Local and global minimality issues for a nonlocal isoperimetric problem in $\mathbb{R}^N$

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Origins

This kind of energies come from the theory of domain patterns in systems with competing short-range attractive interactions and long range repulsive Coulomb interactions ($\alpha = 1$, $N = 3$) (*e.g.* modeling of microphase separation for A/B diblock copolymer melts, Gamov's liquid drop model). See [6] for more on physical background.

Critical sets

We call $E \subset \mathbb{R}^N$ a regular critical set if

- E is of class C^1
- the equation $H_{\partial E}(x) + 2\gamma v_E(x) = \lambda$ of null first variation holds weakly on ∂E, where we set $v_E(x) := \int_E \frac{1}{|x-y|^\alpha}\,\mathrm{d}y$.

Our spaces

- $\widetilde{H}^1(\partial E)$: functions of $H^1(\partial E)$ with null average
- $T^\perp(\partial E)$: functions $\varphi \in \widetilde{H}^1(\partial E)$ s.t. $\int_{\partial E} \varphi \nu_E^i \,\mathrm{d}\mathcal{H}^{N-1} = 0$ for each $i = 1, \dots, N$.

Notice that, by the translation invariance of $\mathcal{F}$, for each $i = 1, \dots, N$ we have
$$\partial^2\mathcal{F}(E)[\nu_E^i] = 0.$$

Main references

[1] E. Acerbi, N. Fusco, M. Morini, *Minimality via second variation for a nonlocal isoperimetric problem.* Accepted Paper: Commun. in Mathematical Physics (2011).

[2] M. Bonacini, R. Cristoferi, *Local and global minimality issues for a nonlocal isoperimetric problm in $\mathbb{R}^N$.* In preparation.

[3] M. Cicalese, E. Spadaro, *Droplet minimizers of an isoperimetric problem with long-range interactions.* Preprint (2013).

[4] V. Julin, G. Pisante, *Minimality via second variation for microphase separation of diblock copolymer melts.* Preprint (2013).

[5] H. Knüpfer, C.B. Muratov, *On an isoperimetric problem with a competing non-local term. II. The general case.* To be published in Commun. Pure Appl. Math.

[6] C.B. Muratov, *Theory of domain patterns in systems with long-range interaction of Coulomb type.* Phys. Rev. E **66** (2002), 1-25.

The problem

We are interested in the behaviour of the following volume constraint minimization problem

$$\min\left\{ \mathcal{F}(E) := \mathcal{P}(E) + m^{\frac{N-\alpha+1}{N}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y \;:\; |E| = 1 \right\}, \quad N \geq 2,$$

perimeter:
- minimized by the ball
- favours compact configurations

nonlocal term:
- maximized by the ball
- favours scattered configurations

with respect to the variation of the parameters $\alpha \in (0, N-1)$ and $m > 0$.

Previous results

See [1] and [4] for a second variation approach with $\alpha = 1$ in a periodic setting and in a general one respectively, see [5] for global issues in $N \leq 7$, and see [3] for a detailed description of the geometry of single droplet patterns in a bounded domain with $\alpha = 1$.

Second variation

Let $E \subset \mathbb{R}^N$ be a regular critical set. We define the quadratic form $\partial^2\mathcal{F}(E) : \widetilde{H}^1(\partial E) \to \mathbb{R}$ by

$$\partial^2\mathcal{F}(E)[\varphi] = \int_{\partial E}\left(|D_\tau\varphi|^2 - |B_{\partial E}|^2\varphi^2\right)\mathrm{d}\mathcal{H}^{N-1} + 2m^{\frac{N-\alpha+1}{N}}\left(\int_{\partial E}(\partial_{\nu_E} v_E)\varphi^2\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial E}\int_{\partial E}\frac{\varphi(x)\varphi(y)}{|x-y|^\alpha}\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\right).$$

Here D_τ denotes the tangential gradient, ∂_{ν_E} the normal derivative and $|B_{\partial E}|^2$ the sum of the squares of the principal curvatures of ∂E. If X is a regular volume preserving vector field, then $\partial^2\mathcal{F}(E)[X\cdot\nu_E]$ is the second variation of $\mathcal{F}$ at E along the flow associated to X.

Main result

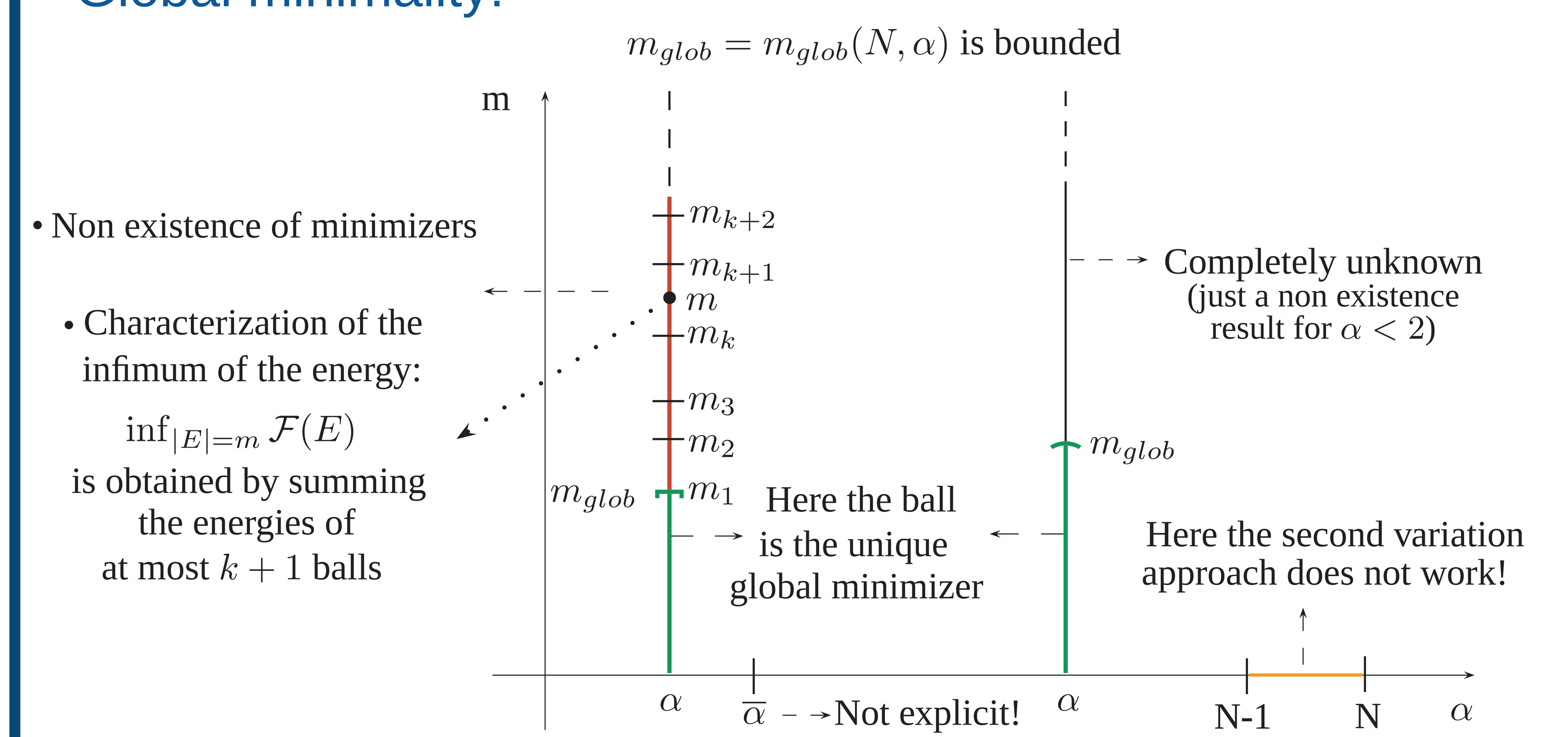
Let E be a regular critical set for $\mathcal{F}$ such that $\partial^2\mathcal{F}(E)[\varphi] > 0$ for all $\varphi \in T^\perp(\partial E)\setminus\{0\}$. Then there exist $\delta > 0$ and $C > 0$ such that

$$\mathcal{F}(F) \geq \mathcal{F}(E) + C\big(\alpha(E,F)\big)^2$$

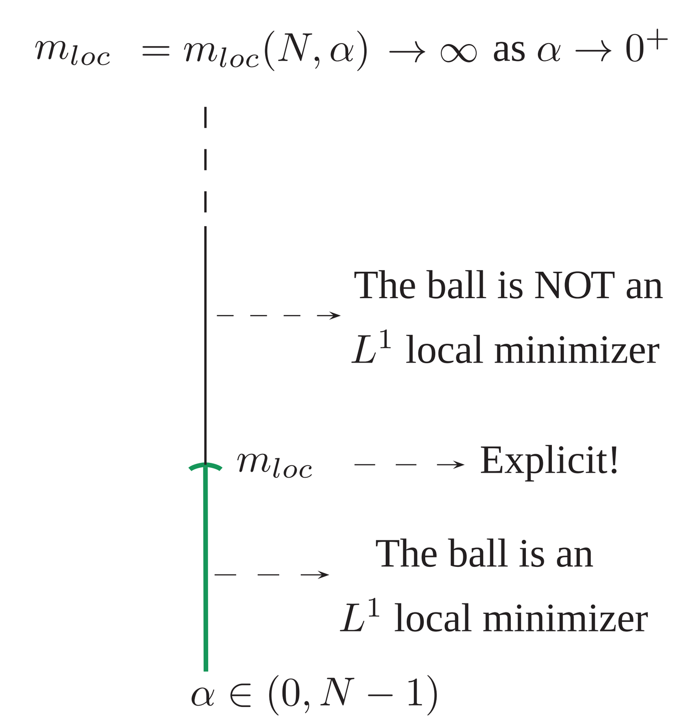
for every $F \subset \mathbb{R}^N$ such that $|F| = 1$ and $\alpha(E,F) := \min_{x\in\mathbb{R}^N} |E \triangle (x+F)| < \delta$.

Local and global minimality of the ball

Global minimality:



Local minimality:



The proof (sketch)

The proof has two main steps:

- $W^{2,p}$ local minimality: a non degenerating property and a $W^{2,p}$ continuity of the second variation allow to prove a local minimality w.r.t. $W^{2,p}$ perturbations.
- L^1 local minimality: arguing by absurd it is possible to construct a sequence of quasi area minimizers that converges in $W^{2,p}$ to E and do not satisfy the estimate of the theorem.

Open problems

- existence and non existence in the case $\alpha \in [\overline{\alpha}, N-1)$: are there other minimizers than the ball? Is the existence set an interval?
- the case $\alpha \in [N-1, N)$, where other techniques seem to be required.