A second order minimality criterion for free discontinuity problems

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Second variation and local minimality

Investigation of sufficiency minimality conditions based on the second variation, for functionals involving competing terms (bulk energies and surface energies):

- [6, 4]: Mumford-Shah functional
- [9, 3]: epitaxial growth of elastic films
- [1]: diblock copolymers
- [7]: cavities in elastic bodies

Here we develop this approach for a 3-dimensional variational model with a nonlinear elastic energy, arising in the context of epitaxial growth of strained elastic films.

Previous related results

- In [5, 8] a two-dimensional variational model for epitaxial growth of strained elastic films, in the framework of linearized elasticity, is introduced by relaxation methods
- In [8] regularity properties of minimizers are studied
- In [9] the local and global minimality of the flat configuration is discussed by means of a second variation approach to local minimality

The model

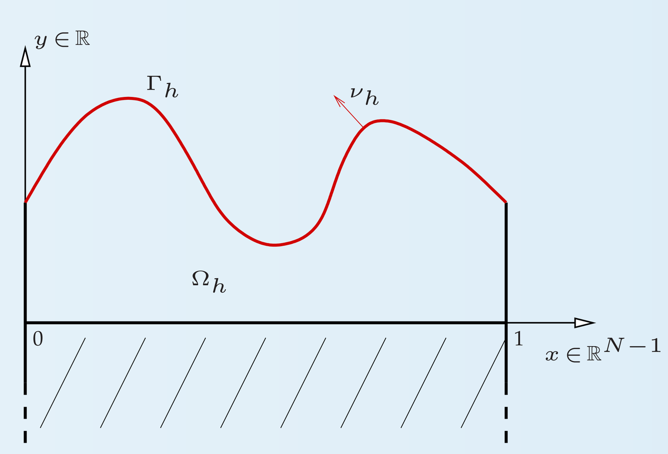
Denote $Q=(0,1)^{N-1}$, $(x,y)\in\mathbb{R}^{N-1}\times\mathbb{R}$. We prescribe

- $d>0$ (volume constraint)
- $u_0(x,y):=(A[x]+q(x),0)$, with $A\in\mathbb{M}^{N-1}_+$ and $q:\mathbb{R}^{N-1}\to\mathbb{R}^{N-1}$ 1-periodic (mismatch strain at the interface $\{y=0\}$).

Space of admissible pairs: $(h,u)\in X$ if

- $h:\mathbb{R}^{N-1}\to(0,+\infty)$ Lipschitz, 1-periodic in the coordinate directions,
  - $\Omega_h:=\big\{(x,y)\in\mathbb{R}^N : x\in Q,\ 0<y<h(x)\big\}$ (reference configuration),
  - $\Gamma_h:=\big\{(x,h(x))\in\mathbb{R}^N : x\in Q\big\}$ (free profile of the film),
  - $\Omega_h^\#,\ \Gamma_h^\#$ periodic extensions,
- $u:\Omega_h\to\mathbb{R}^N$ (deformation of the film),
- $u-Id-u_0\in\mathcal{V}(\Omega_h)$ (Dirichlet boundary condition + periodicity), where

$$\mathcal{V}(\Omega_h):=\Big\{w\in C^1(\overline{\Omega}_h^\#;\mathbb{R}^N) : w(x,0)=0,\ w(x+e_i,y)=w(x,y)\Big\}.$$



Total energy of the system:

$$F(h,u):=\int_{\Omega_h}W(\nabla u)\,dz+\int_{\Gamma_h}\psi(\nu_h)\,d\mathcal{H}^{N-1}\qquad (h,u)\in X.$$

Assumptions on the elastic energy density and anisotropic surface energy density:

- $W:\mathbb{M}^N_+\to[0,+\infty)$ of class C^3, $W(\xi)\to+\infty$ as $\det\xi\to0^+$,
- $\psi:\mathbb{R}^N\to[0,+\infty)$ of class C^3 (away from the origin), positively 1-homogeneous, $m|z|\le\psi(z)\le M|z|$ for all $z\in\mathbb{R}^N$, and

$$\nabla^2\psi(v)[w,w]>c\,|w|^2\quad\text{for all } v\in S^{N-1}\text{ and } w\perp v.$$

Local minimizers

An admissible pair (h,u) is a local minimizer if $F(h,u)<+\infty$ and there exists $\delta>0$ such that

$$F(h,u)\le F(g,v)$$

for all $(g,v)\in X$ such that

- $\|g-h\|_\infty<\delta,\quad |\Omega_g|=|\Omega_h|$,
- $\|\nabla v\circ\Phi_g-\nabla u\|_{L^\infty(\Omega_h;\mathbb{M}^N)}<\delta$.

Here $\Phi_g:\overline{\Omega}_h\to\overline{\Omega}_g$ is a change of variables such that

$$\|\Phi_g-Id\|_\infty\le\|g-h\|_\infty.$$

Critical pairs

An admissible pair $(h,u)\in X$ is a regular critical pair if:

(H1) $h\in C^2,\ u\in C^2,\ \det\nabla u>0$ in $\overline{\Omega}_h$

(H2) u is a critical point for the elastic energy in Ω_h, that is,

$$\begin{cases}\operatorname{div}[W_\xi(\nabla u)]=0 & \text{in } \Omega_h\\ W_\xi(\nabla u)[\nu_h]=0 & \text{on } \Gamma_h\end{cases}\qquad(1)$$

(H3) $W(\nabla u)+H^\psi=const$ on Γ_h, with $H^\psi:=\operatorname{div}_{\Gamma_h}(\nabla\psi\circ\nu_h)$

In addition to these conditions, we will also assume that the second variation of the elastic energy is uniformly positive:

(H4) for every $w\in\mathcal{V}(\Omega_h)$

$$\int_{\Omega_h}W_{\xi\xi}(\nabla u)\nabla w:\nabla w\,dz\ge c_0\|w\|^2_{H^1(\Omega_h;\mathbb{R}^N)}$$

Second variation

Assume that $(h,u)\in X$ satisfies (H1), (H2), (H4).
Given $\varphi\in C^2_{per}(\Gamma_h)$ with $\int_{\Gamma_h}\varphi\,d\mathcal{H}^{N-1}=0$, we set

$$h_t:=h+t\phi,\qquad \phi(x):=\varphi(x,h(x))\sqrt{1+|\nabla h(x)|^2}.$$

Thank to (H4) and to the Implicit Function Theorem, if t is sufficiently small we can consider a deformation u_t satisfying (H2) in Ω_{h_t} (and the Dirichlet condition).

Then the function $\dot u=\frac{\partial u_t}{\partial t}\big|_{t=0}$ belongs to $\mathcal{V}(\Omega_h)$ and satisfies the equation

$$\int_{\Omega_h}W_{\xi\xi}(\nabla u)\nabla\dot u:\nabla w\,dz=\int_{\Gamma_h}\operatorname{div}_{\Gamma_h}(\varphi\,W_\xi(\nabla u))\cdot w\,d\mathcal{H}^{N-1}\qquad\text{for all } w\in\mathcal{V}(\Omega_h),$$

and the second variation of F at (h,u) along the direction φ is given by

$$\begin{aligned}\frac{d^2}{dt^2}F(h_t,u_t)|_{t=0}=&-\int_{\Omega_h}W_{\xi\xi}(\nabla u)\nabla\dot u:\nabla\dot u\,dz+\int_{\Gamma_h}(\nabla^2\psi\circ\nu_h)[\nabla_{\Gamma_h}\varphi,\nabla_{\Gamma_h}\varphi]\,d\mathcal{H}^{N-1}\\&+\int_{\Gamma_h}\big(\partial_{\nu_h}(W\circ\nabla u)-\operatorname{trace}(\mathbf{B}^\psi\mathbf{B})\big)\varphi^2\,d\mathcal{H}^{N-1}\\&-\int_{\Gamma_h}(W\circ\nabla u+H^\psi)\operatorname{div}_{\Gamma_h}\left[\left(\frac{(\nabla h,|\nabla h|^2)}{\sqrt{1+|\nabla h|^2}}\circ\pi\right)\varphi^2\right]d\mathcal{H}^{N-1},\end{aligned}$$

where $\mathbf{B}=D\nu_h$, $\mathbf{B}^\psi=D(\nabla\psi\circ\nu_h)$.

Main result

Introduce the subspace of $H^1(\Gamma_h)$

$$\widetilde{H}^1_\#(\Gamma_h):=\Big\{\vartheta\in H^1_{loc}(\Gamma_h^\#) : \vartheta(x+e_i,h(x+e_i))=\vartheta(x,h(x)),\ \int_{\Gamma_h}\vartheta\,d\mathcal{H}^{N-1}=0\Big\}.$$

Note that the last integral in the expression of the second variation vanishes at a regular critical pair, due to (H3).
We define the quadratic form $\partial^2F(h,u):\widetilde{H}^1_\#(\Gamma_h)\to\mathbb{R}$

$$\begin{aligned}\partial^2F(h,u)[\varphi]:=&-\int_{\Omega_h}W_{\xi\xi}(\nabla u)\nabla v_\varphi:\nabla v_\varphi\,dz+\int_{\Gamma_h}(\nabla^2\psi\circ\nu_h)[\nabla_{\Gamma_h}\varphi,\nabla_{\Gamma_h}\varphi]\,d\mathcal{H}^{N-1}\\&+\int_{\Gamma_h}\big(\partial_{\nu_h}(W\circ\nabla u)-\operatorname{trace}(\mathbf{B}^\psi\mathbf{B})\big)\varphi^2\,d\mathcal{H}^{N-1},\end{aligned}$$

where $v_\varphi\in\widetilde{\mathcal{V}}(\Omega_h)$ is the unique solution to

$$\int_{\Omega_h}W_{\xi\xi}(\nabla u)\nabla v_\varphi:\nabla w=\int_{\Gamma_h}\operatorname{div}_{\Gamma_h}(\varphi W_\xi(\nabla u))\cdot w\,d\mathcal{H}^{N-1}\qquad\text{for every } w\in\widetilde{\mathcal{V}}(\Omega_h).$$

Theorem. *Let $N=2,3$. Assume that $(h,u)\in X$ is a regular critical pair satisfying (H4) and*

$$\partial^2F(h,u)[\varphi]>0\qquad\text{for every } \varphi\in\widetilde{H}^1_\#(\Gamma_h)\setminus\{0\}.\qquad(2)$$

Then (h,u) is a local minimizer for the functional F.

Two main steps in the proof

1. $W^{2,p}$-local minimality $(N=2,3)$
   - Minimality w.r.t. $W^{2,p}$-perturbations of the profile
   - Estimate carefully all the terms appearing in the expression of the second variation
   - Work in fractional Sobolev spaces

2. $W^{2,p}$-local minimality $\Longrightarrow$ L^∞-local minimality $(N\ge2)$
   - Contradiction argument: $F(h_n,u_n)<F(h,u)$, $h_n\to h$
   - Consider solutions to penalization/obstacle problems
   - Use the regularity theory for quasi-minimizers of the area functional to deduce $C^{1,\alpha}$ convergence
   - Use Euler-Lagrange equations to deduce $W^{2,p}$-convergence

Application: stability of flat morphologies

Flat configuration with volume d:

$$(d,v_0),\qquad v_0(z)=M[z],\quad M\in\mathbb{M}^N_+$$

satisfying (H2), (H4) in $\Omega_d=Q\times(0,d)$.
Then:

- there exists $d_0>0$ such that condition (2) is satisfied at the flat configuration for every $d<d_0$;
- for large values of d, the flat configuration is no longer a local minimizer;
- crystalline case: assume $\psi:\mathbb{R}^N\to[0,+\infty)$ Lipschitz, convex, positively 1-homogeneous, such that the associated Wulff shape

$$W_\psi:=\{z\in\mathbb{R}^N : z\cdot\nu\le\psi(\nu)\text{ for every }\nu\in S^{N-1}\}$$

has a flat horizontal facet intersecting the y-axis. Then for every $d>0$ the flat configuration is a local minimizer for the associated functional (if $N=2,3$).

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