Long-time asymptotics for perturbations of the Toda lattice

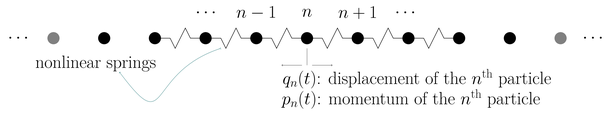
Deniz Bilman, Irina Nenciu

Department of Mathematics, Statistics, and Computer Science. University of Illinois at Chicago

dbilma2@uic.edu

The Toda Lattice

Consider the doubly infinite chain of particles with nearest neighbor interaction,



where the interaction potential is $V_0(r) = e^{-r} + r - 1$, the Toda potential. In this case, the equations of motion read

$$\frac{d}{dt}p_n(t) = -e^{-(q_{n+1}(t)-q_n(t))} + e^{-(q_n(t)-q_{n-1}(t))}, \quad \text{and} \quad \frac{d}{dt}q_n(t) = p_n(t).$$

An infinite dimensional completely integrable system!

- Solvable by direct-inverse scattering
- Infinitely many constants of motion
- Soliton solutions

1-soliton solution is given by $q_n(t) = \log\left(\frac{1+e^{-2kn+2\sigma\sinh(k)t}}{1+e^{-2k(n+1)+2\sigma\sinh(k)t}}\right)$, with speed $c = \frac{\sinh k}{k} > 1$, $k > 0$, and $\sigma = \pm 1$.

Complete integrability

Using the change of variables

$$a_n(t) = \frac{1}{2}e^{-(q_{n+1}(t)-q_n(t))/2}, \qquad b_n(t) = -\frac{1}{2}p_n(t),$$

introduced by Flaschka one obtains a first-order system equivalent to the Toda lattice:

$$\begin{aligned}\frac{d}{dt}a_n(t) &= a_n(t)\big(b_{n+1}(t) - b_n(t)\big),\\ \frac{d}{dt}b_n(t) &= 2\big(a_n(t)^2 - a_{n-1}(t)^2\big).\end{aligned} \tag{1}$$

Then it is seen that there exists a pair of linear operators $(L(t), P(t))$,

$$L = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \ddots & a_{n-2} & 0 & & & \\ 0 & a_{n-2} & b_{n-1} & a_{n-1} & 0 & & \\ & 0 & a_{n-1} & b_n & a_n & 0 & \\ & & 0 & a_n & b_{n+1} & a_{n+1} & 0 \\ & & & 0 & a_{n+1} & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}, \quad P = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \ddots & a_{n-2} & 0 & & & \\ 0 & -a_{n-2} & 0 & a_{n-1} & 0 & & \\ & 0 & -a_{n-1} & 0 & a_n & 0 & \\ & & 0 & -a_n & 0 & a_{n+1} & 0 \\ & & & 0 & -a_{n+1} & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix},$$

such that the Toda lattice equations are equivalent to the Lax equation

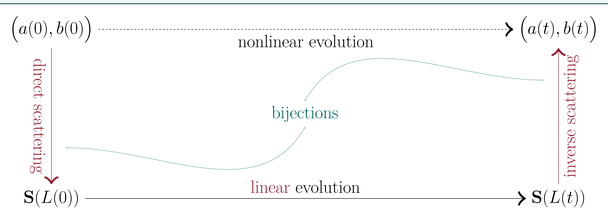
$$\frac{d}{dt}L(t) = [P(t), L(t)] = P(t)L(t) - L(t)P(t),$$

which, in this case, implies that $\sigma(L(t)) = \sigma(L(0))$. In particular, the eigenvalues of L, all of which are real and simple, remain constant in time.

At any time $t \in \mathbb{R}$, $(a(t), b(t))$ is uniquely determined by the scattering data $\mathbf{S}(L(t))$ of $L(t)$:

$$\mathbf{S}(L) = \left\{ \begin{array}{r} \text{Eigenvalues of } L\colon \lambda_j,\ j = 1, 2, \ldots, N. \\ \text{Norming constants}\colon \gamma_j,\ j = 1, 2, \ldots, N. \\ \text{Reflection coefficient}\colon R(\lambda),\ \lambda \in \sigma_{\text{ac}}(L). \end{array} \right\}$$

Initial data with N eigenvalues implies resolution into N 1-solitons as $t \to \infty$.



The perturbed system

We now consider potentials obtained via perturbing the Toda potential: $V_\varepsilon(r) = V_0(r) + \varepsilon u(r)$. Then, the perturbed system reads

$$\begin{aligned}\frac{d}{dt}a_n(t) &= a_n(t)\left(b_{n+1}(t) - b_n(t)\right),\\ \frac{d}{dt}b_n(t) &= 2\left(a_n(t)^2 - a_{n-1}(t)^2\right) + \varepsilon\frac{1}{2}\left(u'\left(\log\left(\frac{1}{4a_{n-1}(t)^2}\right)\right) - u'\left(\log\left(\frac{1}{4a_n(t)^2}\right)\right)\right).\end{aligned}$$

Equivalently, we have

$$\frac{d}{dt}L(t) = [P(t), L(t)] + \varepsilon U(t),$$

where U is a diagonal operator with the diagonal

$$U_n(t) = \frac{1}{2}\left(u'\left(\log\left(\frac{1}{4a_{n-1}(t)^2}\right)\right) - u'\left(\log\left(\frac{1}{4a_n(t)^2}\right)\right)\right).$$

Goal

Understand how long-time asymptotics for these perturbed systems compare to the long-time asymptotics for the Toda lattice.

Ultimately, for a fixed perturbation u and ε^*, we hope to prove that under some conditions on the initial data (a norm threshold), the solutions of the perturbed system will be close to solutions of the Toda lattice, as $t \to \infty$, for any $\varepsilon \in (0, \varepsilon^*)$.

Question: Can one predict the long-time behavior of these nearly-integrable systems by looking at the initial data?

Evolution of scattering data

For the perturbed case, the eigenvalues of L are no longer constant in time. Instead, we have

$$\frac{d}{dt}\lambda(t) = \varepsilon\big(\psi(t), U(t)\psi(t)\big)_{\ell^2(\mathbb{Z})}.$$

We set $\lambda = \frac{1}{2}\left(z + \frac{1}{z}\right)$ implying $\sigma_{ac}(L) = \mathbb{D}$ and $\lambda_j = \frac{1}{2}\left(\zeta_j + \frac{1}{\zeta_j}\right)$, with $\zeta_j \in (-1, 0) \cup (0, 1)$. Considering the problem $L\varphi = \frac{z+z^{-1}}{2}\varphi$, we have the Jost solutions $\varphi_\pm(z)$ uniquely determined by their asymptotics

$$\lim_{n\to\pm\infty}\varphi_\pm(z; n)z^{\mp n} = 1, \quad |z| = 1.$$

Then we have the following evolution equation for the reflection coefficient:

$$R_+(z; t) = R_+(z)e^{(z-z^{-1})t} - \varepsilon\frac{2}{z - z^{-1}}\int_0^t e^{(z-z^{-1})(t-\tau)}\sum_{j=-\infty}^{\infty}\left(U_j(\tau)\left(\frac{\varphi_-(z; j, \tau)}{\Delta(z; \tau)}\right)^2\right)d\tau,$$

with $\Delta(z) = \frac{1}{T(z)}$, where T stands for the transmission coefficient.

If $\zeta \in (-1, 1)$ such that $\lambda = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)$ is an eigenvalue of L, then the right norming constant γ_+ is defined by $\gamma_+(\zeta) = \frac{1}{\|\varphi_+(\zeta;\cdot)\|^2_{\ell^2(\mathbb{Z})}}$, and the evolution equation reads

$$\begin{aligned}\gamma_+(\zeta; t) =& e^{(\zeta-\zeta^{-1})t}\gamma_+(\zeta) +\\ &-\frac{2\varepsilon}{\zeta - \zeta^{-1}}\gamma_+(\zeta, t)^2\int_0^t e^{(\zeta-\zeta^{-1})(t-\tau)}\sum_{n\in\mathbb{Z}}\varphi_+(\zeta; n, \tau)\sum_{j=n}^{\infty}U_j(\tau)\varphi_+(\zeta; j, \tau)K(j, n, \tau)\,d\tau\,,\end{aligned}$$

where $K(j, n) = \varphi_+(\zeta; n)\varphi_+(\zeta^{-1}; j) - \varphi_+(\zeta^{-1}; n)\varphi_+(\zeta; j)$.

For the pure Toda lattice, the velocity of a 1-soliton corresponding to an eigenvalue $\lambda = \frac{\zeta+\zeta^{-1}}{2}$ is determined by the phase of the associated Riemann-Hilbert problem, and it is given by

$$c = v(\zeta) = -\frac{\zeta - \zeta^{-1}}{2\log(|\zeta|)}.$$

Work in progress

1. D. Bilman and I. Nenciu, Perturbed Toda lattices. A numerical study.
2. D. Bilman, Jacobi operators with few bound states.

Acknowledgements

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Results for $u(r) = r^2$, $\varepsilon = 0.05$, with Toda soliton initial data (k=0.6)

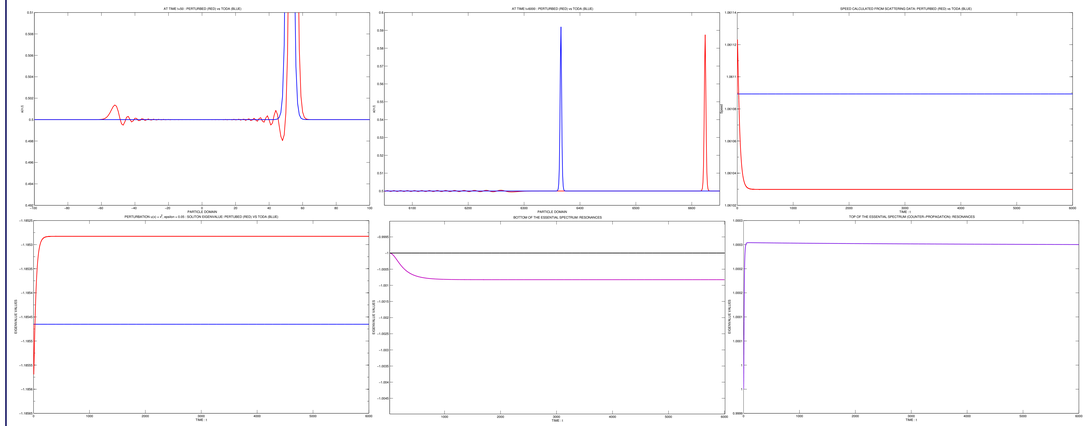

Figure 1: *We first observed that the emerging solitary wave in the perturbed system travels faster than the Toda soliton which is the initial data. Then we computed that the speed of this wave is given by $v(\zeta(t)) + C_u\varepsilon$, where C_u is a constant independent of ε. Note that $C_u>0$ and large enough to compensate for the loss of speed due to the evolution of $\zeta(t)$ under the perturbed dynamics.*

Results for $u(r) = r^3$, $\varepsilon = 0.05$, with Toda soliton initial data (k=0.6)

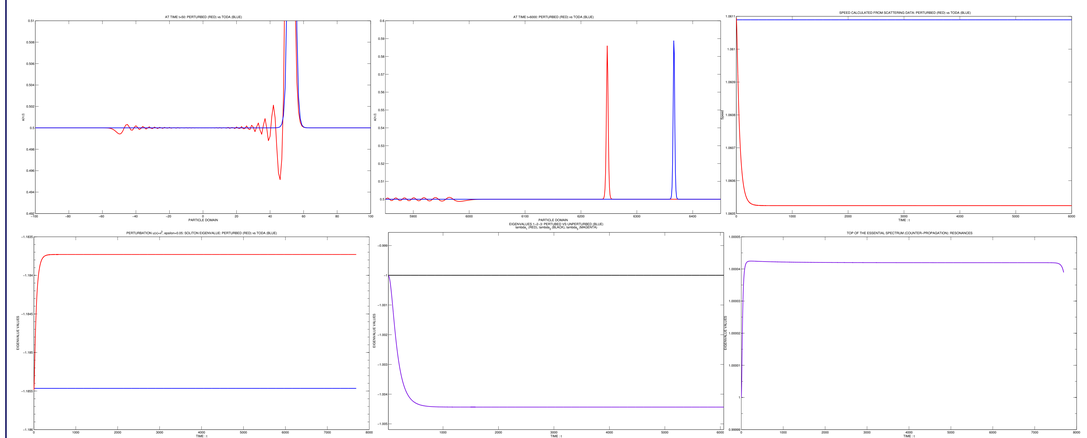

Figure 2: *This time, we observed that the emerging solitary wave in the perturbed system travels slower than the Toda soliton which is the initial data. Then we computed that the speed of this wave is given by $v(\zeta(t)) + C_u\varepsilon$, where C_u is a constant independent of ε. Note that $C_u<0$ in this case.*

References

1. P. Deift and X. Zhou, Perturbation theory for infinite-dimensional integrable systems on the line. A case study. *Acta Math.* **188** (2002), 163–262.
2. H. Flaschka, The Toda Lattice. I. Existence of integrals. *Phys. Rev. B (3)* **9** (1974), 1924–1925.
3. H. Krüger and G. Teschl, Long-time asymptotics for the Toda lattice for decaying initial data revisited. *Rev. Math. Phys.* **21** (2009), 61–109.