Conditional regularity of solutions of the 3D Navier-Stokes equations & implications for intermittency

"Peter Constantin's 60th birthday meeting at CMU"

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Summary of this talk

Consider the **3D Navier-Stokes equations** set in a periodic domain $[0, L]^3$

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f}(\boldsymbol{x}) \qquad \mathrm{div}\,\boldsymbol{u} = \mathrm{div}\,\boldsymbol{f} = 0$$

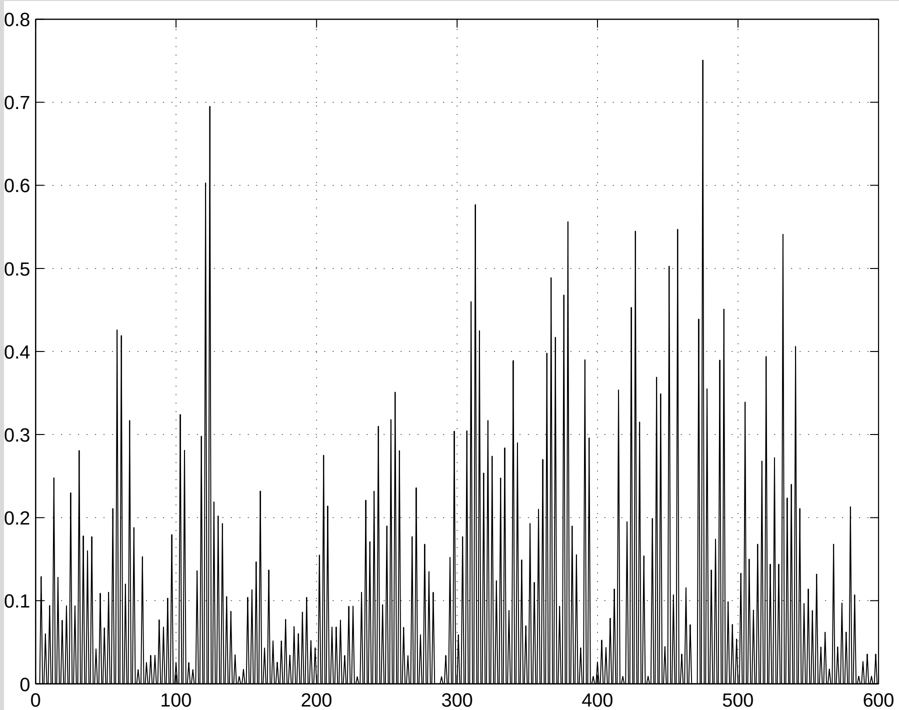
Intermittency may be the key to understanding the NSE :

(a) Intermittent events are manifest as violent spiky surges away from space-time averages in both vorticity & strain. Spectra have non-Gaussian characteristics – see Batchelor & Townsend 1949.

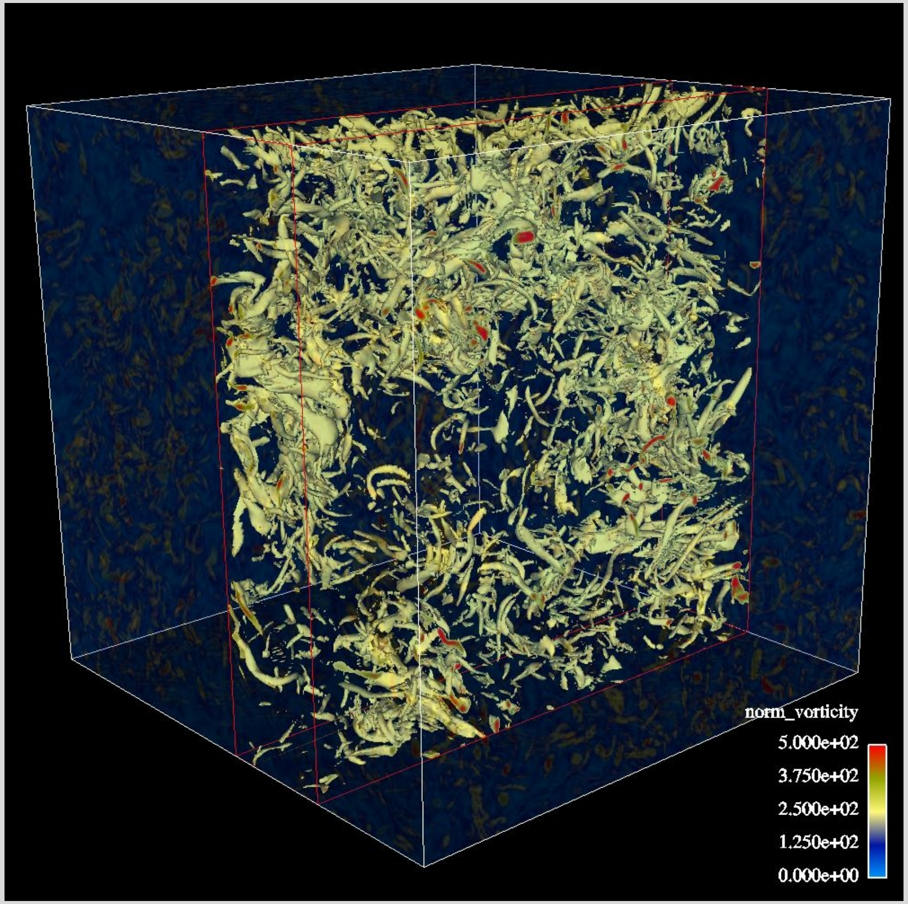
(b) Are the spikes smooth down to some small scale or does vorticity cascade down to scales where the NSE are invalid? Clearly this is connected with the regularity problem.

(i) Estimate resolution lengths for weak solutions.

(ii) Address the problem of conditional regularity in a new way.



Dissipation-range intermittency from wind tunnel turbulence where hot wire anemometry has been used to measure the longitudinal velocity derivative at a single point (D. Hurst & J. C. Vassilicos). The horizontal axis spans 8 integral time scales with $Re_\lambda \sim 200$.



Vorticity iso-surfaces in a 512^3 sub-domain of the LANL decaying 2048^3 NS-simulation at $Re_\lambda \sim 200$. Uneven clustering results in **intermittency**; Batchelor & Townsend (1949); Kuo & Corrsin (1971), Sreenivasan & Meneveau (1988, 1991); Frisch (1995): courtesy of Darryl Holm.

Physicists use **velocity structure functions** to study intermittency

$$\langle\, |u(\boldsymbol{x}+\boldsymbol{r}) - u(\boldsymbol{x})|^p \,\rangle_{ens.av.} \sim r^{\zeta_p}$$

Use of higher moments of vorticity for the NS equations

How might we pick up intermittent behaviour? Consider higher moments of vorticity ($m \geq 1$) as the "frequencies"

$$\Omega_m(t) = \left(L^{-3} \int_{\mathcal{V}} |\boldsymbol{\omega}|^{2m} dV \right)^{1/2m} + \varpi_0$$

The basic frequency associated with the domain is given by $\varpi_0 = \nu L^{-2}$.

$$\Omega_1^2 = L^{-3} \int_{\mathcal{V}} |\boldsymbol{\omega}|^2 dV + \varpi_0 \qquad\qquad H_1\text{-norm}$$

is the enstrophy/unit volume which is related to the energy dissipation rate. **The higher moments will naturally pick up events at smaller scales**

$$\varpi_0 \leq \Omega_1(t) \leq \Omega_2(t) \leq \ldots \leq \Omega_m(t) \leq \Omega_{m+1}(t) \leq \ldots$$

Estimates in terms of Re

Traditionally, most NS-estimates have been found in terms of the Grashof number Gr ($f_{rms}^2 = L^{-3}\|\boldsymbol{f}\|_2^2$) of the divergence-free forcing $\boldsymbol{f}(\boldsymbol{x})$ but it would be more helpful to express these in terms of the Reynolds number Re to facilitate comparison with the results of statistical physics.

$$Gr = L^3 f_{rms}\nu^{-2}\,, \qquad\qquad Re = U_0 L\nu^{-1}\,.$$

Doering and Foias 2002 used the idea of defining U_0 as

$$U_0^2 = L^{-3}\left<\|\boldsymbol{u}\|_2^2\right>_T$$

where the time average $\left<\,\cdot\,\right>_T$ over an interval $[0,\,T]$ is defined by

$$\left<g(\cdot)\right>_T = \limsup_{g(0)} \frac{1}{T}\int_0^T g(\tau)\,d\tau\,.$$

Gr is fixed provided $\boldsymbol{f}$ is L^2-bounded, while Re is the system response.

Leray's energy inequality shows that

$$\tfrac{1}{2}\frac{d}{dt}\int_{\mathcal{V}} |\boldsymbol{u}|^2\, dV \leq -\nu \int_{\mathcal{V}} |\boldsymbol{\omega}|^2\, dV + \|\boldsymbol{f}\|_2 \|\boldsymbol{u}\|_2\,,$$

$$\left\langle \Omega_1^2 \right\rangle_T \leq \varpi_0^2 Gr\, Re + O\left(T^{-1}\right)\,.$$

Doering & Foias (2002) showed that NS-solutions obey $Gr \leq c\, Re^2$ provided the spectrum of $\boldsymbol{f}$ is concentrated in a narrow-band around a single frequency or its spectrum is bounded above & below

$$\left\langle \Omega_1^2 \right\rangle_T \leq c\, \varpi_0^2 Re^3 + O\left(T^{-1}\right)\,.$$

$\nu \left\langle \Omega_1^2 \right\rangle_T$ is the time-averaged energy dissipation rate over $[0,\, T]$ and the Kolmogorov length scale λ_k^{-1} is estimated as

$$\lambda_k^{-4} = \frac{\nu \left\langle \Omega_1^2 \right\rangle_T}{\nu^3} \qquad \Rightarrow \qquad L\lambda_k^{-1} \leq c\, Re^{3/4} + O\big(T^{-1/4}\big)\,.$$

Weak solution result

Theorem 1 : *Weak solutions of the 3D-Navier-Stokes equations satisfy*

$$\left\langle \left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m}\right\rangle_T \leq c\,Re^3 + O\left(T^{-1}\right), \qquad 1 \leq m \leq \infty\,,$$

where $\varpi_0 = \nu L^{-2}$, c *is a uniform constant and*

$$\alpha_m = \frac{2m}{4m-3}\,.$$

Remark : The exponent $\alpha_m = \frac{2m}{4m-3}$ appears to be a natural scaling, consistent with the application of Hölder & Sobolev inequalities.

Proof : (JDG 2011, CMS) The proof is based on a result of Foias, Guillopé and Temam (1983), *New a priori estimates for Navier-Stokes equations in Dimension 3*, Comm. Partial Diff. Equat., 6, 329–359, 1981 (their Theorem 3.1) for weak solutions.

When modified in the manner of Doering & Foias (02) the FGT result becomes

$$\left\langle H_N^{\frac{1}{2N-1}} \right\rangle_T \leq c_N L^{-1} \nu^{\frac{2}{2N-1}} Re^3 + O\left(T^{-1}\right) ,$$

where

$$H_N = \int_{\mathcal{V}} \left|\nabla^N \boldsymbol{u}\right|^2 dV = \int_{\mathcal{V}_k} k^{2N} \left|\hat{\boldsymbol{u}}\right|^2 d^3k \,,$$

where $H_1 = \int_{\mathcal{V}} |\nabla \boldsymbol{u}|^2 \, dV = \int_{\mathcal{V}} |\boldsymbol{\omega}|^2 \, dV$. An interpolation between $\|\boldsymbol{\omega}\|_{2m}$ and $\|\boldsymbol{\omega}\|_2$ is found using H_N

$$\|\boldsymbol{\omega}\|_{2m} \leq c_{N,m} \|\nabla^{N-1}\boldsymbol{\omega}\|_2^a \, \|\boldsymbol{\omega}\|_2^{1-a} \,, \qquad a = \frac{3(m-1)}{2m(N-1)} \,,$$

for $N \geq 3$. $\|\boldsymbol{\omega}\|_{2m}$ is now raised to the power α_m, to be determined.

In effect, we are translating from derivatives to L^{2m}-norms.

$$
\begin{aligned}
\left\langle \|\boldsymbol{\omega}\|_{2m}^{\alpha_m}\right\rangle_T &\leq c_{N,m}^{\alpha_m}\left\langle \|\nabla^{N-1}\boldsymbol{\omega}\|_2^{a\alpha_m}\|\boldsymbol{\omega}\|_2^{(1-a)\alpha_m}\right\rangle_T \\
&\leq c_{N,m}^{\alpha_m}\left\langle H_N^{\frac{1}{2N-1}}\right\rangle_T^{\frac{1}{2}a\alpha_m(2N-1)}\left\langle H_1^{\frac{(1-a)\alpha_m}{2-a\alpha_m(2N-1)}}\right\rangle_T^{1-\frac{1}{2}a\alpha_m(2N-1)}
\end{aligned}
$$

An explicit upper bound in terms of Re is available only if the exponent of H_1 within the average is unity ; that is

$$
\frac{(1-a)\alpha_m}{2-a\alpha_m(2N-1)} = 1\,.
$$

This determines α_m, uniformly in N, as

$$
\alpha_m = \frac{2m}{4m-3}\,.
$$

The constant $c_{N,m}$ can be minimized by choosing $N=3$ which is finite even when $m=\infty$; thus we take the largest value of $c_{3,m}^{\alpha_m}$ and call this c. ■

A continuum of length scales

Based on the definition of the inverse Kolmogorov length λ_k^{-1}, **a generalization of this to a hierarchy of inverse lengths λ_m^{-1} suggests:**

$$\left(L\lambda_m^{-1}\right)^{2\alpha_m} := \left\langle\left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m}\right\rangle_T$$

with $\alpha_m = \frac{2m}{4m-3}$ and where $\varpi_0 = \nu L^{-2}$:

$$L\lambda_m^{-1} \leq c\,Re^{3/2\alpha_m} + O\left(T^{-1/2\alpha_m}\right) \qquad 1 \leq m \leq \infty$$

m	1	9/8	3/2	2	3	...	∞
$3/2\alpha_m$	3/4	1	3/2	15/8	9/4	...	3

Values of the Re-exponent $3/2\alpha_m = 3\left(1 - \frac{3}{4m}\right)$.

Conditional 3D-NS regularity : a very brief history

Older history : Leray (1934) ; Prodi (1959), Foias & Prodi (1962), Serrin (1963) & Ladyzhenskaya (1964) : every Leray-Hopf solution $\boldsymbol{u}$ of the $3D$-NSE with $\boldsymbol{u} \in L^r\left((0,\, T)\,; L^s\right)$ is regular on $(0,\, T]$ provided $2/r + 3/s = 1$ with $s \in (3,\, \infty]$ or if $\boldsymbol{u} \in L^\infty\left((0,\, T)\,; L^p\right)$ with $p > 3$. When $s = 3$: von Wahl (1983) & Giga (1986) first proved the regularity in the space $C\left((0,\, T]\,; L^3\right)$: Kozono & Sohr (1997) & Escauriaza, Seregin & Sverák (2003).

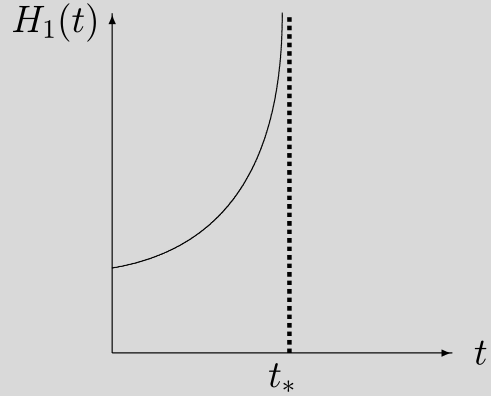
Books : Constantin & Foias 1988 ; Foias, Manley, Rosa & Temam 2001.

Modern developments : Assumptions on the pressure or one velocity derivative : Kukavic & Ziane (2006, 2007), Zhou (2002), Cao & Titi (2008, 2010), Cao (2010), Cao, Qin & Titi (2008), & Chen & Gala (2011) : on the direction of vorticity : BKM (1984), Constantin & Fefferman (1993), & Constantin, Fefferman & Majda (1993) ; Vasseur (2008), Cheskidov & Shvydkoy (2010).

Physical assumptions corresponding to conditional regularity?

- $\|\boldsymbol{u}\|_p$ $(p \geq 3)$ assumed to be bounded. **Drawback :** no physical interpretation.
- Likewise, assumptions on bounds on the pressure or single derivatives of the velocity field have no physical interpretation.
- The global enstrophy $H_1 = \int_{\mathcal{V}} |\boldsymbol{\omega}|^2 dV$ is assumed to be bounded pointwise in time : **Drawback :** assumes the answer we're seeking. It also ignores intermittency, which is important.

- Short time regularity : the upper bound on H_1 blows up at t^*.



To explain intermittency, how do we negotiate with the NSE?

What is the origin of intermittency illustrated by "thin sets" in earlier pictures?

Two possible options are :

1. **The $3D$-NS equations might possess only weak solutions – which exist but are not unique. Resolution difficulties might be due to this.**
2. **Solutions of the $3D$-NS equations are regular (un-proven) but nevertheless allow 'extreme events'.**

Another look at conditional regularity: arXiv/1108.4651 [nlin.CD]

Lemma 1: *With* $1 \leq m < \infty$, *and* $\alpha_m = \frac{2m}{4m-3}$; $\beta_m = \frac{4}{3}m(m+1)$ *and* $n = \frac{1}{2}(m+1)$, *then if* $\exists$ *a solution,* $\Omega_m(t)$ *formally satisfies*

$$\dot{\Omega}_m \leq \varpi_0 \Omega_m \left\{ -\frac{1}{c_{1,m}} \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{\beta_m} + c_{2,m} \left(\varpi_0^{-1} \Omega_n \right)^{2\alpha_n} + c_{3,m} Gr \right\} \qquad (*)$$

Proof: requires just two remarks: see JDG (2010)

(i) For the Laplacian term: need to show that ($A_m = \omega^m$)

$$\frac{1}{2m}\frac{d}{dt}\int_{\mathcal{V}} |\boldsymbol{\omega}|^{2m}\, dV \leq -c_m \int_{\mathcal{V}} |\nabla A_m|^2\, dV$$

plus a Sobolev inequality $\|A_m\|_{\frac{2(m+1)}{m}} \leq c_m \|\nabla A_m\|_2^{3/2(m+1)} \|A_m\|_2^{(2m-1)/2(m+1)}$

(ii) For the nonlinear term: a Hölder inequality gives ($n = \frac{1}{2}(m+1)$)

$$\frac{1}{2m}\frac{d}{dt}\int_{\mathcal{V}} |\boldsymbol{\omega}|^{2m}\, dV \leq \Omega_{m+1}^{2m} \|\nabla \boldsymbol{u}\|_{m+1} \leq c_m\, \Omega_{m+1}^{2m}\, \Omega_n$$

$$y_m = \left(\varpi_0^{-1}\Omega_m\right)^{-2\alpha_n} \qquad\qquad F_m = \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m} - c_{1,m}c_{3,m}Gr$$

Using the fact that $\Omega_n \leq \Omega_m$, (*) linearizes to $(\tau = 2\varpi_0\alpha_n c_{1,m}^{-1}t)$

$$\frac{dy_m}{d\tau} \geq F_m\, y_m - c_{2,m}\,,$$

and integrates to

$$\left\{\varpi_0^{-1}\Omega_m(\tau)\right\}^{2\alpha_n} \leq \frac{\exp\left(-\int_0^\tau F_m d\xi\right)}{y_{m,0} - c_{2,m}\int_0^\tau \exp\left(-\int_0^\xi F_m d\xi'\right)\, d\xi} \qquad (**)$$

where the initial value $y_{m,0} = y_m(0)$.

1. Can the denominator develop a zero in a finite time?
2. Control of $\Omega_m(\tau)$ from above for **any** $m \geq 1$ will control the H_1-norm.

Lower bounds on the dissipation (JDG arXiv/1108.4651, 2011)

Choose a set of parameters μ_m such that with $\alpha_m = \frac{2m}{4m-3}$ & $\beta_m = \frac{4}{3}m(m+1)$

$$\alpha_m\left(\frac{1-\mu_m}{\mu_m}\right) = \beta_m\,, \qquad \Rightarrow \qquad \mu_m = \frac{3}{(2m-1)(4m+3)}$$

α_m	$\frac{2m}{4m-3}$
$\alpha_m - \alpha_{m+1}$	$\frac{6}{(4m+1)(4m-3)}$
β_m	$\frac{4}{3}m(m+1)$
μ_m	$\frac{3}{(2m-1)(4m+3)}$
$\frac{1-\mu_m}{\mu_m} = \frac{\beta_m}{\alpha_m}$	$\frac{2}{3}(m+1)(4m-3)$
$\frac{\alpha_m}{\alpha_{m+1}}$	$\frac{1+\mu_m}{1-\mu_m}$
$\left(\frac{\alpha_m}{\alpha_{m+1}} - 1\right)\left(\frac{1-\mu_m}{\mu_m}\right)$	2

Define the average notation as $\langle\cdot\rangle_{(\tau)} = \frac{1}{\tau}\int_0^\tau \cdot\, d\xi$.

Lemma 2 :

$$\left\langle\left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m}\right\rangle_{(\tau)} \geq \left(\frac{\left\langle\left(\varpi_0^{-1}\Omega_{m+1}\right)^{\alpha_{m+1}}\right\rangle_{(\tau)}^{\frac{\alpha_m}{\alpha_{m+1}}}}{\left\langle\left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m}\right\rangle_{(\tau)}}\right)^{\frac{\beta_m}{\alpha_m}} .$$

Proof of Lemma 2 : Clearly μ_m lies in the range $0 < \mu_m < 1$ so a Hölder inequality gives

$$\begin{aligned}\int_0^\tau \Omega_{m+1}^{\alpha_m(1-\mu_m)} d\xi &= \int_0^\tau \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\alpha_m(1-\mu_m)} \Omega_m^{\alpha_m(1-\mu_m)} d\xi \\ &\leq \left(\int_0^\tau \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\alpha_m\left(\frac{1-\mu_m}{\mu_m}\right)} d\xi\right)^{\mu_m} \left(\int_0^\tau \Omega_m^{\alpha_m} d\xi\right)^{1-\mu_m}\end{aligned}$$

thus leading to

$$\int_0^\tau \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m} d\xi \geq \left(\frac{\int_0^\tau \Omega_{m+1}^{\alpha_m(1-\mu_m)} d\xi}{\left(\int_0^\tau \Omega_m^{\alpha_m} d\xi\right)^{1-\mu_m}}\right)^{1/\mu_m} .$$

It is easily checked that

$$\begin{aligned}\alpha_m(1-\mu_m) = \mu_m\beta_m &= \frac{4m(m+1)}{8m^2+2m-3} \\ &> \frac{2(m+1)}{4m+1} = \alpha_{m+1}\end{aligned}$$

so a further Hölder inequality on the numerator of gives

$$\int_0^\tau \Omega_{m+1}^{\alpha_{m+1}} d\xi \leq \tau^{\frac{\mu_m}{1+\mu_m}} \left(\int_0^\tau \Omega_{m+1}^{\alpha_m(1-\mu_m)} d\xi\right)^{\frac{1}{1+\mu_m}} .$$

The following theorem formally expresses the main result :

Theorem 2 : *If there exists a value of m lying in the range $1 \leq m < \infty$, with initial data $\Omega_{m,0} < C_m \varpi_0 Gr^{\Delta_m/2\alpha_n}$, for which the integral lies on or above the critical value*

$$c\left(\tau\, Gr^{2\delta_{m+1}} + \eta_2\right) \leq \int_0^\tau \left(\varpi_0^{-1}\Omega_{m+1}\right)^{\alpha_{m+1}} d\xi\,,$$

where $\eta_2 \geq \eta_1 Gr^{2(\delta_{m+1}-1)}$, and where δ_{m+1} lies in the range

$$\frac{\alpha_{m+1}}{\alpha_m}\left(1 + \frac{\alpha_m}{2\beta_m}\right) < \delta_{m+1} < 1\,,$$

then $\Omega_m(\tau)$ remains bounded for all time.

Proof : The lower bound on the dissipation is

$$\int_0^\tau \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m} d\xi \;\geq\; c_m\tau^{-1}\left\{\frac{\left(\tau Gr^{2\delta_{m+1}} + \eta_1 Gr^{2(\delta_{m+1}-1)}\right)^{\alpha_m/\alpha_{m+1}}}{(\tau Gr^2 + \eta_1)}\right\}^{\beta_m/\alpha_m}$$

$$\geq\; c_m\tau^{\left(\frac{\alpha_m}{\alpha_{m+1}}-1\right)\frac{\beta_m}{\alpha_m}-1} Gr^{2\left\{\delta_{m+1}\frac{\alpha_m}{\alpha_{m+1}}-1\right\}\frac{\beta_m}{\alpha_m}}$$

$$=\; c_m\,\tau\, Gr^{\Delta_m}$$

where

$$\Delta_m = 2\left\{\delta_{m+1}\frac{\alpha_m}{\alpha_{m+1}} - 1\right\}\frac{\beta_m}{\alpha_m}\,.$$

$$\int_0^\tau F_m d\xi \geq \left\{c_m Gr^{\Delta_m} - c_3 Gr\right\}\tau\,.$$

To have the dissipation greater than forcing ($\Delta_m > 1$) raises the lower bound on δ_{m+1} away from $\frac{\alpha_{m+1}}{\alpha_m}$.

$$\int_0^\tau \exp\left(-\int_0^\xi F_m d\xi'\right) d\xi \;\leq\; Gr^{-\Delta_m}\left[1 - \exp\left(-\tau Gr^{\Delta_m}\right)\right] \;,$$

and so the denominator of (**) satisfies

Denominator of (**) $\geq y_{m,0} - c_{1,m}c_{2,m}Gr^{-\Delta_m}\left(1 - e^{-\tau Gr^{\Delta_m}}\right)$

This can never go negative if $y_{m,0} > c_{1,m}c_{2,m}Gr^{-\Delta_m}$, which means large initial data is restricted by

$$\left(\varpi_0^{-1}\Omega_{m,0}\right)^{2\alpha_n} < C_m Gr^{\Delta_m}$$

where $1 < \Delta_m \leq 4$. ■

Intermittency

1. A feature of intermittent flows lies in the strong excursions of the vorticity away from average with periods of inactivity between the spikes. How does the critical lower bound imposed as an assumption in Theorem 2 lead to this?

2. If $\left\langle \Omega_{m+1}^{\alpha_{m+1}} \right\rangle_{(\tau)}$ lies above critical then not only Ω_m cannot blow up but it actually collapses exponentially. Curiously, & counter-intuitively, if the value of this integral drops below critical then the occurrence of a singular event must still formally be considered.

3. **Experimentally, signals go through cycles of growth/collapse : thus it is not realistic to expect the critical lower bound to hold for all τ.**

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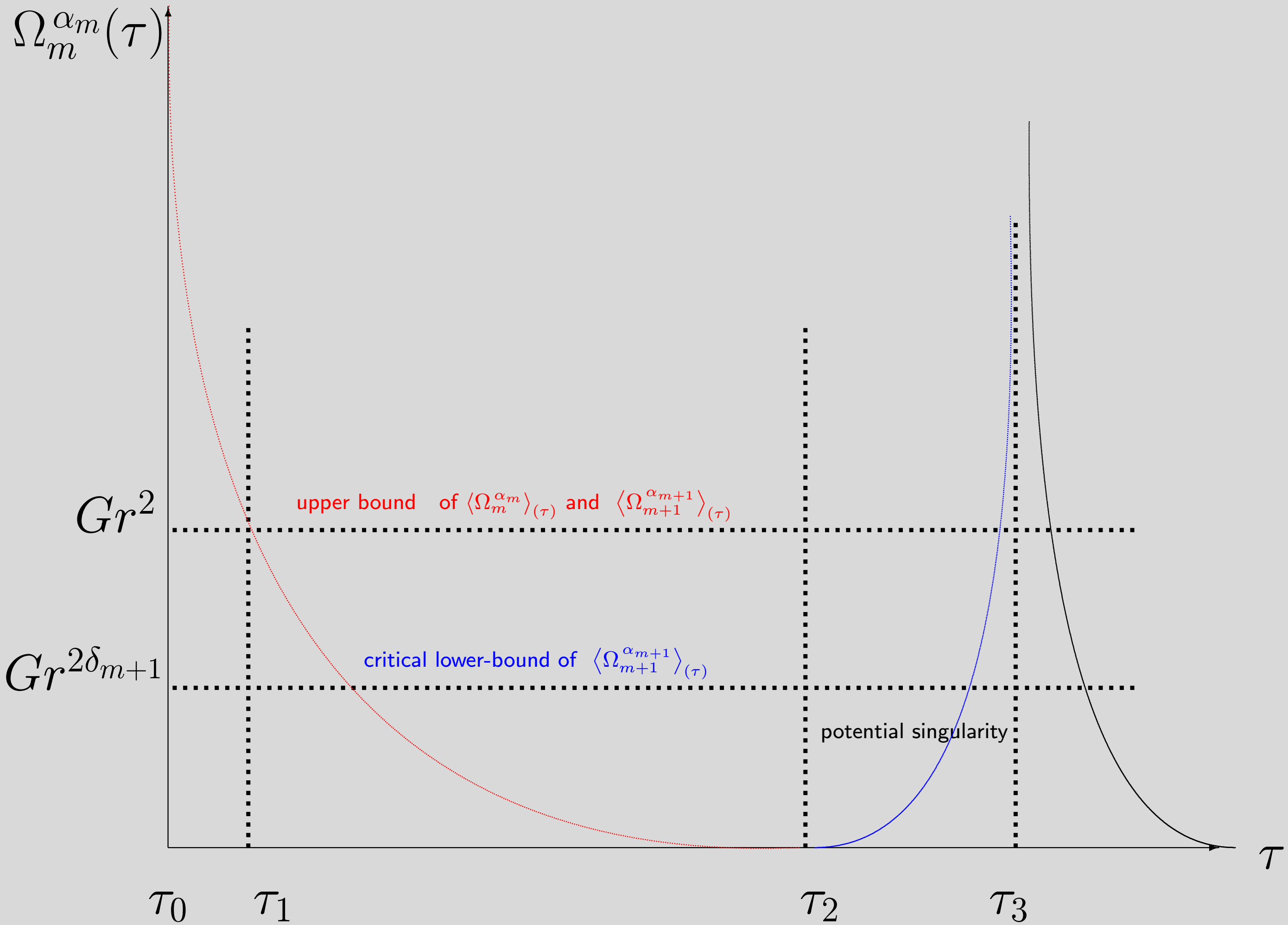
$\Omega_m^{\alpha_m}(\tau)$
Gr^2
upper bound of $\langle \Omega_m^{\alpha_m} \rangle_{(\tau)}$ and $\langle \Omega_{m+1}^{\alpha_{m+1}} \rangle_{(\tau)}$
$Gr^{2\delta_{m+1}}$
critical lower-bound of $\langle \Omega_{m+1}^{\alpha_{m+1}} \rangle_{(\tau)}$
potential singularity
τ
τ_0
τ_1
τ_2
τ_3

Cascade??

It is always true that $\Omega_m \leq \Omega_{m+1}$ but $\alpha_m > \alpha_{m+1}$.

$$\Omega_m^{\alpha_m} \lesseqgtr \Omega_{m+1}^{\alpha_{m+1}} \qquad\qquad ??$$

Now use our definition of a length scale

$$\left(L\lambda_m^{-1}\right)^{2\alpha_m} = \left\langle\left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m}\right\rangle_{(\tau)}$$

although $\lambda_m \lesseqgtr \lambda_{m+1}$?? Our lower bound becomes

$$\left\langle\left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m}\right\rangle_{(\tau)} \geq \left(\frac{\lambda_m}{\lambda_{m+1}}\right)^{2\beta_m} .$$

We could enforce the assumption that there exists an ordered cascade of length scales $\lambda_m > \lambda_{m+1}$, decreasing sufficiently fast with m to take advantage of the quadratic nature of β_m.

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