• Laplace's method
  \[ \lim_{\gamma \to \infty} \Psi(q(\gamma), 0) = \Psi(q_0, 0) \]

The long time average velocity field over the phase space differs when \( F \) is below, at or above the threshold \( F_0 \).

- The long time average velocity of the Smoluchowski equation converges to that of the Smoluchowski equation in the over-damped limit (\( m \to 0 \))
- The long time average velocity of the Smoluchowski equation converges to that of the corresponding deterministic equation as noise goes to zero and the convergence rate differs when \( F \) is below, at or above the threshold \( F_0 \).

- We proved:
  - The long time average velocity of the Langevin equation converges to that of the Smoluchowski equation in the over-damped limit (\( m \to 0 \))
  - The long time average velocity of the Smoluchowski equation converges to that of the corresponding deterministic equation as noise goes to zero and the convergence rate differs when \( F \) is below, at or above the threshold \( F_0 \).

\[ \dot{q} = -\frac{\Psi'(q)}{\gamma} + \sqrt{2\gamma \beta W(t)}, \quad q(0) = q_0. \]

**Langevin equation**

\[ m\ddot{q} = F - \nabla \Psi(q) - \gamma \dot{q} + \sqrt{2\gamma \beta W(t)}, \quad q(0) = q_0, \quad \dot{q}(0) = \dot{p}_0. \]

- \( F \)—the external force
- \( \Psi \)—the smooth periodic potential
- \( \gamma \)—the friction coefficient
- \( \beta \)—the inverse temperature
- \( W \)—the Brownian motion

\( F_0 \) is the threshold of the external force.

**Tilted periodic potential**

**Long time average velocity**

1. Motivation

**Particle diffusion in tilted periodic potentials**

The long time average velocity as a function of the external force \( F \)

The threshold \( F_0 \) of the external force \( F \)

The scaling law of the long time average velocity \( V_F \)

**Langevin equation:**

\[ m\ddot{q} = F - \nabla \Psi(q) - \gamma \dot{q} + \sqrt{2\gamma \beta W(t)}, \quad q(0) = q_0, \quad \dot{q}(0) = \dot{p}_0. \]

- \( F \) is the external force
- \( \Psi \) is the smooth periodic potential
- \( \gamma \) is the friction coefficient
- \( \beta \) is the inverse temperature
- \( W \) is the Brownian motion

**Pinning and de-pinning**

**Charge density waves**

**Phase boundary versus applied load**

**Josephson junction**

**Charge density waves**

**Phase boundary versus applied load**

**Josephson junction**

**1. Motivation**

• Josephson effect
  WHAT is the effective voltage on a Josephson junction?

• Phase boundary propagation
  HOW to estimate evolvement of the earth crack in the long run?

• Charge density wave
  WHY do the charge-density wave appear non-Ohmic conduction when the applied field is small enough?

**2. Model—pinning and de-pinning**

• Particle diffusion in tilted periodic potentials

The long time average velocity as a function of the external force \( F \)

The threshold \( F_0 \) of the external force \( F \)

The scaling law of the long time average velocity \( V_F \)

**Langevin equation:**

\[ m\ddot{q} = F - \nabla \Psi(q) - \gamma \dot{q} + \sqrt{2\gamma \beta W(t)}, \quad q(0) = q_0, \quad \dot{q}(0) = \dot{p}_0. \]

- \( F \) is the external force
- \( \Psi \) is the smooth periodic potential
- \( \gamma \) is the friction coefficient
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**Tilted periodic potential**

**Long time average velocity**

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**Josephson junction**

**Charge density waves**

**Phase boundary versus applied load**

**Josephson junction**

**3. Over-damped limit (\( \gamma \to \infty \) or \( m \to 0 \))—results**

\[ \dot{q} = F - \nabla \Psi(q) + \sqrt{2\gamma \beta W(t)}, \quad q(0) = q_0. \]

**Smoluchowski equation**

**We proved:**

- The long time average velocity of the Langevin equation converges to that of the Smoluchowski equation in the over-damped limit (\( m \to 0 \))
- The long time average velocity of the Smoluchowski equation converges to that of the corresponding deterministic equation as noise goes to zero and the convergence rate differs when \( F \) is below, at or above the threshold \( F_0 \).

\[ q_0 = \frac{1}{\gamma} H(q_0, \dot{q}_0), \quad \dot{p}_0 = \frac{1}{\gamma} H(p_0, \dot{p}_0) + b(q_0, \dot{q}_0) + W, \]

where \( H(q, \dot{q}) \) is the Hamiltonian function.

\( q \) and \( \dot{q} \) are the position and velocity process respectively.

\[ H(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 + \Psi(q) \]

\( q_0, \dot{q}_0 \) are the initial values.

\( q, \dot{q} \) are the position and velocity process respectively.

\( m \) is the mass.

**Rescaled random dynamical system**

\[ \dot{q} = \frac{1}{\gamma} H(q, \dot{q}), \quad \dot{p} = \frac{1}{\gamma} H(p, \dot{p}) + b(q, \dot{q}) + W, \]

where \( H(q, \dot{q}) \) is the Hamiltonian function.

**4. Over-damped limit (\( \gamma \to \infty \) or \( m \to 0 \))—methods**

**Approximate the second order process by pieces of the first order process**

**Sample path approximation**

We use pieces of the first order Smoluchowski process to approximate the position process of the second order Langevin equation and show that the total approximation error, i.e., the sum of deviations of each piece, vanishes as \( m \to 0 \).

**Ergodicity**

The long time average velocity equals to the integration of the velocity field over the phase space w.r.t. an invariant measure

**Laplace’s method**

\[ V_F = \int \dot{q} \, dQ, \quad Q \text{ is the phase space.} \]

**Convergence of \( V_F \) in the vanishing mass limit (\( m \to 0 \))**

**Convergence of \( V_F \) in the vanishing noise limit (\( \beta \to \infty \))**

**5. Under-damped limit (\( \gamma \to 0 \))—results**

\[ \dot{q} = F - \nabla \Psi(q) + \sqrt{2\gamma \beta W(t)}, \quad q(0) = q_0. \]

**6. Under-damped limit (\( \gamma \to 0 \))—methods**

**Graph \( \Gamma \) and level sets of \( H(q, p) \)**

**Rescaled random dynamical system**

**7. Undergoing and future work**

**Bistability phenomenon, i.e., the pinning and running states coexist in the system**

**We obtained:**

- Derivation of the bi-stability thresholds
- Asymptotics of the mean return time of the pinning or running state in the vanishing noise limit (\( \beta \to \infty \))

**Bi-stability phenomenon, i.e., the pinning and running states coexist in the system**

**Undergoing work:**

- Two dimensional gradient systems in tilted periodic potentials
  - Goal:
    - Scaling law of the long time average velocity
  - Method:
    - Structural stability theory and bifurcation theory

**Future work:**

- Diffusion in random potentials
- Jump-diffusion driven by more general noise