

1. PDE FORMULATION

- Given an initial ball of ice $B_{\Lambda_0} \subset \mathbb{R}^3$ and (radially symmetric) initial temperature distribution $u(0, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}$, find $\{\Lambda_t\}_{t>0}$ and $\{u(t, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}\}_{t>0}$ such that

$$\partial_t u(t, z) = \frac{1}{2} \Delta_z u(t, z), \quad z \in \mathbb{R}^3 \setminus \partial B_{\Lambda_t}, \quad t > 0, \quad (1)$$

$$u(t, z) = -\gamma/\Lambda_t, \quad z \in \partial B_{\Lambda_t}, \quad (2)$$

$$-\dot{\Lambda}_t = \frac{1}{2} \nabla_z u(t, z) \cdot n_t^+(z) + \frac{1}{2} \nabla_z u(t, z) \cdot n_t^-(z), \quad z \in \partial B_{\Lambda_t}, \quad (3)$$

where n_t^+ and n_t^- are the unit normals along ∂B_{Λ_t} pointing outside and inside of B_{Λ_t} .

- Removal of surface tension.** Using the fact that $\gamma/|z|$ solves heat equation, we consider $v(t, z) := u(t, z) + \gamma/|z|$ and solve

$$\partial_t v(t, z) = \frac{1}{2} \Delta_z v(t, z), \quad z \in \mathbb{R}^3 \setminus \partial B_{\Lambda_t}, \quad t > 0, \quad (4)$$

$$v(t, z) = 0, \quad z \in \partial B_{\Lambda_t}, \quad (5)$$

$$-\dot{\Lambda}_t = \frac{1}{2} \nabla_z v(t, z) \cdot n_t^+(z) + \frac{1}{2} \nabla_z v(t, z) \cdot n_t^-(z), \quad z \in \partial B_{\Lambda_t}. \quad (6)$$

- Integral form of growth condition.** Integrating (6) and using (4)–(5), we obtain

$$\frac{1}{3} (\Lambda_0^3 - \Lambda_t^3) = \alpha \int_{\mathbb{R}^3} v_0(z) dz - \alpha \int_{\mathbb{R}^3} v(t, z) dz,$$

where α^{-1} is the area of S^2 .

- Standard approach.** Show a contraction-type property (after normalization) of the mapping: from the cubed **bdry** Λ^3 to the temperature **energy** $3\alpha \int_{\mathbb{R}^3} v(t, z) dz$, where v solves (4)–(5).

- Observations/challenges.**

- If v_0 is sufficiently large, $t \mapsto \Lambda_t$ cannot be smooth (and may jump).
- Then, the bdry condition (5) **cannot be satisfied at all** $t > 0$.
- If $\Lambda_s = \tilde{\Lambda}_s$ for $s \in [0, t]$, then, for small $\delta > 0$, one may have

$$3\alpha \sup_{s \in [t, t+\delta]} \left| \int_{\mathbb{R}^3} v(s, z) dz - \int_{\mathbb{R}^3} \tilde{v}(s, z) dz \right| \approx v(t^-, \cdot) \Big|_{\partial B_{\Lambda_{t-}}} \sup_{s \in [t, t+\delta]} |\Lambda_s^3 - \tilde{\Lambda}_s^3|.$$

- If $v_0 < 1$, the comparison principle yields desired contraction.
- But, if not, the **contraction does not hold**.

2. PROBABILISTIC FORMULATION

- Itô's formula** yields: $v(t, \cdot)$ is the **density of BM** started from v_0 and **killed at hitting** $(\Lambda_s)_{s \in [0, t]}$.

- **Probabilistic growth condition:**

$$\frac{1}{3}(\Lambda_{0-}^3 - \Lambda_t^3) = \alpha \int_{\mathbb{R}^3} v_0(z) dz - \alpha \int_{\mathbb{R}^3} v(t, z) dz = \alpha \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \leq t) v_0(z) dz,$$

$$\tau^z(\Lambda) := \inf\{t \geq 0 : (|z + W_t| - \Lambda_t)(|z| - \Lambda_{0-}) < 0\}.$$

- **Jump condition.** The potential jumps of Λ introduce ambiguity. We define the notion of a “physical” solution by assuming that the jumps are the smallest possible:

$$\Lambda_{t-} - \Lambda_t = \inf \left\{ y \in (0, \Lambda_{t-}] : \text{Leb}(B_{\Lambda_{t-}} \setminus B_{\Lambda_{t-}-y}) > \int_{\Lambda_{t-}-y \leq |z| \leq \Lambda_{t-}} v(t-, z) dz \right\}$$

3. PROOF OF UNIQUENESS

- We argue by contradiction and assume that there exist two non-increasing right-continuous solutions Λ , $\tilde{\Lambda}$ and $t_0 \geq 0$ s.t. $\Lambda_s = \tilde{\Lambda}_s$ for $s < t_0$, and $\sup_{s \in [t_0, t_0 + \varepsilon]} |\Lambda_s - \tilde{\Lambda}_s| > 0$ for all small enough $\varepsilon > 0$.
- W.l.o.g. $t_0 = 0$.
- **Probabilistic formulation of desired contraction-type property:**

$$\|\Gamma(\Lambda^3) - \Gamma(\tilde{\Lambda}^3)\| \leq F(\|\Lambda^3 - \tilde{\Lambda}^3\|), \quad F(x) < x \text{ for } x > 0,$$

$$\Gamma_t(\Lambda^3) := \Lambda_{0-}^3 - 3\alpha \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \leq t) v_0(z) dz.$$

- **Challenge.** Recall that, for small $\delta > 0$, one may have

$$\begin{aligned} & \|\Gamma(\Lambda^3) - \Gamma(\tilde{\Lambda}^3)\| \\ &= 3\alpha \sup_{s \in [0, \delta]} \left| \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \leq s) v(0^-, z) dz - \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq s) v(0^-, z) dz \right| \\ &\approx v(0^-, \cdot) \Big|_{\partial B_{\Lambda_{0-}}} \cdot \sup_{s \in [0, \delta]} |\Lambda_s^3 - \tilde{\Lambda}_s^3|. \end{aligned}$$

- **Proposition 1 (difficult).** If $v_0(z)$ is piecewise monotone as a function of $|z|$, then so are $v(t, \cdot)$ and $v(t^-, \cdot)$ for all $t \geq 0$.

- The above prop. and jump condition imply that $\Lambda_0 = \tilde{\Lambda}_0$, $v(0, z) = \tilde{v}(0, z)$, yielding:

$$\sup_{s \in [0, \delta]} |\Gamma_s(\Lambda^3) - \Gamma_s(\tilde{\Lambda}^3)| = 3\alpha \sup_{s \in [0, \delta]} \left| \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \leq s) v(0, z) dz - \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq s) v(0, z) dz \right|.$$

- Then, we expect

$$\sup_{s \in [0, \delta]} |\Gamma_s(\Lambda^3) - \Gamma_s(\tilde{\Lambda}^3)| \approx v(0, \cdot) \Big|_{\partial B_{\Lambda_0}} \cdot \sup_{s \in [0, \delta]} |\Lambda_s^3 - \tilde{\Lambda}_s^3|,$$

where $v(0, \cdot) \Big|_{\partial B_{\Lambda_0}} \leq 1$.

- Since the case $v(0, \cdot) \Big|_{\partial B_{\Lambda_0}} < 1$ is easy, we focus on $v(0, \cdot) \Big|_{\partial B_{\Lambda_0}} = 1$.
- The jump condition yields, for small enough $\Lambda_0 - |z| > 0$:

$$v(0, z) = 1 - \psi(\Lambda_0 - |z|), \quad \psi \uparrow, \quad \psi(0^+) = 0.$$

- Consider t s.t. $\Lambda_t^3 - \tilde{\Lambda}_t^3 = \sup_{s \leq t} |\Lambda_s^3 - \tilde{\Lambda}_s^3|$ and deduce

$$\begin{aligned} \frac{\Lambda_t^3}{3} - \frac{\tilde{\Lambda}_t^3}{3} &= \alpha \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \leq s) v(0, z) dz - \alpha \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq s) v(0, z) dz \\ &\leq \alpha \int_{B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq t, \tau^z(\Lambda) > t) v(0, z) dz + \alpha \int_{\mathbb{R}^3 \setminus B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq t, \tau^z(\Lambda) > t) v(0, z) dz. \end{aligned}$$

Goal: obtain a contradiction to the above, for small enough $t > 0$.

- **Lemma 1 (easy).** There exists a constant C_1 s.t. $f(|z|) := |z|^2 v(t, z) \leq C_1(|z| - \Lambda_0)$ for $|z| \geq \Lambda_0$ and all $t \geq 0$.
- **Coupling 1.** Herein, we estimate the blue term.

(1) Considering Bessel process $R^{|z|} \sim |z + W|$,

$$dR_t^x = \frac{1}{R_t^x} dt + dB_t, \quad R_0^x = x,$$

we can write

$$\begin{aligned} &\alpha \int_{\mathbb{R}^3 \setminus B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq t, \tau^z(\Lambda) > t) v(0, z) dz \\ &= \int_{\Lambda_0}^{\infty} \mathbb{P}(\min_{0 \leq s \leq t} (R_s^x - \tilde{\Lambda}_s) \leq 0, \min_{0 \leq s \leq t} (R_s^x - \Lambda_s) > 0) f(x) dx \\ &= \mathbb{E} \int_{\Lambda_0}^{\infty} \mathbf{1}_{\{\min_{0 \leq s \leq t} (R_s^x - \tilde{\Lambda}_s) \leq 0, \min_{0 \leq s \leq t} (R_s^x - \Lambda_s) > 0\}} f(x) dx. \end{aligned}$$

- (2) **Note:** $\{R_s^x\}_x$ are coupled via the same path of BM B .
(3) Then, there is $C_2 > 0$ s.t.

$$R_t^x - R_t^y \geq C_2(x - y) \text{ for all } x \geq y \text{ and } 0 \leq t \leq \tau^y(\Lambda).$$

- (4) The above and continuity of $x \mapsto R^x$ yield:

$$\begin{aligned} \{x \geq \Lambda_0 : \min_{0 \leq s \leq t} (R_s^x - \tilde{\Lambda}_s) \leq 0, \min_{0 \leq s \leq t} (R_s^x - \Lambda_s) > 0\} &= (\underline{X}, \overline{X}], \\ \overline{X} - \underline{X} &\leq C_3 \sup_{s \leq t} |\Lambda_s - \tilde{\Lambda}_s| \leq C_4 (\Lambda_t^3 - \tilde{\Lambda}_t^3). \end{aligned}$$

(5) The above and Lemma 1 yield:

$$\begin{aligned} & \int_{\Lambda_0}^{\infty} \mathbf{1}_{\{\min_{0 \leq s \leq t} (R_s^x - \tilde{\Lambda}_s) \leq 0, \min_{0 \leq s \leq t} (R_s^x - \Lambda_s) > 0\}} f(x) dx \\ & \leq C_5 \int_{\underline{X}}^{\overline{X}} (x - \Lambda_0) dx \leq C_6 (\overline{X} - \Lambda_0) (\Lambda_t^3 - \tilde{\Lambda}_t^3). \end{aligned}$$

(6) Noticing that

$$0 = \min_{0 \leq s \leq t} (R_s^{\overline{X}} - \tilde{\Lambda}_s) = \overline{X} + \min_{0 \leq s \leq t} \left(\int_0^s \frac{1}{R_r^{\overline{X}}} dr + B_s - \tilde{\Lambda}_s \right) \Rightarrow \overline{X} \leq \max_{0 \leq s \leq t} (-B_s + \tilde{\Lambda}_s),$$

we conclude

$$\alpha \int_{\mathbb{R}^3 \setminus B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq t, \tau^z(\Lambda) > t) v(0, z) dz \leq C_7 \mathbb{E} \max_{0 \leq s \leq t} (B_s + \tilde{\Lambda}_s - \Lambda_0)$$

• **Coupling 2.** Herein, we estimate the red term.

(1) **Couple via the same path of 3-dim BM W** and recall that $v(0, z) = 1 - \psi(\Lambda_0 - |z|)$:

$$\begin{aligned} & \alpha \int_{B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq t, \tau^z(\Lambda) > t) v(0, z) dz \\ & = \alpha \mathbb{E} \int_{B_{\Lambda_0}} \mathbf{1}_{\sup_{r \in [0, t]} (|z + W_r| - \tilde{\Lambda}_r) \geq 0, \sup_{r \in [0, t]} (|z + W_r| - \Lambda_r) < 0} (1 - \psi(\Lambda_0 - |z|)) dz \\ & = \alpha \mathbb{E}(J - K). \end{aligned}$$

(2) We estimate the leading term on $A_\varepsilon := \{\omega : \sup_{s \in [0, t]} |W_s| \leq \varepsilon\}$:

$$J := \int_{B_{\Lambda_0}} \mathbf{1}_{\sup_{r \in [0, t]} (|z + W_r| - \tilde{\Lambda}_r) \geq 0, \sup_{r \in [0, t]} (|z + W_r| - \Lambda_r) < 0} dz =: \text{Leb}(D(W)).$$

Lemma 2 (difficult). For small enough $\varepsilon, t > 0$: $\text{Leb}(D(W)) \leq \sup_{s \leq t} \text{Leb}(B_{\Lambda_s} \Delta B_{\tilde{\Lambda}_s}) = \alpha^{-1} \frac{1}{3} (\Lambda_t^3 - \tilde{\Lambda}_t^3)$.

(3) We estimate the residuals:

Lemma 3. There exists $C_8 > 0$ s.t., for small enough $\varepsilon, t > 0$,

$$\alpha \mathbb{E}(J \mathbf{1}_{A_\varepsilon^c} - K) \leq -C_8 \mathbb{E} \psi \left(\max_{0 \leq s \leq t} (B_s - \Lambda_s + \Lambda_0) \right)$$

(4) Thus,

$$\begin{aligned} & \alpha \int_{B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq t, \tau^z(\Lambda) > t) v(0, z) dz \\ & \leq \frac{1}{3} (\Lambda_t^3 - \tilde{\Lambda}_t^3) - C_8 \mathbb{E} \psi \left(\max_{0 \leq s \leq t} (B_s - \Lambda_s + \Lambda_0) \right) \end{aligned}$$

- Collecting the blue and red parts, we obtain

$$\begin{aligned} \frac{1}{3}(\Lambda_t^3 - \tilde{\Lambda}_t^3) &\leq \frac{1}{3}(\Lambda_t^3 - \tilde{\Lambda}_t^3) - C_8 \mathbb{E} \psi \left(\max_{0 \leq s \leq t} (B_s - \Lambda_s + \Lambda_0) \right) + C_7 \mathbb{E} \max_{0 \leq s \leq t} (B_s + \tilde{\Lambda}_s - \Lambda_0), \\ \mathbb{E} \psi \left(\max_{0 \leq s \leq t} (B_s - \Lambda_s + \Lambda_0) \right) &\leq C_9 \mathbb{E} \max_{0 \leq s \leq t} (B_s + \tilde{\Lambda}_s - \Lambda_0) \end{aligned}$$

- To show that the above is impossible, we first establish the **comparison**:

Lemma 3 (easy). There exists a solution $\hat{\Lambda}$ to 1-phase Stefan problem with the same ψ (up to multiplicative constant) and with $\hat{\Lambda}_0 = \Lambda_0$, s.t. $\hat{\Lambda} \geq \Lambda \vee \tilde{\Lambda}$.

- Then,

$$\begin{aligned} \mathbb{E} \psi \left(\max_{0 \leq s \leq t} (B_s - \Lambda_s + \Lambda_0) \right) &\geq \mathbb{E} \psi \left(\max_{0 \leq s \leq t} (B_s - \hat{\Lambda}_s + \Lambda_0) \right), \\ \mathbb{E} \max_{0 \leq s \leq t} (B_s + \hat{\Lambda}_s - \Lambda_0) &\geq \mathbb{E} \max_{0 \leq s \leq t} (B_s + \tilde{\Lambda}_s - \Lambda_0), \end{aligned}$$

and it suffices to show that, for any $\bar{C} > 0$, we have

$$\mathbb{E} \psi \left(\max_{0 \leq s \leq t} (B_s - \hat{\Lambda}_s + \Lambda_0) \right) > \bar{C} \mathbb{E} \max_{0 \leq s \leq t} (B_s + \hat{\Lambda}_s - \Lambda_0),$$

for all small enough $t > 0$.

- **Time-reversal.** For simplicity, we prove the above assuming that $\hat{\Lambda}$ is a solution to **1-dim Stefan problem**.

(1) **Growth condition** for $\hat{\Lambda}$ yields:

$$\begin{aligned} \Lambda_0 - \hat{\Lambda}_t &= \mathbb{E} \int_0^{\Lambda_0} \mathbf{1}_{\sup_{s \leq t} (x + B_s - \hat{\Lambda}_s) \geq 0} (1 - \psi(\Lambda_0 - x)) dx \\ &= \mathbb{E} \int_{\inf_{s \leq t} (\hat{\Lambda}_s - B_s)}^{\Lambda_0} (1 - \psi(\Lambda_0 - x)) dx = \mathbb{E} \int_0^{\sup_{s \leq t} (B_s - \hat{\Lambda}_s + \Lambda_0)} (1 - \psi(x)) dx \\ &= \mathbb{E} \sup_{s \leq t} (B_s - \hat{\Lambda}_s + \Lambda_0) - \mathbb{E} \Psi \left(\sup_{s \leq t} (B_s - \hat{\Lambda}_s + \Lambda_0) \right), \quad \Psi(x) := \int_0^x \psi(y) dy, \\ \mathbb{E} \Psi \left(\sup_{s \leq t} (B_s - \hat{\Lambda}_s + \Lambda_0) \right) &= \mathbb{E} \sup_{s \leq t} (B_s - \hat{\Lambda}_s + \hat{\Lambda}_t) = \mathbb{E} \sup_{s \leq t} (B_s - B_t - \hat{\Lambda}_s + \hat{\Lambda}_t) \\ &= \mathbb{E} \sup_{s \leq t} (B_{t-s} - \hat{\Lambda}_s + \hat{\Lambda}_t) = \mathbb{E} \sup_{s \leq t} (B_s - \hat{\Lambda}_{t-s} + \hat{\Lambda}_t). \end{aligned}$$

(2) For small enough $t > 0$,

$$\mathbb{E} \psi \left(\sup_{s \leq t} (B_s - \hat{\Lambda}_s + \Lambda_0) \right) > \bar{C} \mathbb{E} \Psi \left(\sup_{s \leq t} (B_s - \hat{\Lambda}_s + \Lambda_0) \right) = \bar{C} \mathbb{E} \sup_{s \leq t} (B_s - \hat{\Lambda}_{t-s} + \hat{\Lambda}_t),$$

which implies that it suffices to show

$$\mathbb{E} \sup_{s \leq t} (B_s - \hat{\Lambda}_{t-s} + \hat{\Lambda}_t) \geq \mathbb{E} \sup_{s \leq t} (B_s + \hat{\Lambda}_s - \Lambda_0),$$

- (3) The above follows from “**semi-convexity**”:

$$\hat{\Lambda}_t - \hat{\Lambda}_{t-s} \leq \Lambda_0 - \hat{\Lambda}_s, \quad s \in [0, t].$$

- (4) The above follows from the **scaling** property:

Lemma 4 (easy) For any $q \in (0, 1]$:

$$\hat{\Lambda}_{qt} - \Lambda_0 \geq \sqrt{q} (\hat{\Lambda}_t - \Lambda_0) \geq q (\hat{\Lambda}_t - \Lambda_0)$$