A quasistatic evolution model for perfectly plastic plates derived by Γ-convergence

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Dimension reduction problems for thin structures

The rigorous derivation of lower dimensional models for thin structures is a question of great interest in mechanics and its applications. In the early 90's a rigorous approach to dimension reduction has emerged in the stationary framework and in the context of nonlinear elasticity. This approach is based on Γ-convergence and, starting from the seminal paper [3, 4], has led to establish a hierarchy of limit models for plates, rods, and shells. More recently, the Γ-convergence approach to dimension reduction has gained attention also in the evolutionary framework: in nonlinear elasticity, crack propagation, elastoplasticity with hardening, and delamination problems.

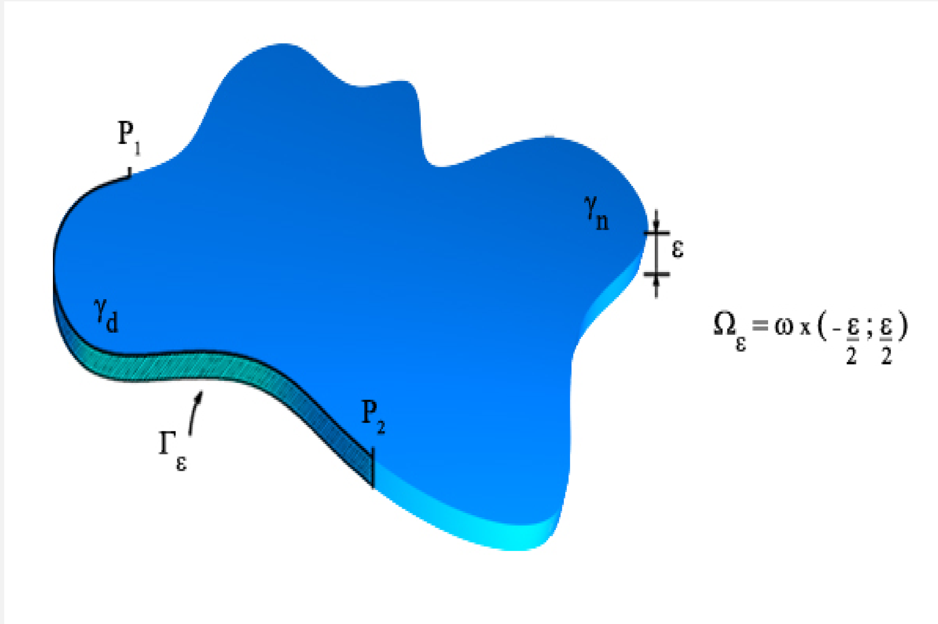
Here we focus on the rigorous justification of a quasistatic evolution model for a thin plate in perfect plasticity. More precisely, we show that solutions to the 3d quasistatic evolution problem converge, as the thickness parameter tends to zero, to a solution to a quasistatic evolution problem associated to a limit model derived via Γ-convergence.

The setting

We consider a **linearly elastic - perfectly plastic** thin plate of reference configuration

$$\Omega_\varepsilon := \omega \times (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2})$$

where ω is a domain in $\mathbb{R}^2$ with a C^2 boundary and $\varepsilon > 0$ is the thickness parameter.



The classical formulation

The quasistatic evolution problem in perfect plasticity on Ω_ε can be formulated as follows. Let $u^\varepsilon(t)$ denote the displacement field at time t and let $Eu^\varepsilon(t)$ denote the infinitesimal strain tensor at t, that is, the symmetric part of $Du^\varepsilon(t)$. Let $\sigma^\varepsilon(t)$ be the stress tensor at t and let $e^\varepsilon(t)$ and $p^\varepsilon(t)$ (a deviatoric symmetric matrix) be the elastic and plastic strain tensors at t. Assume that the plate is subjected to a **time-dependent boundary condition** $w^\varepsilon(t)$ **prescribed on a subset** $\Gamma_\varepsilon := \gamma_d \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ **of the lateral boundary of** Ω_ε and that for simplicity there are no applied loads. The classical formulation of the quasistatic evolution problem on a time interval $[0, T]$ consists in finding $u^\varepsilon(t)$, $e^\varepsilon(t)$, $p^\varepsilon(t)$, and $\sigma^\varepsilon(t)$ such that the following conditions are satisfied for every $t \in [0, T]$:

(cf1) *kinematic admissibility:* $Eu^\varepsilon(t) = e^\varepsilon(t) + p^\varepsilon(t)$ in Ω_ε and $u^\varepsilon(t) = w^\varepsilon(t)$ on Γ_ε;

(cf2) *constitutive law:* $\sigma^\varepsilon(t) = \mathbb{C}e^\varepsilon(t)$ in Ω_ε, where $\mathbb{C}$ is the elasticity tensor;

(cf3) *equilibrium:* $\operatorname{div} \sigma^\varepsilon(t) = 0$ in Ω_ε and $\sigma^\varepsilon(t)\nu_{\partial\Omega_\varepsilon} = 0$ on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$, where $\nu_{\partial\Omega_\varepsilon}$ is the outer unit normal to $\partial\Omega_\varepsilon$;

(cf4) *stress constraint:* $\sigma^\varepsilon_D(t) \in K$, where σ^ε_D is the deviatoric part of σ^ε and K is a given convex and compact subset of deviatoric 3×3 matrices, representing the set of admissible stresses;

(cf5) *flow rule:* $\dot{p}^\varepsilon(t) = 0$ if $\sigma^\varepsilon_D(t) \in \operatorname{int} K$, while $\dot{p}^\varepsilon(t)$ belongs to the normal cone to K at $\sigma^\varepsilon_D(t)$ if $\sigma^\varepsilon_D(t) \in \partial K$.

Condition (cf5) can also be written in the equivalent form:

(cf5′) *maximum dissipation principle:* $H(\dot{p}^\varepsilon(t)) = \sigma^\varepsilon_D(t) : \dot{p}^\varepsilon(t)$, where H is the support function of K, i.e., $H(p) := \sup\{\sigma : p \ : \ \sigma \in K\}$.

The static result: a weak formulation

We consider a boundary displacement w^ε independent of time, we introduce the functional

$$\mathcal{E}_\varepsilon(u, e, p) := \tfrac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}e : e\, dx + \int_{\Omega_\varepsilon} H(p)\, dx$$

defined on the class $\mathcal{A}(\Omega_\varepsilon, w^\varepsilon)$ of all triples (u, e, p) satisfying $Eu = e + p$ in Ω_ε and $u = w^\varepsilon$ on Γ_ε, and we study its limit, as $\varepsilon \to 0$, in the sense of Γ-convergence. As pointed out in [1], because of the linear growth of H, the functional $\mathcal{E}_\varepsilon$ is not coercive in any Sobolev norm. **The natural setting for a weak formulation is the space** $BD(\Omega_\varepsilon)$ **of functions with bounded deformation in** Ω_ε **for the displacement** u **and the space** $M_b(\Omega_\varepsilon \cup \Gamma_\varepsilon; \mathbb{M}^{3\times3}_D)$ **of** $\mathbb{M}^{3\times3}_D$**-valued bounded Borel measures on** $\Omega_\varepsilon \cup \Gamma_\varepsilon$ **for the plastic strain** p. This is also natural from a mechanical point of view, because it is well known that in absence of hardening displacements may develop jump discontinuities along so-called slip surfaces, on which plastic strain concentrates. Since $p \in M_b(\Omega_\varepsilon \cup \Gamma_\varepsilon; \mathbb{M}^{3\times3}_D)$, the functional

$$\int_{\Omega_\varepsilon} H(p)\, dx$$

has to be interpreted according to the theory of convex functions of measures, developed in [5, 8], as

$$\int_{\Omega_\varepsilon \cup \Gamma_\varepsilon} H\Big(\frac{dp}{d|p|}\Big)\, d|p|,$$

where $dp/d|p|$ is the Radon-Nicodym derivative of p with respect to its total variation $|p|$. Moreover, the boundary condition is relaxed by requiring that

$$p = (w^\varepsilon - u) \odot \nu_{\partial\Omega_\varepsilon}\mathcal{H}^{\in} \quad \text{on } -_\varepsilon, \tag{1}$$

where $\odot$ denotes the symmetric tensor product. The mechanical interpretation of (1) is that u may not attain the boundary condition: in this case a plastic slip is developed along Γ_ε, whose amount is proportional to the difference between the prescribed boundary value and the actual value.

The class $\mathcal{A}_{KL}(w)$

Conditions (2)–(3) imply that u **is a Kirchhoff-Love displacement in** $BD(\Omega)$, that is, u_3 **belongs to the space** $BH(\omega)$ of functions with bounded Hessian in ω and there exists $\bar{u} \in BD(\omega)$ such that

$$u(x) = (\bar{u}_1(x') - x_3\partial_1 u_3(x'),\ \bar{u}_2(x') - x_3\partial_2 u_3(x'),\ u_3(x')) \quad \text{for a.e. } x = (x', x_3) \in \Omega.$$

Moreover,

$$(Eu)_{\alpha\beta} = (E\bar{u})_{\alpha\beta} - x_3\partial^2_{\alpha\beta}u_3 \quad \text{for } \alpha, \beta = 1, 2.$$

We note that the averaged tangential displacement $\bar{u}$ may exhibit jump discontinuities, while, because of the embedding of $BH(\omega)$ into $C(\overline{\omega})$, the normal displacement u_3 is continuous, but its gradient may have jump discontinuities. Moreover, the second equality in (2), together with the second condition in (3), implies that u_3 satisfies the boundary condition $u_3 = w_3$ on γ_d. Since the dependence of u on x_3 is affine, we can conclude that in the limit model slip surfaces are vertical surfaces whose projection on ω is the union of the jump set of $\bar{u}$ and the jump set of ∇u_3. **Conditions** (2)–(3) **do not imply, in general, that** e **and** p **are affine with respect to** x_3. However, one can prove that e and p admit the following decomposition:

$$e = \bar{e} + x_3\hat{e} + e_\perp, \quad p = \bar{p} \otimes \mathcal{L}^1 + \hat{p} \otimes x_3\mathcal{L}^1 - e_\perp,$$

where $\bar{e}, \hat{e} \in L^2(\omega; \mathbb{M}^{2\times2}_{sym})$, $e_\perp \in L^2(\Omega; \mathbb{M}^{2\times2}_{sym})$, and $\bar{p}, \hat{p} \in M_b(\omega \cup \gamma_d; \mathbb{M}^{2\times2}_{sym})$. **Since it may be energetically convenient to have** $e_\perp \neq 0$, **we cannot in general express the limit functional** $\mathcal{J}$ **in terms of two-dimensional quantities only.**

The variational formulation

The first existence result of a quasistatic evolution in perfect plasticity has been proved in [7] by means of viscoplastic approximations. More recently, in [1] the problem has been reformulated within the framework of the **variational theory for rate-independent processes**, developed in [6]. The variational formulation reads as follows: to find a triple $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ such that for every $t \in [0, T]$ we have

(qs1) *global stability:* $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ satisfies $Eu^\varepsilon(t) = e^\varepsilon(t) + p^\varepsilon(t)$ in Ω_ε, $u^\varepsilon(t) = w^\varepsilon(t)$ on Γ_ε, and minimizes

$$\tfrac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}f : f\, dx + \int_{\Omega_\varepsilon} H(q - p^\varepsilon(t))\, dx$$

among all kinematically admissible triples (v, f, q);

(qs2) *energy balance:*

$$\tfrac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}e^\varepsilon(t) : e^\varepsilon(t)\, dx + \int_0^t \int_{\Omega_\varepsilon} H(\dot{p}^\varepsilon(s))\, dx ds = \tfrac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}e^\varepsilon(0) : e^\varepsilon(0)\, dx + \int_0^t \int_{\Omega_\varepsilon} \mathbb{C}e^\varepsilon(s) : E\dot{w}^\varepsilon(s)\, dx ds.$$

The existence of a quasistatic evolution according to the previous formulation and the extent to which this is equivalent to the original formulation is the main focus of [1].

The static result: the limit model

Setting $\Gamma_d := \gamma_d \times (-\frac{1}{2}, \frac{1}{2})$, we show that the Γ-limit of $\mathcal{E}_\varepsilon$ is finite only on the class $\mathcal{A}_{KL}(w)$ of triples (u, e, p) such that $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$, $p \in M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$, and

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w - u) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_d, \tag{2}$$

$$e_{i3} = 0 \quad \text{in } \Omega, \qquad p_{i3} = 0 \quad \text{in } \Omega \cup \Gamma_d, \qquad i = 1, 2, 3, \tag{3}$$

where w is a suitable limit boundary datum and $\nu_{\partial\Omega}$ is the outer unit normal to $\partial\Omega$. Note that, owing to (3), we can identify e with a function in $L^2(\Omega; \mathbb{M}^{2\times2}_{\rm sym})$ and p with a measure in $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{2\times2}_{\rm sym})$.

Theorem 1. *On $\mathcal{A}_{KL}(w)$ the Γ-limit is given by the functional*

$$\mathcal{J}(u, e, p) := \tfrac{1}{2}\int_\Omega \mathbb{C}_r e : e\, dx + \mathcal{H}_r(p)$$

where

$$\mathcal{H}_r(p) := \int_{\Omega \cup \Gamma_d} H_r\Big(\frac{dp}{d|p|}\Big)\, d|p|,$$

and the tensor $\mathbb{C}_r$ and the function H_r are defined through pointwise minimization formulas.

The main result

Theorem 2. *Let $w^\varepsilon(t)$ be a prescribed boundary datum of Kirchhoff-Love type on Γ_ε, and consider a compact sequence of initial data $(u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0)$. For every $\varepsilon > 0$, let $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ be a quasistatic evolution in the sense of (qs1)–(qs2) for the boundary datum $w^\varepsilon(t)$ and the initial datum $(u^\varepsilon_0, e^\varepsilon_0, p^\varepsilon_0)$, then, (up to a suitable scaling) $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ converges, as $\varepsilon \to 0$, to a limit triple $(u(t), e(t), p(t))$ that solves the quasistatic evolution problem associated to the limit model in Theorem 1.*

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