On the Dimension of a Certain Measure

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Notation and Terminology

Fix p, $1 < p < \infty$ and let $f : \mathbb{C} \setminus \{0\} \to (0, \infty)$ be homogeneous of degree p on $\mathbb{C} \setminus \{0\}$. That is,

$$f(\eta) = |\eta|^p f(\frac{\eta}{|\eta|}) > 0 \text{ when } \eta \in \mathbb{C} \setminus \{0\}.$$

We also assume that ∇f is $\delta-$monotone on $\mathbb{C}$ for some $0 < \delta \leq 1$. By definition, this means that $f \in W^{1,1}(B(0,R))$ for each $R > 0$ and for almost every $\eta, \eta' \in \mathbb{C}$ (with respect to two dimensional Lebesgue measure)

$$\langle \nabla f(\eta) - \nabla f(\eta'), \eta - \eta' \rangle \geq \delta |\nabla f(\eta) - \nabla f(\eta')||\eta - \eta'|.$$

In my thesis, we realized that $\delta-$monotonicity and the following are equivalent for a.e $\eta \in \mathbb{C}$ and all ζ with $|\zeta| = 1$

$$\frac{1}{c}|\eta|^{p-2} \leq f_{\zeta\zeta}(\eta) = \zeta^T D^2 f \zeta \leq c|\eta|^{p-2}.$$

Euler-Lagrange Equation

Let Ω be a bounded region in the complex plane. Let $z = x_1 + \mathrm{i}x_2$. Next, given $h \in W^{1,p}(\Omega)$ let $\mathfrak{A} = \{h + \phi : \phi \in W_0^{1,p}(\Omega)\}$. It is well known from [HKM, Chapter 5] that there is $u \in \mathfrak{A}$ satisfying

$$\inf_{w \in \mathfrak{A}} \int_\Omega f(\nabla w) \mathrm{d}\nu = \int_\Omega f(\nabla u) \mathrm{d}\nu \text{ for some } u \in \mathfrak{A}.$$

Also u is a weak solution at $z \in \Omega$ to the Euler-Lagrange equation,

$$0 = \nabla \cdot (\nabla f(\nabla u(z))) = \sum_{k,j=1}^{2} f_{\eta_k \eta_j}(\nabla u(z))\, u_{x_k x_j}(z) \qquad (1)$$

That is, $u \in W^{1,p}(\Omega)$ and

$$\int_\Omega \langle \nabla f(\nabla u(z)), \nabla \phi(z) \rangle \mathrm{d}\nu = 0 \text{ whenever } \phi \in W_0^{1,p}(\Omega).$$

[HKM] Juha Heinonen, Tero Kilpeläinen, and Olli Martio, *Nonlinear potential theory of degenerate elliptic equations*, Dover Publications Inc., 2006.

Existence of Measure μ

Next, suppose $\Omega \subset \mathbb{C}$ is a bounded simply connected domain, N is a neighborhood of $\partial\Omega$, and $u > 0$ is a weak solution to (1) in $\Omega \cap N$. Also assume that $u = 0$ on $\partial\Omega$ in the $W^{1,p}(\Omega \cap N)$ sense. Under this scenario it follows from [HKM, Chapter 21] that there exists a unique finite positive Borel measure μ with support on $\partial\Omega$ satisfying

$$\int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla \phi \rangle \mathrm{d}\nu = -\int_{\partial\Omega} \phi \mathrm{d}\mu \qquad (2)$$

whenever $\phi \in C_0^\infty(N)$.
We remark from (2) that if $\partial\Omega$ and f are smooth enough then

$$\mathrm{d}\mu = \frac{f(\nabla u)}{|\nabla u|} \mathrm{d}H^1|_{\partial\Omega}.$$

Hausdorff Measure and Dimension

Let $\lambda > 0$ be defined on $(0, r_0)$ with $\lim_{r \to 0^+} \lambda(r) = 0$ for some fixed r_0. We define the H^λ measure of a set $E \subset \mathbb{C}$ as follows;
For fixed $0 < \epsilon < r_0$, let $\{B(z_i, r_i)\}$ be a cover of E with $0 < r_i < \epsilon$, $i = 1, 2, \ldots$, and set

$$\phi_\epsilon^\lambda(E) = \inf \sum_i \lambda(r_i).$$

where the infimum is taken over all possible covers of E.
Then the Hausdorff H^λ measure of E is

$$H^\lambda(E) = \lim_{\epsilon \to 0} \phi_\epsilon^\lambda(E).$$

When $\lambda(r) = r^\alpha$ we write H^α for H^λ. Next we define the Hausdorff dimension of the measure μ obtained in (2) as

$$\text{H-dim } \mu = \inf\{\alpha : \exists \text{Borel } E \subset \partial\Omega \text{ with } H^\alpha(E) = 0 \text{ and } \mu(E) = \mu(\partial\Omega)\}.$$

Results on Harmonic Measure

Let Ω be simply connected and $\mu = \omega$ in (2) be harmonic measure with respect to a point in $z_0 \in \Omega$ (the case when $f(\eta) = |\eta|^2$, and u is solution to Laplace's equation in $\Omega \setminus \{z_0\}$), and let

$$\tilde{\lambda}(r) = r \exp\{A\sqrt{\log\frac{1}{r} \log\log\log\frac{1}{r}}\}, \quad 0 < r < 10^{-6} \qquad (3)$$

then in [M], Makarov proved that

Theorem[Makarov]

a) There exists an absolute constant $A > 0$ such that ω is absolutely continuous with respect to $H^{\tilde{\lambda}}$ measure.
b) If $A > 0$ is small enough, then there is an example of a simply connected domain Ω and a corresponding harmonic measure ω that is singular with respect to $H^{\tilde{\lambda}}$ measure.

Corollary

If Ω is a simply connected domain in the plane then H-dim $\omega = 1$.

[M] N. G. Makarov, *On the distortion of boundary sets under conformal mappings*, Proc. London Math. Soc. (3) 51(1985).

Results on $P-$Harmonic Measure

In [BL], Bennewitz and Lewis obtained the following result for μ defined as in (2) for fixed p, $1 < p < \infty$, relative to $f(\eta) = |\eta|^p$. In this case the corresponding pde (1) becomes

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0, \qquad (4)$$

which is called the $p-$Laplace equation. Moreover a weak solution of (4) is called a $p-$harmonic function.

Theorem[Bennewitz, Lewis]

Let $\Omega \subset \mathbb{C}$ be a domain bounded by a quasi circle and let N be a neighborhood of $\partial\Omega$. Fix $p \neq 2$, $1 < p < \infty$, and suppose u is p-harmonic in $\Omega \cap N$ with boundary value 0 in the $W^{1,p}(\Omega \cap N)$ Sobolev sense. If μ is the measure corresponding to u as in (2) relative to $f(\eta) = |\eta|^p$, then H-dim $\mu \leq 1$ for $2 < p < \infty$ while H-dim $\mu \geq 1$ for $1 < p < 2$. Moreover, if $\partial\Omega$ is the von Koch snowflake then strict inequality holds for H-dim μ.

In [LNP], Lewis, Nyström, and Poggi-Corradini proved that

Theorem[Lewis, Nyström, Poggi-Corradini]

Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and N a neighborhood of $\partial\Omega$. Fix $p \neq 2, 1 < p < \infty$, and let u be p harmonic in $\Omega \cap N$ with boundary value 0 on $\partial\Omega$ in the $W^{1,p}(\Omega \cap N)$ Sobolev sense. Let μ be the measure corresponding to u as in (2), relative to $f(\eta) = |\eta|^p$ and put

$$\lambda(r) = r \exp\left[A\sqrt{\log\tfrac{1}{r} \log\log\tfrac{1}{r}}\right] \text{ for } 0 < r < 10^{-6}.$$

a) If $p > 2$, there exists $A = A(p) \leq -1$ such that μ is concentrated on a set of $\sigma-$finite H^λ measure.
b) If $1 < p < 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to H^λ measure.

Recently, in [L], Lewis proved that

Theorem[Lewis]

Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and N be a neighborhood of $\partial\Omega$. Fix $p \neq 2$, $1 < p < \infty$, and let u be $p-$harmonic in $\Omega \cap N$ with boundary value 0 on $\partial\Omega$ in the $W^{1,p}(\Omega \cap N)$ Sobolev sense. Let μ be the measure corresponding to u as in (2), relative to $f(\eta) = |\eta|^p$ and let $\tilde{\lambda}(r)$ be as in (3).

a) If $p > 2$, then μ is concentrated on a set of $\sigma-$finite H^1 measure.
b) If $1 < p < 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to $H^{\tilde{\lambda}}$ measure.

[BL] Björn Bennewitz and John L. Lewis, *On the dimension of P-harmonic measure*, Ann. Acad. Sci. Fenn. Math. 30 (2005).
[LNP] J. Lewis, and K. Nyström, and P. Poggi-Corradini, *P-harmonic measure in simply connected domains*, Ann. Inst. Fourier, 61(2011).
[L] John L. Lewis, *P-harmonic measure in simply connected domains revisited*, Transactions of the AMS (To appear).

Main Results

Theorem 1 [A1]

Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and N be a neighborhood of $\partial\Omega$. Fix p, $1 < p < \infty$, let f be homogeneous of degree p and let ∇f be δ monotone for some $0 < \delta \leq 1$. Let $u > 0$ be a weak solution to (1) in $\Omega \cap N$ with boundary value 0 on $\partial\Omega$ in the $W^{1,p}(\Omega \cap N)$ Sobolev sense. Let μ be the measure corresponding to u as in (2) and put

$$\lambda(r) = r \exp\left[A\sqrt{\log\frac{1}{r} \log\log\frac{1}{r}}\right] \text{ for } 0 < r < 10^{-6}.$$

a) If $p \geq 2$, there exists $A = A(p) \leq -1$ such that μ is concentrated on a set of $\sigma-$finite H^λ Hausdorff measure.
b) If $1 < p \leq 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to H^λ Hausdorff measure.

Corollary

Let u and μ be as in Theorem 1. Then H-dim $\mu \leq 1$ when $2 < p < \infty$, H-dim $\mu = 1$ when $p = 2$, and H-dim $\mu \geq 1$ when $1 < p < 2$.

Theorem 2 [A2]

Let u and μ be as in the Theorem 1. If $\partial\Omega$ is a Wolff snowflake and $1 < p < 2$, then H-dim $\mu > 1$ while if $2 < p < \infty$ then H-dim $\mu < 1$.

[A1] Murat Akman, *On the dimension of a certain measure in the plane*, submitted, arXiv:1301.5860
[A2] Murat Akman *Hausdorff dimension of A-harmonic measure on Wolff snowflakes.*, work in progress.

Future Work

Let μ be as in the Theorem 1 and $\Omega \subset \mathbb{R}^2$.

1. Does our result hold when the measure function

$$\tilde{\lambda}(r) = r \exp\left[A\sqrt{\log\frac{1}{r} \log\log\log\frac{1}{r}}\right] \text{ for } 0 < r < 10^{-6}.$$

2. It is know that H-dim $\mu = 1$ when f is homogenous of degree 2 and ∇f is $\delta-$monotone. What is H-dim μ when $p \neq 2$ (even when $\partial\Omega$ is von Koch snowflake)?

Let μ be as in the Theorem 1 and $\Omega \subset \mathbb{R}^n$, $n \geq 3$. In [B], Bourgain proved that

Theorem[Bourgain]

If $\Lambda = \mathbb{R}^n \setminus E$ is a domain in $\mathbb{R}^n$, then H-dim $\omega_E < n$.

Recently, in [LNV], Lewis, Nyström, Vogel proved that

Theorem[Lewis, Nyström, Vogel]

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a $(\tilde{\delta}, \tilde{r}_0)$ Reifenberg flat domain, $w \in \partial\Omega$, and p fixed, $n \leq p < \infty$. Let $u > 0$ be p harmonic in Ω with $u = 0$ continuously on $\partial\Omega$. Let μ be the measure associated with u as in (2). There exists, $\hat{\delta} = \hat{\delta}(p, n) > 0$, such that if $0 < \tilde{\delta} \leq \hat{\delta}$, then μ is concentrated on a set of $\sigma-$finite H^{n-1}measure

They also conjectured that;

Conjecture[Lewis, Nyström, Vogel]

Let D and Ω be domains in $\mathbb{R}^n$ with $D \subset \Omega$, $\partial D \subset \partial\Omega$ and $\partial D \cap \Omega$ a compact subset of Ω. Let $u > 0$ be $p-$harmonic in D, with $u = 0$ on $\partial\Omega$. If $p \geq n$ then H-dim $\mu \leq n - 1$.

1. Is this conjecture true for measure μ as in Theorem 1?
2. What can be said about the H-dim μ when $1 < p < n$?

[B] J. Bourgain, *On the Hausdorff dimension of harmonic measure in higher dimension*, Inventiones Mathematicae, 87 (1987)
[LNV] John L. Lewis and Kaj Nyström and Andrew Vogel, *On the dimension of P-harmonic measure in space*, submited.