Let $\Omega$ be a bounded connected open set in $\mathbb{R}^2$, with $\partial \Omega = \partial \Omega^D \cup \partial \Omega^N$. By time-discretization approach, we prove that the existence of a continuous-time evolution that satisfies an energy inequality and a stability criterion. The energy balance also takes into account the energy dissipated at the front of the fracture. The stability criterion is stated in the framework of Griffith's theory, in terms of the energy release rate, when the crack grows at least at one point of its front.

### The class of admissible cracks
Let $\Omega$ be a bounded connected open set in $\mathbb{R}^2$, with $\partial \Omega = \partial \Omega^D \cup \partial \Omega^N$ Lipschitz. Fix $\theta \in (0, \pi)$ and $\gamma > 0$. Let $S$ be the class of sets of the form

$$S = \left\{ \Gamma \subset \Omega \mid \text{any crack set } \Gamma \text{ where } \Gamma \subset \partial \Omega \text{ and } \Gamma \cap \partial \Omega \text{ is bounded from above by } \gamma/\theta \right\},$$

satisfying the following assumptions:

- open ball condition: $K_i$ is a curve of class $C^{1,1}$ and for every $x \in K_i$, there exist two open balls $B_1, B_2$ of radius $\gamma$, such that $(B_1 \cup B_2) \cap (K_i \setminus K_i) = \emptyset$ and $\partial B_1 \cap \partial B_2 = \emptyset$, i.e. at any point $x \in K_i$, the curvature of $K_i$ is bounded from above by $1/\gamma$;
- intersection condition: if $\Gamma_i \cap \Gamma_j = \emptyset$ for some $i \neq j$, then they intersect in their endpoints in $\partial \Omega$;
- angle condition: if $K_i \cap K_j \neq \emptyset$ for some $i, j$, then they intersect in an angle greater than or equal to $\theta$;
- bounded number of branches: there exists $M \in \mathbb{N}$ such that $M \leq M_0$ for every $\Gamma \in S$.

**Theorem 1.** The class $S$ is compact with respect to the Hausdorff metric. If $\Gamma_0 \subset S$ and $\Gamma_n \to \Gamma$ in the Hausdorff metric, then $H^1(\Gamma_n) \to H^1(\Gamma)$.

We divide the points of a set $\Gamma \in S$ in three groups:

- the set $\Gamma_i$ of crack tip points: $x \in \Gamma$ belongs to $\Gamma_i$ if there exists $r > 0$ such that $\Gamma \cap B_r(x)$ is a $C^{1,1}$ curve with endpoint $x$;
- the set $\Gamma_\ell$ of singular points: $x \in \Gamma$ belongs to $\Gamma_\ell$ if there exist two unit vectors $v_1, v_2$, tangent to $\Gamma$ at $x$, such that $v_1, v_2 \neq \pm 1$;
- the set $\Gamma_2$ of regular points: $x \in \Gamma$ belongs to $\Gamma_2$ if there exists $r > 0$ such that $\Gamma \cap B_r(x)$ is a $C^{1,1}$ curve and $\Gamma$ is in the relative interior of $\Gamma_2$.

**Theorem 3.** Let $\omega \in H^1(\Omega, \mathbb{R}^2)$ and any boundary displacement $w \in H^1(\Omega)$, the bulk energy of the elastic body $\Omega \setminus \Gamma$ is given by $E(w, u) = \inf \{ \| \nabla w \|^2 + E(\omega, \Gamma) : w \in H^1(\Omega)^n, \omega = w \text{ on } \partial \Omega \}$, where $w$ are admissible elastic displacement. Fix a crack tip $p \in \Gamma$. We define the energy release rate at $p$ as $G^\Gamma(p, \rho) = \lim_{\rho \to 0} E(w, u) = \lim_{\rho \to 0} E(w, \Gamma)$, where $\Gamma^\rho$ is any $C^{1,1}$ extension of $\Gamma$ at $p$, such that $H^1(\Gamma^\rho) = \rho$.

**Proposition 4.** For any $0 \leq i < j \leq [\Gamma(T)]$ the following inequality holds:

$$\|\nabla w_\ell\|^2 \geq H^1(\Gamma_\ell) + \frac{1}{2} \sum_{k=i}^{j-1} \sum_{c=1}^k \left( M^2 \right)^2 \left( \frac{H^1(\Gamma_c)}{H^1(\Gamma_\ell)} \right)^2 \leq \left( \frac{H^1(\Gamma_\ell)}{H^1(\Gamma_\ell)} \right)^2 \|\nabla w_i\|^2 + \|\nabla w_j\|^2 + 2 \left( \|\nabla w_i\|^2 + H^1(\Gamma_\ell) \right).$$

In addition, the discrete Griffith’s principle is satisfied, for every $t \in [0, T]$ and $p \in \Gamma_\ell(T)$,

$$v(t, \rho) \geq \inf_{\Gamma \in \Gamma_\ell(T)} \left\{ \frac{1}{2} \sum_{k=i}^{j-1} \sum_{c=1}^k \left( M^2 \right)^2 \left( \frac{H^1(\Gamma_c)}{H^1(\Gamma_\ell)} \right)^2 \right\} \|\nabla w_i\|^2 + \|\nabla w_j\|^2 + 2 \left( \|\nabla w_i\|^2 + H^1(\Gamma_\ell) \right) + \frac{1}{2} \left( \|\nabla w_i\|^2 \right)^2.$$

Therefore, we recover the continuous-time rate-dependent evolution $E(t, u(t))$ as the limit of the discrete-time evolution $E(t, u(t))$ when the incremental step $\tau \to 0$. The velocity $v(t)$ at time $t$ is given by

$$v(t) = \frac{1}{\mu(t)} \left( u(t) - u(\sigma) \right)$$

References