Notation and Terminology

Fix $p, 1 < p < \infty$ and let $f : C \rightarrow (0, \infty)$ be homogeneous of degree $p$ on $C \setminus \{0\}$. That is,

\[ |f(\lambda x)| = |\lambda|^p |f(x)| \text{ for all } \lambda \in \mathbb{R} \text{ and } x \in C \setminus \{0\}. \]

We also assume that \( \nabla f \) is \( \delta\)-monotone on $C$ for some $0 < \delta \leq 1$. By definition, this means that \( f \in W^{1,\delta}(\mathbb{R}_+; \mathbb{R}_+^n) \) for each $R > 0$ and for almost every $\eta, \eta' \in C$ (with respect to two-dimensional Lebesgue measure)

\[ \langle \nabla \eta, \nabla \eta' \rangle \cdot (\nabla \eta - \nabla \eta') \geq \delta \langle \nabla \eta, \nabla \eta' \rangle \| \eta - \eta' \|. \]

In this way, we relax that \( \delta\)-monotonicity and the following are equivalent for $u - v \in C$ and all $\epsilon > 0$.

\[ 2 \epsilon^2 |u - v|^2 \leq \int_0^\epsilon \int_0^\epsilon \langle \nabla \eta, \nabla \eta' \rangle \cdot (\nabla \eta - \nabla \eta') \, d\eta \, d\eta' \leq \epsilon^2 |u - v|^2. \]

Lagrange Equations

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded region in the complex plane. Let $z = x + iy$. Next, given $\epsilon > 1$, $\Omega \subseteq \{ |z| + \epsilon \} \cap \{ x = \frac{1}{\epsilon} \}$. It is well known from [8], Chapter 2 that there is a $u \in \mathcal{A}$ satisfying

\[ \int_0^i \int_0^i \langle \nabla w, \nabla w \rangle \, dw = \int_0^i \langle \nabla v, \nabla v \rangle \, dw \text{ for all } \epsilon > 0. \]

Also $u$ is a weak solution at $z = 0$ to the Euler-Lagrange equation,

\[ 0 = \nabla \cdot (\nabla \psi(z)) - \sum_{k=1}^n \lambda_k \psi(z) \nu_k(z), \]

which is called the $p$-Laplace equation. Moreover a weak solution of (4) is called a $p$-harmonic function.

Results on Harmonic Measure

Let $\Omega$ be simply connected and $u - \mu \omega(\Omega (2))$ be harmonic measure with respect to a point $z_0 \in \Omega$ (the case when $f(y) = |y|^2$) and $u$ is a solution to Laplace’s equation in $\Omega \setminus \{0\}$, and let

\[ \lambda(\epsilon) = \exp \left( \frac{1}{\epsilon} \log \log \log \frac{1}{\epsilon} \right), \quad 0 < \epsilon < 10^{-6}. \]

Then in [19], Makavey proved that

**Theorem (Makavey)**

a) There exists an absolute constant $A > 0$ such that $\omega$ is absolutely continuous with respect to $\mu^A$ measure.

b) If $\Omega$ is small enough, then there is an example of a simply connected domain $\Omega$ and a corresponding harmonic measure $\mu$ that is singular with respect to $\mu^A$ measure.

Corollary

If $\Omega$ is a simply connected domain in the plane then $\text{H-dim} \mu = 1$.

Results on $P-$Harmonic Measure

In [18], Bennewitz and Lewis obtained the following result for $p$ defined as in (2) for fixed $p, 1 < p < \infty$, relative to $f(y) = |y|^p$. In this case the corresponding pde (1) becomes

\[ \nabla \cdot (\nabla \omega(z)^p - \nabla \omega(z)) = 0, \]

which is called the $p$-Laplace equation. Moreover a weak solution of (4) is called a $p$-harmonic function.

Theorem (Bennewitz, Lewis)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain bounded by a quasicycle and let $N$ be a neighborhood of $\partial \Omega$. Fix $p, 1 < p < \infty$, and suppose $u$ is $p$-harmonic in $\mathbb{R}^n \setminus \{0\}$ with boundary value $0$ on $\mathbb{R}^n \setminus \{0\}$. Then $u$ is $p$-harmonic corresponding to $u \in (2)$ relative to $f(y) = |y|^p$ then $\text{H-dim} \mu \leq 2$ for $2 < p < \infty$ while $\text{H-dim} \mu \geq 2$ for $p < 1$. Moreover, if $\Omega$ is the von Koch meander then strict inequality holds for $\text{H-dim} \mu$.

In [17], Lewis, Nystrom, and Poggi-Corradini proved that

**Theorem (Lewis, Nystrom, Poggi-Corradini)**

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded simply connected domain and $N$ be a neighborhood of $\partial \Omega$. Fix $p, 1 < p < \infty$, and let $u$ be $p$-harmonic in $\mathbb{R}^n \setminus \{0\}$ with boundary value $0$ on $\mathbb{R}^n \setminus \{0\}$. Then $u$ is $p$-harmonic corresponding to $u \in (2)$ relative to $f(y) = |y|^p$ and put

\[ \lambda(\epsilon) = \exp \left( \frac{1}{\epsilon} \log \log \log \frac{1}{\epsilon} \right), \quad 0 < \epsilon < 10^{-6}. \]

a) If $2 < p$, there exists $A = A(p) \geq 1$ such that $\mu$ is concentrated on a set of $\mu^A$ measure.

b) If $1 < p < 2$, there exists $A = A(p) \geq 1$ such that $\mu$ is absolutely continuous with respect to $\mu^A$ measure.

Recently, in [1], Lewis proved that

**Theorem (Lewis)**

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded simply connected domain and $N$ be a neighborhood of $\partial \Omega$. Fix $p, 1 < p < \infty$, and let $u$ be $p$-harmonic in $\mathbb{R}^n \setminus \{0\}$ with boundary value $0$ on $\mathbb{R}^n \setminus \{0\}$. Then $u$ is $p$-harmonic corresponding to $u \in (2)$ relative to $f(y) = |y|^p$ and put

\[ \lambda(\epsilon) = \exp \left( \frac{1}{\epsilon} \log \log \log \frac{1}{\epsilon} \right), \quad 0 < \epsilon < 10^{-6}. \]

a) If $2 < p$, there exists $A = A(p) \geq 1$ such that $\mu$ is concentrated on a set of $\mu^A$ measure.

b) If $1 < p < 2$, there exists $A = A(p) \geq 1$ such that $\mu$ is absolutely continuous with respect to $\mu^A$ measure.

Future Work

Let $\mu$ and $\mu$ be as in the Theorem 1. Then $\text{H-dim} \mu \leq 1$ when $2 < p < \infty$ and $\text{H-dim} \mu = 1$ when $p = 2$, and $\text{H-dim} \mu \geq 2$ when $1 < p < 2$.

**Theorem (Bougain)**

Let $\mu = \omega$ as in the Theorem 1. If $\text{H-dim} \mu = 1$ when $f$ is homogeneous of degree 2 and $\text{H-dim} \mu$ is finite.

Recently, in [18], Lewis, Nystrom, Vogel proved that

**Theorem (Lewis, Nystrom, Vogel)**

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a (3,8) Reifenberg flat domain, $\omega \in \mathbb{P}$, and fixed, $n_0 \in \mathbb{N}$. Let $u > 0$ be $p$-harmonic in $\mathbb{R}^n$ with $u = 0$ continuously on $\partial \Omega$. Let $\mu$ be the measure associated with $u$ as in (2). Then $\mu$ satisfies

\[ \lambda(\mu) = A \mu \Rightarrow \lambda(\mu) = B \mu. \]

They also conjectured that

**Conjecture (Lewis, Nystrom, Vogel)**

Let $D$ and $\Omega$ be domains in $\mathbb{R}^n$ with $D \subseteq \Omega \subseteq \mathbb{R}^n$ and $\Omega$ a compact subset of $D$. Let $\mu$ be $p$-harmonic in $D$, with $\mu = 0$ on $\partial \Omega$. If $p \geq n$ then $\text{H-dim} \mu = n - 1$.

1. Is this conjecture true for measure $\mu$ as in Theorem 1?
2. What can be said about the $\text{H-dim} \mu$ when $1 < p < n$?