A model for crack growth with branching and kinking

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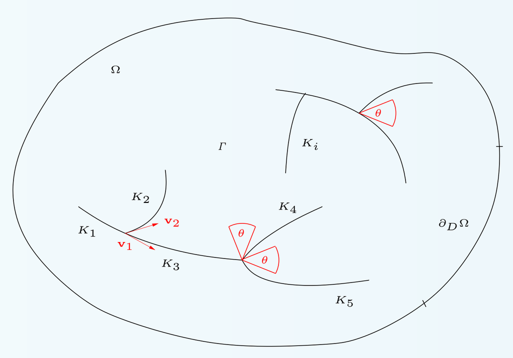
Abstract

We study an evolution model for fractured elastic materials in the 2-dimensional antiplane case, for which the crack path is not assumed to be known *a priori*. We introduce some general assumptions on the structure of the fracture sets suitable to allow for kinking and branching of the crack to develop. In addition we define the front of the fracture and its velocity.

By means of a time-discretization approach, we prove the existence of a continuous-time evolution that satisfies an energy inequality and a stability criterion. The energy balance also takes into account the energy dissipated at the front of the fracture. The stability criterion is stated in the framework of Griffith's theory, in terms of the energy release rate, when the crack grows at least at one point of its front.

The class of admissible cracks



Let Ω be a bounded connected open set in $\mathbb{R}^2$, with $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$ Lipschitz.
Fix $\theta \in (0,\pi)$ and $\eta > 0$. Let $\mathcal{S}$ be the class of sets of the form

$$\Gamma = \bigcup_{i=1}^{M} K_i$$

satisfying the following assumptions:

- *open ball condition*: K_i is a curve of class $C^{1,1}$ and for every $x \in K_i$ there exist two open balls B_1, B_2 of radius η, such that $(B_1 \cup B_2) \cap (K_i \cup \partial\Omega) = \emptyset$ and $\overline{B}_1 \cap \overline{B}_2 = \{x\}$ (*i.e.* at any point $x \in K_i$ the curvature of K_i is bounded from above by $1/\eta$);
- *intersection condition*: if $K_i \cap K_j \neq \emptyset$ for some $i \neq j$, then they intersect in their endpoints;
- *angle condition*: if $K_i \cap K_j \neq \emptyset$ for some $i \neq j$, then they intersect with an angle greater than or equal to θ;
- *bounded number of branches*: there exists $M_0 \in \mathbb{N}$ such that $M \leq M_0$ for every $\Gamma \in \mathcal{S}$.

Theorem 1. *The class $\mathcal{S}$ is compact with respect to the Hausdorff metric. If $\Gamma_n \in \mathcal{S}$ and $\Gamma_n \to \Gamma$ in the Hausdorff metric, then*

$$\mathcal{H}^1(\Gamma_n) \to \mathcal{H}^1(\Gamma)\,.$$

We divide the points of a set $\Gamma \in \mathcal{S}$ in three groups:

- the set T_Γ of *crack tip points*: $x \in \Gamma$ belongs to T_Γ if there exists $r > 0$ such that $\Gamma \cap \overline{B_r(x)}$ is a $C^{1,1}$ curve with endpoint x;
- the set S_Γ of *singular points*: $x \in \Gamma$ belongs to S_Γ if there exist two unit vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ tangent to Γ at x such that $\mathbf{v}_1 \cdot \mathbf{v}_2 \neq \pm 1$;
- the set R_Γ of *regular points*: $x \in \Gamma$ belongs to R_Γ if there exists $r > 0$ such that $\Gamma \cap \overline{B_r(x)}$ is a $C^{1,1}$ curve and x is in the relative interior of Γ.

Velocity of crack tips

Let $t \mapsto \Gamma(t)$ be a 1-parameter family such that

- $\Gamma(t) \in \mathcal{S}$ for every $t \in [0,T]$
- $\Gamma(\tau) \subset \Gamma(t)$ if $\tau \leq t$
- the function $t \mapsto \mathcal{H}^1(\Gamma(t))$ is absolutely continuous in $[0,T]$.

Proposition 2. *Set $\mu(t) := \mathcal{H}^1 \llcorner \Gamma(t)$. Then $\mu \in AC([0,T];\mathcal{M}_b(\Omega))$ and for a.e. $t \in [0,T]$ there exists*

$$\dot\mu(t) := w^*-\lim_{s\to t} \frac{\mu(s)-\mu(t)}{s-t}\,.$$

In addition

$$\text{supp}\,\dot\mu(t) \subset T_{\Gamma(t)}\,. \qquad (1)$$

We introduce the concept of *velocity* at crack tip using the above *variational* approach: from (1)

$$\dot\mu(t) = \sum_{p \in T_{\Gamma(t)}} v(p,t)\delta_p\,.$$

We call $v(p,t)$ the *velocity* of $p \in T_{\Gamma(t)}$.

Existence result

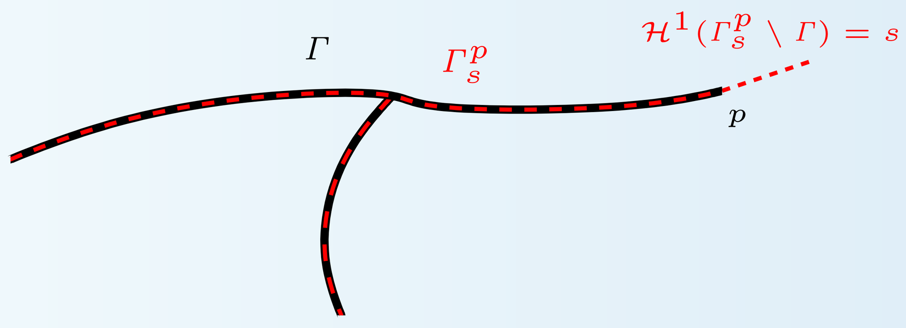
For any crack set $\Gamma \in \mathcal{S}$ and any boundary displacement $w \in H^1(\Omega)$, the *bulk energy* of the elastic body $\Omega \setminus \Gamma$ is given by

$$E(w,\Gamma) := \inf\left\{\|\nabla u\|^2_{L^2} :\, u \in H^1(\Omega\setminus\Gamma), u = w \text{ on } \partial_D\Omega\right\}\,,$$

where u are admissible antiplane *elastic displacements*.
Fix a crack tip $p \in T_\Gamma$. We define the *energy release rate at* p as

$$\mathcal{G}(w,\Gamma,p) := -\lim_{s\to 0} \frac{E(w,\Gamma^p_s) - E(w,\Gamma)}{s}\,,$$

where Γ^p_s is any $C^{1,1}$ estension of Γ at p, such that $\mathcal{H}^1(\Gamma^p_s \setminus \Gamma) = s$.



Theorem 3. *Let $w \in H^1(0,T;H^1(\Omega))$ and $\Gamma_0 \in \mathcal{S}$. Then there exists an evolution $(\Gamma(\cdot),u(\cdot)) : [0,T] \to \mathcal{S} \times L^2(\Omega)$ with the following properties:*

(a) for every $t \in [0,T]$ the function $u(t)$ belongs to $H^1(\Omega\setminus\Gamma(t))$ and solves

$$\begin{cases} \Delta u(t) = 0 & \text{in } \Omega\setminus\Gamma(t) \\ \frac{\partial u(t)}{\partial\nu} = 0 & \text{on } \Gamma(t)\cup\partial\Omega\setminus\partial_D\Omega \\ u(t) = w(t) & \text{on } \partial_D\Omega \end{cases}$$

(b) energy inequality: for all $0 \leq a < b \leq T$

$$\|\nabla u(b)\|^2_{L^2} + \mathcal{H}^1(\Gamma(b)) + \int_a^b \sum_{p\in T_{\Gamma(t)}} v(t,p)^2\,dt$$
$$\leq \|\nabla u(a)\|^2_{L^2} + \mathcal{H}^1(\Gamma(a)) + 2\int_a^b \langle\nabla u(t),\nabla\dot w(t)\rangle\,dt$$

(c) Griffith's principle:

- *$v(p,t) \geq 0$ for every $t \in [0,T]$ and for every $p \in T_{\Gamma(t)}$;*
- *there exists an open set $A \subset [0,T]$ such that for every $t \in A$, if $p \in T_{\Gamma(t)} \setminus \bigcup_{\tau<t} T_{\Gamma(\tau)}$, then*

$$\mathcal{G}(w(t),\Gamma(t),p) \leq 1 + v(t,p) \qquad (2)$$
$$[-\mathcal{G}(w(t),\Gamma(t),p) + 1 + v(t,p)]\,v(t,p) = 0. \qquad (3)$$

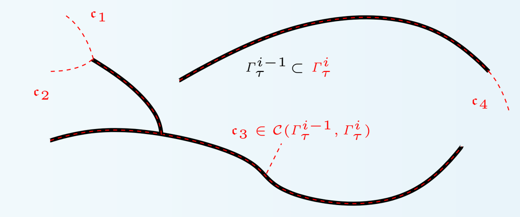
Proof strategy: time discretization

Given $\Gamma_1, \Gamma_2 \in \mathcal{S}$ with $\Gamma_1 \subset \Gamma_2$, let $\mathcal{C}(\Gamma_1,\Gamma_2)$ be the set of connected components of $\Gamma_2 \setminus \Gamma_1$.
We now construct the discrete-time evolution with time incremental step $\tau > 0$. Define

- $u^0_\tau := u_0$ and $\Gamma^0_\tau := \Gamma_0$;
- recursively u^i_τ and Γ^i_τ as minimizers of

$$\|\nabla u\|^2_{L^2} + \mathcal{H}^1(\Gamma) + \underbrace{\frac{1}{\tau}\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma^{i-1}_\tau,\Gamma)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2}_{\text{approximation of the velocity term}}$$

under the constraints $\Gamma \in \mathcal{S}$, $\Gamma^{i-1}_\tau \subset \Gamma$, $u \in H^1(\Omega\setminus\Gamma)$ and $u = w(i\tau)$ on $\partial_D\Omega$.



Notice that at each incremental step the fracture is permitted to grow simultaneously at many tips and to develop new branches.

Proposition 4. *For any $0 \leq i < j \leq [T/\tau]$ the following inequality holds:*

$$\|\nabla u^j_\tau\|^2_{L^2} + \mathcal{H}^1(\Gamma^j_\tau) + \frac{1}{\tau}\sum_{h=i}^{j-1}\sum_{\mathfrak{c}\in\mathcal{C}(\Gamma^h_\tau,\Gamma^{h+1}_\tau)} \left(\mathcal{H}^1(\mathfrak{c})\right)^2$$
$$\leq \|\nabla u^i_\tau\|^2_{L^2} + \mathcal{H}^1(\Gamma^i_\tau) + 2\int_{i\tau}^{j\tau} \langle\nabla u_\tau(t),\nabla\dot w(t)\rangle\,dt + o(\tau).$$

In addition, the following discrete Griffith's principle is satisfied: for every $t \in (0,T)$ and $p_\tau \in T_{\Gamma_\tau(t)}$

$$v_\tau(t,p_\tau) \geq 0$$
$$\mathcal{G}(w_\tau(t),\Gamma_\tau(t),p_\tau) \leq 1 + v_\tau(t,p_\tau)$$
$$[-\mathcal{G}(w_\tau(t),\Gamma_\tau(t),p_\tau) + 1 + v_\tau(t,p_\tau)]\,v_\tau(t,p_\tau) = 0\,,$$

where $w_\tau(t) = w(i\tau)$ and $\Gamma_\tau(t) = \Gamma^i_\tau$ for $i\tau \leq t < (i+1)\tau$, and

$$v_\tau(t,p_\tau) = \begin{cases} 0 & \text{if } p_\tau \in T_{\Gamma_\tau(t)} \cap T_{\Gamma_\tau(t-\tau)} \\ \frac{1}{\tau}\mathcal{H}^1(\mathfrak{c}) & \text{if } p_\tau \in \mathfrak{c} \in \mathcal{C}(\Gamma_\tau(t-\tau),\Gamma_\tau(t))\,. \end{cases}$$

We recover the continuous-time rate-dependent evolution $(\Gamma(t),u(t))$ as limit of the discrete-time evolution $(\Gamma_\tau(t),u_\tau(t))$ when the incremental step $\tau \to 0$.

Remarks on result and proof

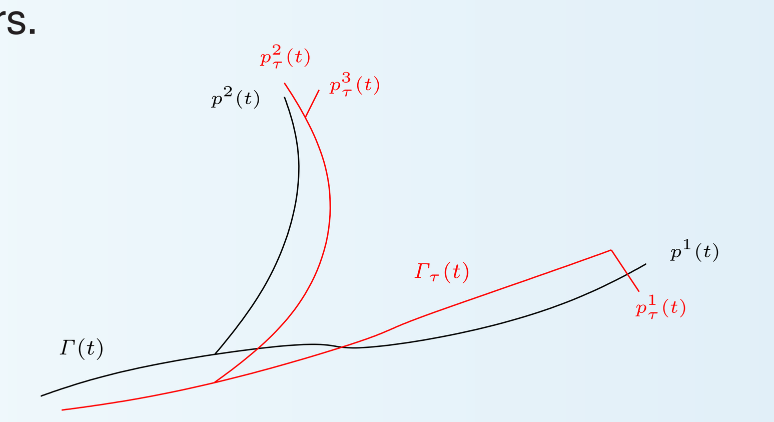
(i) The key step in order to prove the energy inequality *(b)* from its discrete version is the inequality

$$\int_a^b \sum_{p\in T_{\Gamma(t)}} v(t,p)^2\,dt \leq \liminf_{\tau\to 0} \int_a^b \sum_{p_\tau\in T_{\Gamma_\tau(t)}} v_\tau(t,p_\tau)^2\,dt$$

for $(a,b) \subset [0,T]$.

(ii) Griffith's principle *(c)* for the continuous-time evolution provides a partial characterization of the process. It is obtained from the correspondent discrete version in Proposition 4 as the time-step $\tau \to 0$. Note that, while the discrete version holds true *for every* $t \in [0,T]$, in the continuous-time case it is satisfied only in an open set $A \subset [0,T]$, due to the following difficulties in the approximation procedure:

- bad news: in case of a *static* tip $p \in T_{\Gamma(t_0)}$ at time t_0, *i.e.* if there exists $t' < t_0$ such that $p \in T_{\Gamma(t)}$ for $t \in [t',t_0]$, then we are not able to garantee that it is approximated by static tips of the discrete-time evolutions or to avoid other problematic behaviours.



For example, a static tip might be approximated by a discrete-time sequence of cracks that bifurcate or kink near the tip. The approximation procedure suggests that, in this situation, many direction of growth for the crack tip (of the continuous-time evolution) are possible: which would be the preferred one? How to deal with the energy release rate $\mathcal{G}$, which, as proved by Negri [7], depends on the kinking angle?

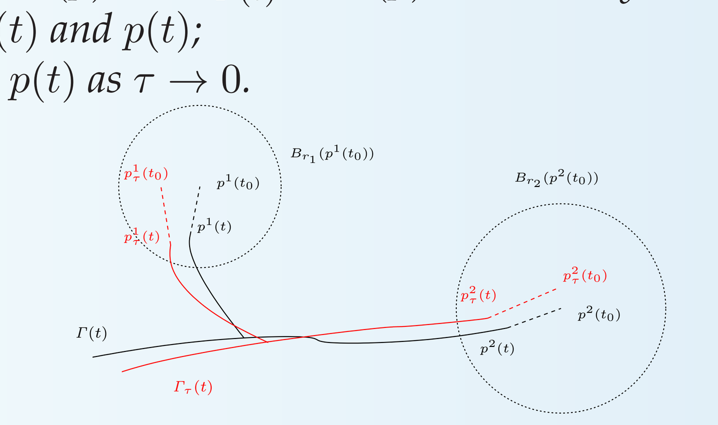
- good news: in case of a *moving* tip $p \in T_{\Gamma(t_0)}$ at time t_0, *i.e.*

$$p \in T_{\Gamma(t_0)} \setminus \bigcup_{t<t_0} T_{\Gamma(t)}\,,$$

the following local result holds:

Proposition 5. *There exist $\alpha(t_0) < t_0$ and $r = r(t_0,p)$ such that for all $t \in (\alpha(t_0),t_0]$ and for τ small enough*

- *$\Gamma_\tau(t) \cap \overline{B_r(p)}$ and $\Gamma(t) \cap \overline{B_r(p)}$ are $C^{1,1}$ curves;*
- *$T_{\Gamma_\tau(t)} \cap B_r(p)$ and $T_{\Gamma(t)} \cap B_r(p)$ have exactly one element each, labelled $p_\tau(t)$ and $p(t)$;*
- *$p_\tau(t) \to p(t)$ as $\tau \to 0$.*



Hence any ambiguity about kinking angle or multiple approximating tips is avoided, so that in this situation we are able to deduce conditions (2) and (3) in *(c)* by the discrete Griffith's principle in Proposition 4.

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