Vortices in stochastically perturbed Ginzburg-Landau equations

LEHRSTUHL I FÜR MATHEMATIK

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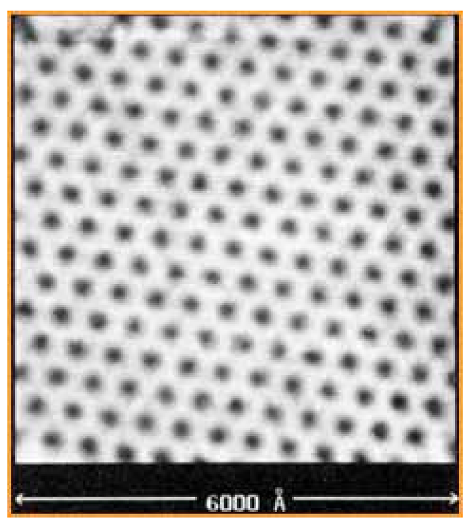
Motivation

We want to describe the behavior of point singularities of the solution to the Ginzburg-Landau equation.



Vortex lattice in NbSe2

Our physical inspiration comes from the theory of superconductivity. In certain situations, the superconducting matter forms *Abrikosov vortices* (⇝ Nobel Prize in Physics, 2003). The nature of the phenomenon is essentially probabilistic. The existing mathematical theory does not take it into account. Our problem is a toy model that helps us to understand, what kind of difficulties can we expect.

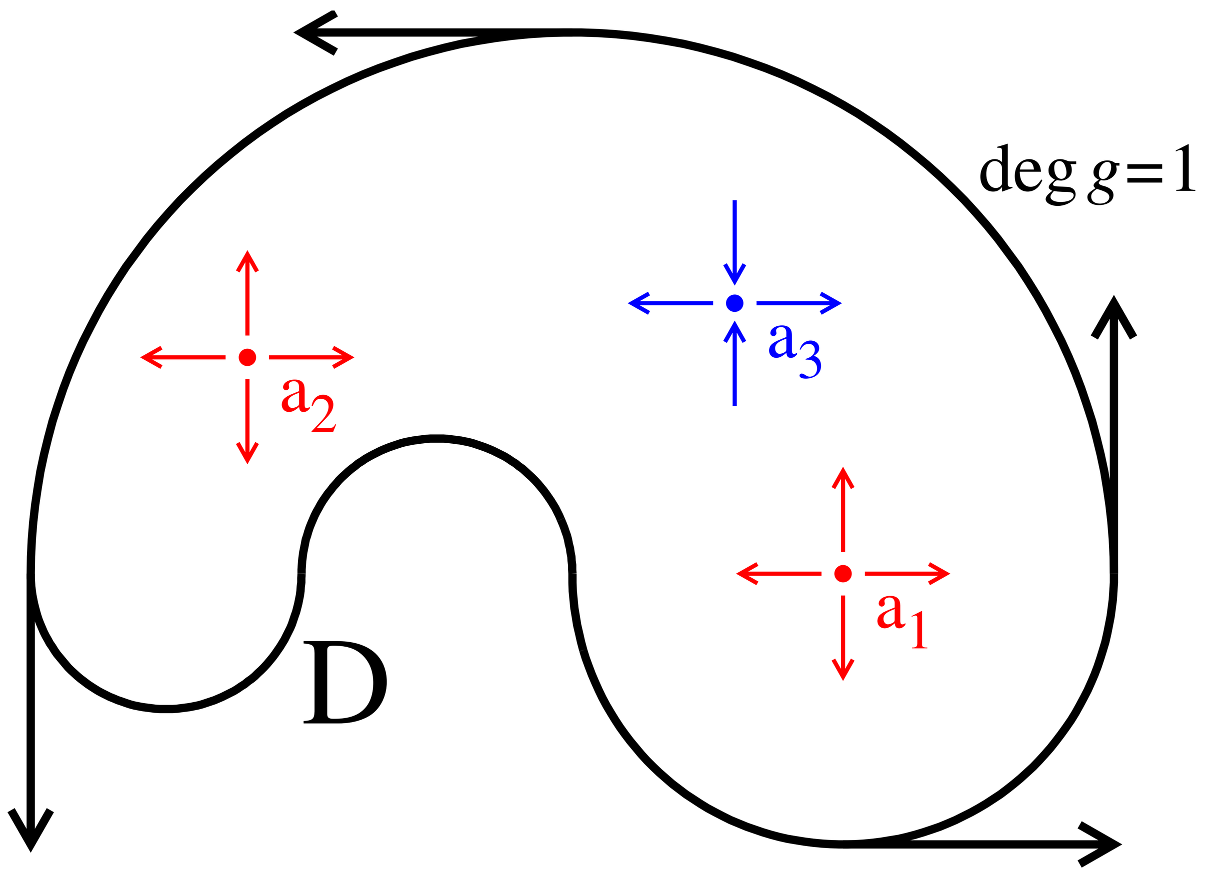
Ginzburg-Landau vortices

Evolution equations related to the *Ginzburg-Landau energy*:

$$E_\varepsilon(u_\varepsilon) = \int_D \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2$$

$$u_\varepsilon|\partial D = g, \quad |\deg g| = n \neq 0$$

Energy bound $E(u_\varepsilon) \lesssim \pi n \log\frac{1}{\varepsilon}$ as $\varepsilon \to 0$ gives rise to an $\mathbb{S}^1$-valued map with a finite number of point singularities $a = \{a_k\}$ with non-trivial winding number (*vortices*). Energy concentrates at the a_k of the order $\pi\log(1/\varepsilon)$ when the winding numbers $d_k = \pm 1$.



Energy expansion (Bethuel et al):

$$E_\varepsilon(u_\varepsilon) = n\left(\pi\log\frac{1}{\varepsilon} + \gamma\right) + W(a,d) + o(1)$$

Here γ is a global constant and $W(a,d)$ is the *renormalized energy*.

ODE for vortex paths

Theorem 1. *For the solutions of*

$$\text{(1)} \qquad \frac{1}{\log\frac{1}{\varepsilon}}\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon + \frac{1}{\log\frac{1}{\varepsilon}}\nabla u_\varepsilon \cdot F(x,t)$$

the vortex dynamics is determined by an ODE

$$\dot{a}_k = -\frac{1}{\pi}\frac{\partial W}{\partial a_k} - F(a_k,t).$$

Other equations: Jerrard and Soner, Lin, Sandier and Serfaty, etc.

Important ingredients

(A) How many vortices: upper bounds on the energy

$$E_\varepsilon(u_\varepsilon) \leqslant \pi n \log\frac{1}{\varepsilon} + C.$$

(B) Where they are: compactness of the *rescaled energy measure*

$$\mu_\varepsilon(u_\varepsilon(t)) := \frac{e_\varepsilon(u_\varepsilon(t))}{\log\frac{1}{\varepsilon}} \xrightarrow[\varepsilon\to 0]{} \pi\sum_{k=1}^{n}\delta_{a_k(t)}.$$

(C) How do they move for our equation: *conservation laws*

$$\partial_t e_\varepsilon(u_\varepsilon) = -\frac{1}{\log\frac{1}{\varepsilon}}|\partial_t u_\varepsilon|^2 + \nabla\cdot(\partial_t u_\varepsilon\cdot\nabla u_\varepsilon) + \frac{1}{\log\frac{1}{\varepsilon}}(\partial_t u_\varepsilon\cdot\nabla u_\varepsilon)\cdot F.$$

Stochastic Ginzburg-Landau equations

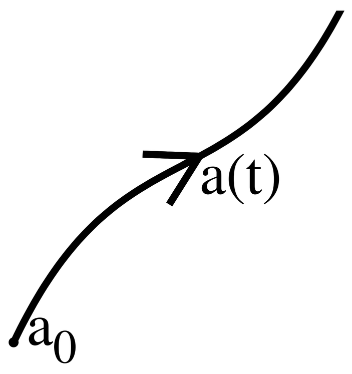
If in (1) the force is *random*:

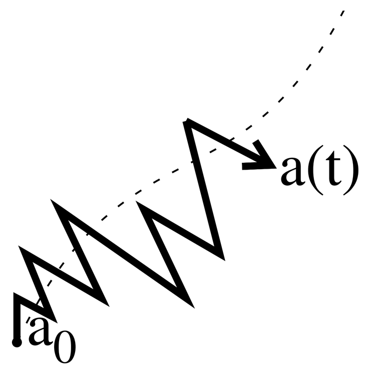
$$\text{(2)} \qquad du_\varepsilon = \log\frac{1}{\varepsilon}\left(\Delta u_\varepsilon + \frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon\right)dt + \beta_\varepsilon\nabla u_\varepsilon\cdot F(x)\circ dB_t$$

Our goal is the vortex motion governed by

$$\text{(3)} \qquad da_k = -\frac{1}{\pi}\frac{\partial W}{\partial a_k}dt - F(a_k)\circ dB_t.$$

We expect to obtain instead of this... something like this





Challenges

- How to understand? - No derivatives, no conservation laws ⇝ Stochastic calculus
- How to control? - One more tricky parameter ω from the probability space ⇝ Look at expectations
- What the correct scaling of β_ε is? ⇝ We are looking for the stochastic forcing that is strong enough to make difference but still moderated enough to preserve the general picture.

Existence and uniqueness

The equation (2) makes sense, when considered as an integral equation. The stochastic integral is a well-defined object, whereas a derivative of a stochastic process is not.

We transform (2) into a PDE with random, but nice coefficients and so establish

Theorem 2. *For every $\varepsilon > 0$, the solution $u_\varepsilon(x,t;\omega)$ to the equation (2) exists and is unique for $t \in [0,T]$ for any $T < +\infty$. This solution is a continuous $C^2(D)$-valued semimartingale, almost sure.*

Energy in the stochastic case

Instead of (C), we can derive the *Itô equation* on the energy. That gives us the control similar to (A).

Theorem 3. *There exists C such that the process*

$$Y_\varepsilon(t) := E_\varepsilon(u_\varepsilon(t))\exp\{-Ct\beta_\varepsilon^2\}$$

is a supermartingale.

The quest of tightness

We want to pass to the limit, like in (B). Proving directly the relative compactness for a *stochastic process* is a really hard task. We may check the *tightness* for the family of interest instead, thanks to the Prokhorov theorem!

Theorem 4. *For $\beta_\varepsilon \lesssim \frac{1}{\sqrt{\log\frac{1}{\varepsilon}}}$, the family of the rescaled energy measures $\mu_\varepsilon(u_\varepsilon(t))$ is tight and consequently relatively compact in the space $C(0,T;(C_0^{0,\alpha})^*)$ for every $\alpha \in (0,1)$.*

Further directions

- Effective motion law for the case $\beta_\varepsilon \lesssim \frac{1}{\sqrt{\log\frac{1}{\varepsilon}}}$. The intuition is, it should be deterministic.
- Tightness in the maximal regime $\beta_\varepsilon \sim 1$. That is expected to give (3).

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