \tag{5}$$

where

$$\Omega(\xi) = A\left(-\xi^{2\alpha} D_{\alpha}^{\perp}(\xi) + \xi \int_{\xi}^{\infty} \frac{\omega(r)}{r^2}\, dr + \omega(\xi)\right).$$

Thus potential singularities of the left hand side of (5) can be controlled by $D_{\alpha}^{\perp}(\xi)$.

The supercritical SQG case: finite time regularization

Lemma

Suppose that $u = \nabla^\perp(-\Delta)^{-1/2}\theta$, ω *is a modulus of continuity, and* x, y, ξ *are as above (we are in a breakthrough scenario). Let* $e = \frac{x-y}{|x-y|}$. *Then we have*

$$|(u(x) - u(y)) \cdot e| \le \Omega(\xi), \tag{5}$$

where

$$\Omega(\xi) = A\left(-\xi^{2\alpha} D_\alpha^\perp(\xi) + \xi \int_\xi^\infty \frac{\omega(r)}{r^2}\, dr + \omega(\xi)\right).$$

Thus potential singularities of the left hand side of (5) can be controlled by $D_\alpha^\perp(\xi)$. Now the rest of the proof of the finite time regularization goes through as in the Burgers case.

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For the conservative SQG, no infinite time results. K-Nazarov: if $s > 11$, then for every A there exists θ_0 such that $\|\theta_0\|_{H^s} \leq 1$, but $\limsup_{t\to\infty} \|\theta(\cdot, t)\|_{H^s} \geq A$.

Happy Birthday Peter!