Local and global minimality issues for a nonlocal isoperimetric problem in $\mathbb{R}^N$

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The problem
We are interested in the behaviour of the following volume constraint minimization problem

$$\min \left\{ F(E) := P(E) + m \frac{N-1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x) \chi_E(y)}{|x-y|^\alpha} \, dx \, dy : |E| = 1 \right\}, \quad N \geq 2,$$

perimeter: minimized by the ball
favours compact configurations
nonlocal term: maximized by the ball
favours scattered configurations

with respect to the variation of the parameters $\alpha \in (0, N-1)$ and $m > 0$.

Origins
This kind of energies come from the theory of domain patterns in systems with competing short-range attractive interactions and long range repulsive Coulomb interactions ($\alpha = 1, N = 3$) (e.g. modeling of microphase separation for A/B diblock copolymer melts, Gamov’s liquid drop model). See [6] for more on physical background.

Critical sets
We call $E \subset \mathbb{R}^N$ a regular critical set if

- $E$ is of class $C^1$
- the equation $H_{E_i}(x) + 2\gamma v_E(x) = \lambda$ of null first variation holds weakly on $\partial E$, where we set $v_E(x) := \int_{E_i} \frac{1}{|x-y|^\alpha} \, dy$.

Notice that, by the translation invariance of $F$, for each $i = 1, \ldots, N$ we have $\partial^2 F(E)[v_{E_i}] = 0$.

Second variation
Let $E \subset \mathbb{R}^N$ be a regular critical set. We define the quadratic form $\partial^2 F(E) : \widetilde{H}^1(\partial E) \to \mathbb{R}$ by

$$\partial^2 F(E)[\varphi] = \int_{\partial E} (|D_t \varphi|^2 - |B_{\partial \tilde{E}}|^2 \varphi^2) \, dH^{N-1} + 2m \frac{N-1}{2} \left( \int_{\partial \tilde{E}} (\partial_{\nu E} \varphi)^2 \, dH^{N-1} \right)^2 + \int_{\partial \tilde{E}} \frac{\varphi(x) \varphi(y)}{|x-y|^\alpha} \, dH^{N-1}(x) \, dH^{N-1}(y).$$

Here $D_t$ denotes the tangential gradient, $\partial_{\nu E}$ the normal derivative and $|B_{\partial \tilde{E}}|^2$ the sum of the squares of the principal curvatures of $\partial E$. If $X$ is a regular volume preserving vector field, then $\partial^2 F(E)[X \cdot \nu_E]$ is the second variation of $F$ at $E$ along the flow associated to $X$.

Main result
Let $E$ be a regular critical set for $F$ such that $\partial^2 F(E)[\varphi] > 0$ for all $\varphi \in \tilde{T}(\partial E) \setminus \{0\}$. Then there exist $\delta > 0$ and $C > 0$ such that

$$F(F) \geq F(E) + C (\alpha(E, F))^2$$

for every $F \subset \mathbb{R}^N$ such that $|F| = 1$ and $\alpha(E, F) := \min_{x \in \mathbb{R}^N} |E \Delta (x + F)| < \delta$.

Local and global minimality of the ball

Global minimality:
- Non existence of minimizers
- Characterization of the infimum of the energy:
  $$\inf_{E \subset \mathbb{R}^N} F(E)$$
  is obtained by summing the energies of at most $k+1$ balls

Local minimality:
- Completely unknown (just a non existence result for $\alpha < 2$)
- The ball is NOT an $L^1$ local minimizer
- Here the second variation approach does not work!
- The ball is NOT an $L^1$ local minimizer

Main references

Previous results
See [1] and [4] for a second variation approach with $\alpha = 1$ in a periodic setting and in a general one respectively, see [5] for global issues in $N \leq 7$, and see [3] for a detailed description of the geometry of single droplet patterns in a bounded domain with $\alpha = 1$.

The proof (sketch):
- $W^{2,p}$ local minimality: a non degenerating property and a $W^{2,p}$ continuity of the second variation allow to prove a local minimality w.r.t. $W^{2,p}$ perturbations.
- $L^1$ local minimality: arguing by absurd it is possible to construct a sequence of quasi area minimizers that converges in $W^{2,p}$ to $E$ and do not satisfy the estimate of the theorem.

Open problems
- existence and non existence in the case $\alpha \in [\pi, N-1)$: are there other minimizers than the ball? Is the existence set an interval?
- the case $\alpha \in [N-1, N)$, where other techniques seem to be required.