The Cahn-Hilliard equation:

**How do we have such a weird rate?**

Data fitting indicates $\epsilon M$ the diffusion mobility.

A power law relation $R(t) \sim t^{1/3}$. What else is going on?

**Diffusion mobility dependent coarsening (Sheng et al. 2010)**

**Geometry of interfaces**

The 0-level set of $u$ is a hypersurface $\Gamma(t) \subset \mathbb{R}^d$. Let $\rho(x, t)$ be the signed distance function from $x$ to $\Gamma(t)$ and $z = \frac{\rho}{\epsilon}$. Then $x \mapsto \{s_1, s_2, z\}$.

$$\frac{\partial \rho}{\partial t} = \text{the normal velocity of } \Gamma,$$

$$\nabla z = \sum_{i=1}^2 (\nabla s_i) \frac{\partial}{\partial s_i} + \epsilon^{-1}(\nabla s_i) \frac{\partial}{\partial z},$$

$$\epsilon^2 \Delta z = \partial \rho + \epsilon k \partial z + \epsilon^2(\Delta z + z \kappa_1 \partial z) + + \epsilon^2 \Delta z + O(\epsilon^3),$$

where in 3D:

$$\kappa = k_1 + k_2 = H,$$

$$\kappa_1 = -k_1^2 - k_2^2 = 2K - H^2.$$

Here $k_1$, $k_2$ are principal curvatures and $H$, $K$ are mean and Gaussian curvatures.

**Interface migration laws for the CH with $M = \frac{1+u}{2}$**

- In time scale $t = O(\epsilon^{-1})$, one-sided Mullins-Sekerka (or Hele-Shaw)
  $$\Delta \mu_1 = 0 \quad \text{in } \Omega_+, \quad \mu_1 = H \quad \text{on } \Gamma, \quad V = \partial_\mu \mu_1^+ \quad \text{on } \Gamma. \quad (1)$$
  Disjoint components of $\Omega_+$ do not communicate.

- In time scale $t = O(\epsilon^{-2})$, a quasi-stationary porous medium diffusion process in $\Omega_-$ determines the normal velocity of $\Gamma$.
  $$\nabla \cdot (\mu_1 \nabla \mu_1) = 0 \quad \text{in } \Omega_-, \quad \mu_1 = H \quad \text{on } \Gamma, \quad \nabla = \partial_\mu \mu_1^+ \quad \text{on } \Gamma. \quad (2)$$

- Scaling arguments suggest $R \sim t^{1/3}$ in the $t = O(\epsilon^{-1})$ dynamics and $R \sim t^{1/4}$ in the $t = O(\epsilon^{-2})$ dynamics. So the numerical simulation is in a regime where a hybrid behavior, or a crossover of coarsening occurs.

**Interface migration laws for CH with $M = 1 - u^2$**

- In time scale $t = O(\epsilon^{-1})$ time scale, no migration of interface.

- In time scale $t = O(\epsilon^{-2})$ time scale, the normal velocity of the interface is determined by surface diffusion together with a quasi-stationary porous medium diffusion process in both phases.
  $$\nabla \cdot (\mu_1 \nabla \mu_1) = 0 \quad \text{in } \Omega_{\pm}, \quad \mu_1 = H \quad \text{on } \Gamma, \quad \nabla = \partial_\mu \mu_1^+ \quad \text{on } \Gamma. \quad (4)$$

- Our asymptotic analysis indicates that even though the diffusion mobility is degenerate in both phases, the quasi-stationary porous medium diffusion process provides the communication mechanism for disjoint components of the interface $\Gamma$, which makes coarsening possible.

- This mechanism depends on the fact that the double well potential $W(u)$ is smooth at $u = \pm 1$ with $W''(\pm 1) \neq 0$. 

**Explanation through asymptotic analysis**

The two phases are separated by an interface. We can directly describe the evolution of the interface. Sharp interface models are connected to phase field models.

Free energy is related to interface area by $\Gamma$-limit:

$$\lim_{\epsilon \to 0} \int_\Omega \frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \, dx = \int \alpha_1 \, ds.$$

Under the mass conservation restriction, minimizers of $E$ minimize the interface area.

**Cahn-Hilliard equation for phase separations**

Let $u(x)$ denote the density of components of a binary mixture and $W(u)$ a double-well potential. A transition layer between the two phases has thickness of order $\epsilon$.

$$L(u) = \int_\Omega \frac{\epsilon^2}{2} |\nabla u|^2 + W(u) \, dx.$$ 

The Cahn-Hilliard equation is:

$$H^{-1} \cdot \text{gradient flow}$$

$$u_t = \nabla \cdot M(u) \nabla (-\epsilon^2 \Delta u + W'(u))$$

**Coarsening Rates**

A characteristic length scale $R(t)$ grows in time, the growth rate depends on the diffusion mobility $M$.

1. $M = 1$ : $R(t) \sim t^{1/3}$
   Front migrates according to Mullins-Sekerka model (Pego ‘89).

2. $M = 1 - u^2$ : $R(t) \sim t^{1/4}$
   This rate is well known and is generally attributed to surface diffusion. But surface diffusion is not enough for coarsening to occur when there are disjoint branches of interfaces. What else is going on?

3. $M = \frac{1+u}{2}$ : $R(t) \sim ????

**Numerical simulation: data fitting ($M = \frac{1+u}{2}$, Sheng et al. 2010)**

Assume a power law relation $R(t)^\alpha - R_0^\alpha = kt$.

Data fitting indicates $n = 3.3$, that is $R(t) \sim t^{1/3.3}$.

**Question:** How do we have such a weird rate?