Why GBD?

The classical mathematical formulation of many problems arising from Fracture Mechanics requires the minimization in the space $SBD(\Omega)$ of functionals whose main energy term is given by

$$
\int_\Omega Q(c(u))dx + \mathcal{H}^{n-1}(J_u),
$$

where $Q$ is a positive definite quadratic form. Through the Compactness Theorem in $SBD$ by Bellettini, Coscia, and Dal Maso, we are able to guarantee the existence of solutions to such minimum problems assuming an a priori $L^\infty$-bound for minimizing sequences (see [2]). Regrettably, this requirement in general is not satisfied in spite of the presence of $L^\infty$-boundary data, since the maximum principle does not hold for these vector valued problems. For the same reason we can approximate (1) by integral functionals depending on Sobolev functions only if we assume a priori a bound on the $L^\infty$-norm of the functions, as proved by Chambolle in [3, 4].

The results we present in this poster show that, rather than $SBD(\Omega)$, the proper space where to formulate, solve, and approximate minimum problems related to (1) is the space $GBD(\Omega)$ of Generalised Special functions of Bounded Deformation, introduced by Dal Maso in [6].

Few preliminaries: Definition of GBD

Definition. The space $GBD(\Omega)$ is the space of all $L^\infty$-measurable functions $u : \Omega \to \mathbb{R}^n$ with the following property:

There exists $\lambda \in M_1^+(\Omega)$ such that for every $\xi \in \mathbb{R}^n$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega$ the slice $u_y^\perp := u_y^\perp \xi \in \mathcal{S}_{\mathbb{R}^n}$ belongs to $BV_{\mathcal{M}_1}(\mathbb{R}^n)$ and

$$
\int_\Omega \left( \int_{\mathbb{R}^n} |D(u^\perp)\sigma|^{1/2} + H(\mathcal{S}_{\mathbb{R}^n}) \right) \mathcal{H}^{n-1}(y) \leq \lambda(B)
$$

holds for every Borel set $B \subseteq \Omega$, where we have set $u_y^\perp := u_y^\perp \xi \in \mathcal{S}_{\mathbb{R}^n}$.

The space $GBD(\Omega)$ is the space of all functions $u \in GBD(\Omega)$ such that for every $\xi \in \mathbb{R}^n$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega$ the function $u_y^\perp := u_y^\perp \xi \in \mathcal{S}_{\mathbb{R}^n}$.

Few preliminaries: Fine properties of GBD

Structure Theorem. Let $u \in GBD(\Omega)$, let $\xi \in \mathbb{R}^n$, and let $J_u^\perp := \{ y \in \Omega : |u(x)\xi| \neq 0 \}$.

1. The approximate jump set $J_u^\perp$ is $(\mathcal{H}^{n-1}, n-1)$-rectifiable and $(J_u^\perp)^{\mathcal{H}^{n-1}} = J_u^\perp$.

2. The approximate limits $u^\#$ and $u^\#$ from the two sides of $J_u^\perp$ are $\mathcal{H}^{n-1}$-measurable functions defined $\mathcal{H}^{n-1}$-a.e. on $J_u^\perp$. Moreover, for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega$ and for every $\xi \in \mathbb{R}^n$ we have $u^\#(y + \xi) = (u^\#)^+(y + \xi)$, where the norms to $J_u^\perp$ and $J_u^\perp$ are related so that $\|u^\#\|_{\mathcal{H}^{n-1}} \leq \|u\|_{\mathcal{H}^{n-1}}$.

3. There exists the approximate symmetric gradient $\varepsilon(u) \in L^1(\mathcal{M}_n^+)$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega$ we have $\varepsilon(u)(y) = \varepsilon(u)^0(y) \in \mathbb{R}^{n \times n}$.

Few preliminaries: A compactness theorem

Compactness Theorem. Let $(u_\nu) \subseteq GBD(\Omega)$. Suppose that there exist $M < +\infty$ and two increasing continuous functions $v_0, v_1 : \mathbb{R}^n \to \mathbb{R}^n$, both with $v_0(\cdot) \to +\infty$ and $v_1(\cdot) \to +\infty$ as $x \to +\infty$, such that

$$
\int_\Omega \left( v_0(u_\nu)dx + \int_\Omega |u_\nu|^2 \right) \to +\infty
$$

for every $\nu < \infty$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega$. Then there exists a subsequence, still denoted by $u_\nu$, and a function $u \in GBD(\Omega)$ such that

$$
u \to u \text{ pointwise } L^n(\Omega, \mathcal{M}_1^+),
$$

$$
u \to u \text{ weakly } L^2(\Omega, \mathcal{M}_1^+),
$$

Application 1: The Ambrosio-Tortorelli convergence in the vectorial case

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Theorem 2. Assume that $\Omega$ has Lipschitz boundary. Let $u \in L^2(\Omega, \mathbb{R}^n)$ and let $u_\nu, u_\nu \gamma$ be a minimizer of the problem

$$
\min_{\{v \in L^2(\Omega, \mathbb{R}^n) \cap SBV(\Omega, \mathbb{R}^n) \}} \left( \int_\Omega (Q(c(v)) + \frac{1}{4e_\nu} (\frac{1}{e_\nu} - 1)^2 + \epsilon\nu \|
\n\right. dx,
$$

where $V_\nu := \{ v \in H^1(\Omega, \mathbb{R}^n) : \|v\|_{\mathcal{H}^{n-1}} \leq \lambda^{\nu} \in \mathbb{R}^n \}$, and $Q : \mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous and satisfies the following properties:

1. For every $\xi \in \mathbb{R}^n$, the function $Q(\cdot, \xi)$ is a quadratic form which is positive definite on the subspace $H^1(\Omega, \mathbb{R}^n) \cap \{ v : \|v\|_{\mathcal{H}^{n-1}} \leq \lambda^{\nu} \}$.\n
2. $\epsilon\nu A^2 \leq Q(\cdot, \xi) \leq c_1 A^2$ holds as $\|A\|_{\mathbb{R}^n} \to +\infty$.

Then $u_\nu \gamma \to u$ in $L^2(\Omega, \mathbb{R}^n)$ and a subsequence of $(u_\nu)$ converges in $L^2(\Omega, R^n)$ to a minimizer $u$ of the following problem

$$
\min_{\{v \in L^2(\Omega, \mathbb{R}^n) \cap SBV(\Omega, \mathbb{R}^n) \}} \left( \int_\Omega (Q(c(v)) + \Delta_\varepsilon^{\mathcal{H}^{n-1}} + \frac{1}{e_\nu} \|g - v\|^2 dx, \right)
$$

References


