The Long Time Behavior of Diffusion in Tilted Periodic Potentials

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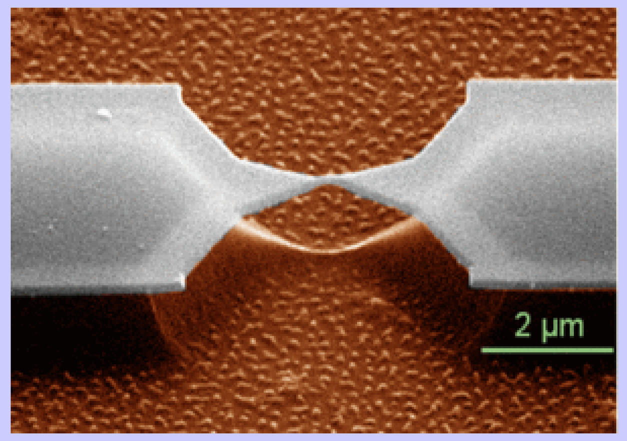
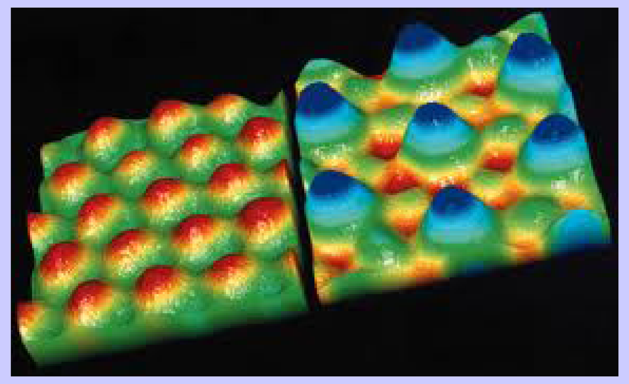
> 1. Motivation

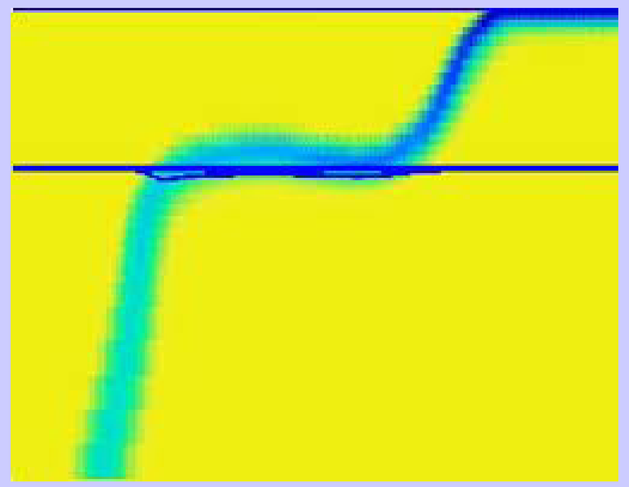
- Josephson effect
  WHAT is the effective voltage on a Josephson junction?
- Phase boundary propagation
  HOW to estimate evolvement of the earth crack in the long run?
- Charge density wave
  WHY does the charge-density wave appear non-Ohmic conduction when the applied field is small enough?



Josephson junction



Charge density waves



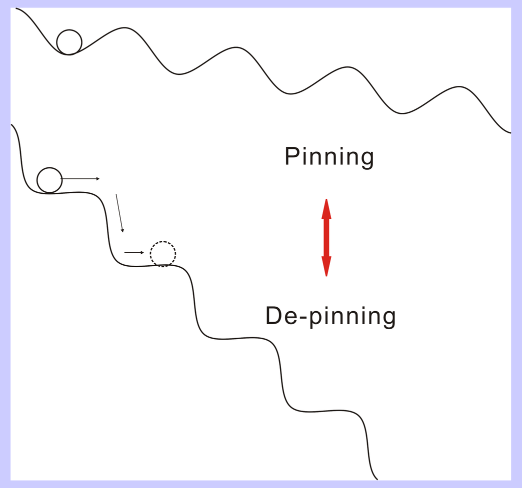
Phase boundary velocity versus applied load

> 2. Model—pinning and de-pinning

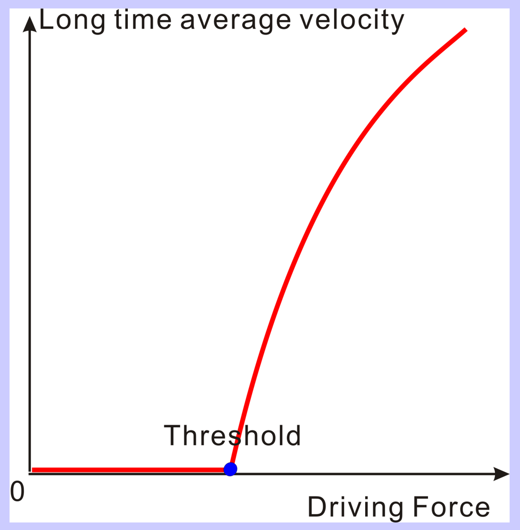
- Particle *diffusion* in *tilted* periodic potentials
- The long time average velocity as a function of the *external force* F
- The threshold F_* of the external force F
- The *scaling law* of the long time average velocity V_F
- *Langevin equation*:

$$m\ddot{q} = F - \Psi'(q) - \gamma\dot{q} + \sqrt{2\gamma\beta^{-1}}\dot{W}(t),$$
$$q(0) = q_0, \quad \dot{q}(0) = p_0,$$

- ◇ F—the external force
- ◇ Ψ—the smooth periodic potential
- ◇ γ—the friction coefficient
- ◇ β—the inverse temperature
- ◇ W—the Brownian motion



Tilted periodic potential



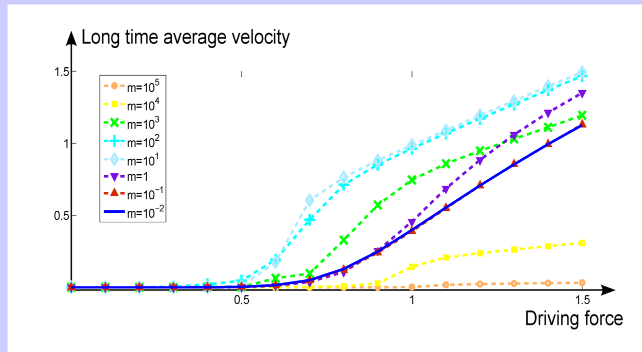
Pinning and de-pinning

> 3. Over-damped limit ($\gamma \to \infty$ or $m \to 0$)—results

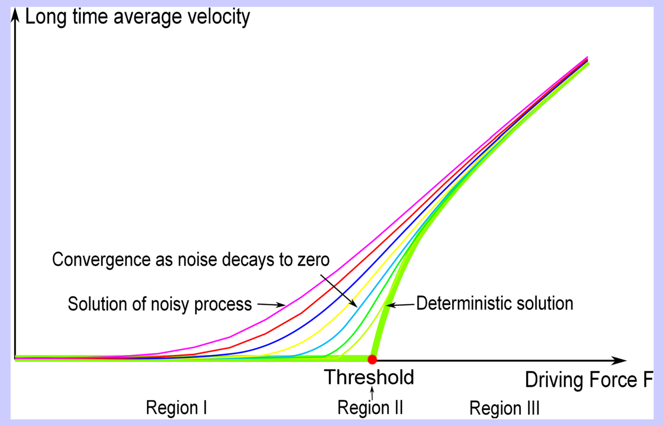
- *Smoluchowski equation*

$$\gamma\dot{q} = F - \Psi'(q) + \sqrt{2\gamma\beta^{-1}}\dot{W},$$
$$q(0) = q_0.$$

- **We proved**:
  - The long time average velocity of the **Langevin equation** converges to that of the **Smoluchowski equation** in the **over-damped limit** ($m \to 0$)
  - The long time average velocity of the **Smoluchowski equation** converges to that of the corresponding **deterministic** equation as noise goes to zero and the **convergence rate** differs when F is below, at or above the threshold F_*



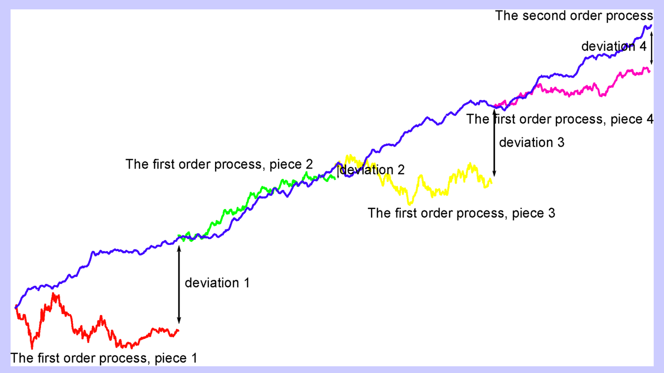
Convergence of V_F in the vanishing mass limit ($m \to 0$)



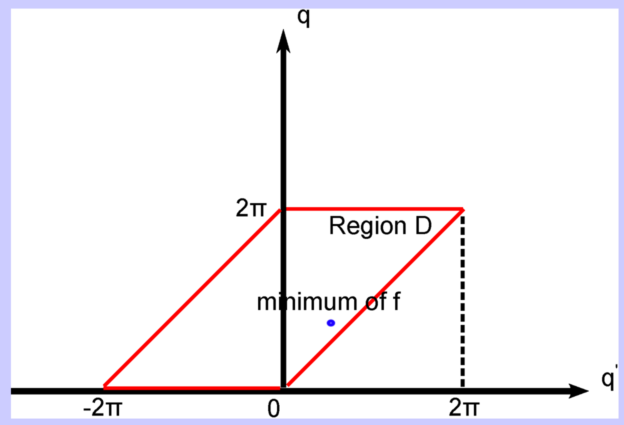
Convergence of V_F in the vanishing noise limit ($\beta \to \infty$)

> 4. Over-damped limit ($\gamma \to \infty$ or $m \to 0$)—methods

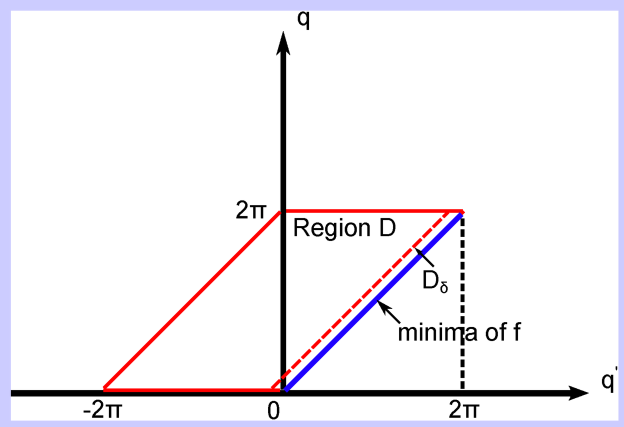
- *Sample path* approximation
  We use pieces of the first order *Smoluchowski* process to approximate the position process of the second order *Langevin equation* and show that the total ***approximation error***, i.e., the sum of deviations of each piece, vanishes as $m \to 0$
- *Ergodicity*
  The long time average velocity equals to the integration of the velocity field over the phase space w.r.t. an invariant measure
- *Laplace's method*



Approximate the second order process by pieces of the first order process



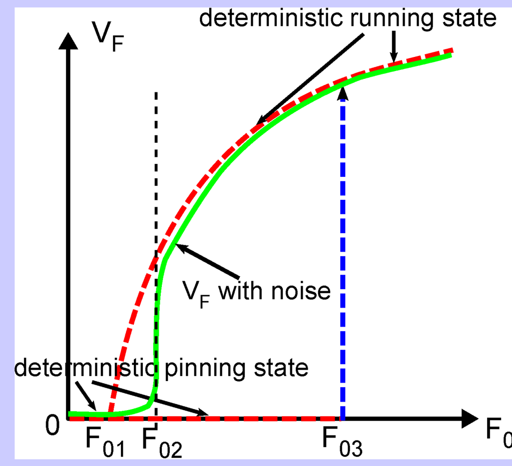
$F < F_*$



$F > F_*$

> 5. Under-damped limit ($\gamma \to 0$)—results

- *Bi-stability phenomenon*, i.e., the pinning and running states coexist in the system
- **We obtained**:
  - Derivation of the **bi-stability thresholds**
  - Asymptotics of the **mean return time** of the pinning or running state in the vanishing noise limit ($\beta \to \infty$)



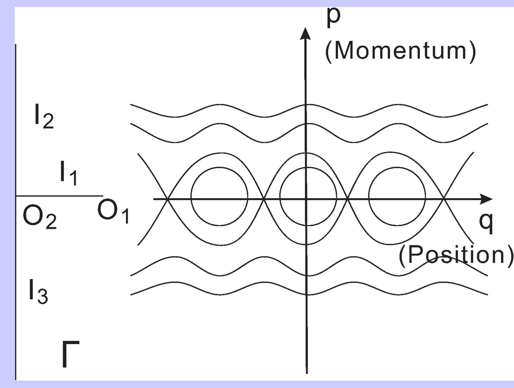
Bistability

> 6. Under-damped limit ($\gamma \to 0$)—methods

- Rescaled *random dynamical system*

$$\dot{q}^\epsilon = \frac{1}{\epsilon}H_p(q^\epsilon, p^\epsilon),$$
$$\dot{p}^\epsilon = -\frac{1}{\epsilon}H_q(q^\epsilon, p^\epsilon) + b(q^\epsilon, p^\epsilon) + \dot{W},$$

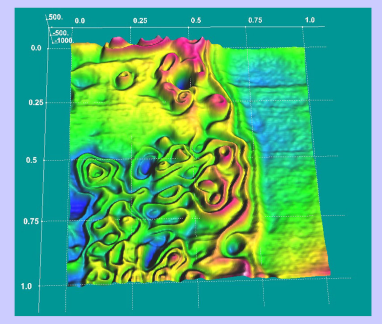
  where $H(q,p)$ is the *Hamiltonian function*
- *Freidlin's Hamiltonian graph* Γ
- *Laplace's method*



Graph Γ and level sets of $H(q,p)$

> 7. Undergoing and future work

- *Undergoing work*:
  - Two dimensional *gradient systems* in tilted periodic potentials
    - ∗ Goal:
      Scaling law of the long time average velocity
    - ∗ Method:
      Structural stability theory and *bifurcation theory*
- *Future work*:
  - Diffusion in *random potentials*
  - *Jump-diffusion* driven by more general noise



Random potential