The Kepler Problem in hyperbolic space

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- Step 1: deduction of the potential: Let $U(r, \theta, \varphi)$ be the potential of the Kepler problem. For $U$ to be physically realistic, it must satisfy:
  
  i) $U$ is a function of $r$ in normal coordinates $U = U(r)$
  
  ii) The gravitational flow through spheres is constant
      
      $$\int_{S_r} \langle \nabla U \cdot n \rangle dA_r = cte. \quad (1)$$

- Using the hypotheses,

      $$U(r) = \int \frac{c}{4\pi \sinh^2(r)} dr = k \coth(r). \quad (2)$$

- Step 2: The movement of a particle with initial conditions $x_0, \dot{x}_0$ is contained in a submanifold isometric to $H^2$.

- Final Statement of the problem: The potential $U(r) = k \coth(r)$ in $H^2$. The corresponding Lagrangian is

      $$L(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2} (\dot{r}^2 + \sinh^2(r)\dot{\theta}^2) + k\coth(r). \quad (3)$$
The Kepler laws

- **First law**: bounded orbits are ellipses with the origin in one of the foci.
- **Second law**: Throughout the solutions, \( \frac{dA}{dt} = \text{cte} \) where

\[
A(t) = \int_{S(t)} d\mu,
\]

and \( d\mu = \cosh(r) dA \)

- **Third Law**: \( T = f(a) \),

\[
T = \frac{1}{\sqrt{k}} \pi \left( \frac{1}{\sqrt{\frac{2}{x} - 2}} - \frac{1}{\sqrt{\frac{2}{x} + 2}} \right),
\]  

(4)

where \( x = \tanh(2a) \),

the variable \( a \) is the major semi axis.