On the size of the Navier – Stokes singular set

Walter Craig
McMaster University

Center for Nonlinear Analysis Seminar
Carnegie Mellon University
February 16 2017
Joint work with:

Maxim Arnold  University of Illinois Urbana – Champaign
Andrei Biryuk  Kuban State University, Krasnodar

Acknowledgements: Fields Institute, NSERC, Canada Research Chairs Program
Outline

▶ Initial value problem for the Navier – Stokes equations
▶ The singular sets $S(u)$ and the energy concentration set $S_{L^2}(u)$
▶ Three estimates on Leray weak solutions
▶ Phase space volume: lower bounds on $WF(u)$ and $WF_{L^2}(u)$
▶ Ideas of the proof:
  ▶ new a priori estimate on the Fourier transform
  ▶ microlocal analysis
  ▶ defect measures
Outline

- Initial value problem for the Navier – Stokes equations
- The singular sets $S(u)$ and the energy concentration set $S^{L^2}(u)$
- Three estimates on Leray weak solutions
- Phase space volume: lower bounds on $WF(u)$ and $WF^{L^2}(u)$

Ideas of the proof:
- new a priori estimate on the Fourier transform
- microlocal analysis
- defect measures
Outline

- Initial value problem for the Navier–Stokes equations
- The singular sets $S(u)$ and the energy concentration set $S_{L^2}(u)$
- Three estimates on Leray weak solutions
  - Phase space volume: lower bounds on $WF(u)$ and $WF_{L^2}(u)$
- Ideas of the proof:
  - new a priori estimate on the Fourier transform
  - microlocal analysis
  - defect measures
Outline

- Initial value problem for the Navier – Stokes equations
- The singular sets $S(u)$ and the energy concentration set $S^{L^2}(u)$
- Three estimates on Leray weak solutions
- Phase space volume: lower bounds on $WF(u)$ and $WF^{L^2}(u)$
- Ideas of the proof:
  - new a priori estimate on the Fourier transform
  - microlocal analysis
  - defect measures
Outline

- Initial value problem for the Navier – Stokes equations
- The singular sets $S(u)$ and the energy concentration set $S^{L^2}(u)$
- Three estimates on Leray weak solutions
- Phase space volume: lower bounds on $WF(u)$ and $WF^{L^2}(u)$
- Ideas of the proof:
  - new a priori estimate on the Fourier transform
  - microlocal analysis
  - defect measures
Navier – Stokes equations for incompressible viscous fluids

\[ \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u \]  
\[ \nabla \cdot u = 0 \]  

For \( t = 0 \) specify an initial velocity field \( u_0(x) \), \( \nabla \cdot u_0 = 0 \)

Finite energy

\[ e(u_0) := \frac{1}{2} \int |u_0(x)|^2 \, dx \]
\[ = \frac{1}{2} \|u_0\|_{L^2}^2 < +\infty \]

Space-time domain

\[ D = \mathbb{R}^3 \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}^+ := Q \]

Alternatively

\[ D = T^3 \quad (x, t) \in T^3 \times \mathbb{R}^+ \]

A domain \( D \subseteq \mathbb{R}^3 \) with smooth boundary – we leave this open.
Navier–Stokes equations for incompressible viscous fluids

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u \\
\nabla \cdot u &= 0
\end{align*}
\]

For \( t = 0 \) specify an initial velocity field \( u_0(x), \nabla \cdot u_0 = 0 \)

Finite energy

\[
e(u_0) := \frac{1}{2} \int |u_0(x)|^2 \, dx
\]

\[
= \frac{1}{2} \| u_0 \|_{L^2}^2 < +\infty
\]

Space-time domain

\[
D = \mathbb{R}^3 \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}^+ := Q
\]

Alternatively

\[
D = T^3 \quad (x, t) \in T^3 \times \mathbb{R}^+
\]

A domain \( D \subseteq \mathbb{R}^3 \) with smooth boundary – we leave this open.
- **Navier – Stokes equations** for incompressible viscous fluids

\[
\begin{align*}
\partial_t u + (u \cdot \nabla)u &= -\nabla p + \nu \Delta u \\
\nabla \cdot u &= 0
\end{align*}
\] (1)

- For \( t = 0 \) specify an initial velocity field \( u_0(x), \nabla \cdot u_0 = 0 \)

Finite energy

\[
e(u_0) := \frac{1}{2} \int |u_0(x)|^2 \, dx
\]

\[
= \frac{1}{2} \|u_0\|_{L^2}^2 < +\infty
\]

- **Space-time domain**

\[
D = \mathbb{R}^3 \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}^+ := Q
\]

Alternatively

\[
D = \mathbb{T}^3 \quad (x, t) \in \mathbb{T}^3 \times \mathbb{R}^+
\]

A domain \( D \subseteq \mathbb{R}^3 \) with smooth boundary – we leave this open.
Question: do solutions exist?

Answer, *yes* in some sense.

If no singularities are formed, then *yes*.

Solutions exist, they are unique, and the mathematical theory is satisfactory.

If singularities form, then *weak solutions* exist. However they may not be unique, they exhibit infinite velocities, and the theory is less than satisfactory.

The original existence theorem is due to Leray (1934).

There have been many further contributions.
Question: do solutions exist?

Answer, *yes* in some sense.

If no singularities are formed, then *yes*.
Solutions exist, they are unique, and the mathematical theory is satisfactory

If singularities form, then *weak solutions* exist. However they may not be unique, they exhibit infinite velocities, and the theory is less than satisfactory

The original existence theorem is due to Leray (1934)
There have been many further contributions
Question: do solutions exist?

Answer, yes in some sense.
If no singularities are formed, then yes.
Solutions exist, they are unique, and the mathematical theory is satisfactory
If singularities form, then weak solutions exist. However they may not be unique, they exhibit infinite velocities, and the theory is less than satisfactory

The original existence theorem is due to Leray (1934)
There have been many further contributions
Leray weak solutions

The usual definition of a weak solution over $t \in [0, T]$:

1. The pair $(u(x, t), p(x, t))$ is a solution of (1) in the sense of distributions

2. Integrability conditions Initial energy $e(u_0) := \frac{1}{2}R^2 < +\infty$

\[
\frac{1}{2} \int |u(x, t)|^2 \, dx < +\infty
\]

\[
\nu \int_0^T \int_D |\nabla u(x, t)|^2 \, dxdt < +\infty
\]

\[
\iint_{loc} |p|^{5/3} \, dxdt < +\infty \quad \text{(Sohr & vonWahl (1986))}
\]

3. The energy inequality is satisfied

\[
\frac{1}{2} \int_D |u(x, T)|^2 \, dx + \nu \int_0^T \int_D |\nabla u(x, t)|^2 \, dxdt \leq \frac{1}{2} \int_D |u_0(x)|^2 \, dx
\]
Leray weak solutions

The usual definition of a weak solution over \( t \in [0, T] \):

1. The pair \((u(x, t), p(x, t))\) is a solution of (1) in the sense of distributions

2. Integrability conditions

   Initial energy \( e(u_0) := \frac{1}{2} R^2 < +\infty \)

   \[
   \frac{1}{2} \int |u(x, t)|^2 \, dx < +\infty \\
   \nu \int_0^T \int_D |\nabla u(x, t)|^2 \, dx \, dt < +\infty \\
   \int \int_{loc} |p|^{5/3} \, dx \, dt < +\infty \quad \text{(Sohr & vonWahl (1986))}
   \]

3. The energy inequality is satisfied

   \[
   \frac{1}{2} \int_D |u(x, T)|^2 \, dx + \nu \int_0^T \int_D |\nabla u(x, t)|^2 \, dx \, dt \leq \frac{1}{2} \int_D |u_0(x)|^2 \, dx
   \]
Leray weak solutions

The usual definition of a weak solution over \( t \in [0, T] \):

1. The pair \((u(x, t), p(x, t))\) is a solution of (1) in the sense of distributions

2. Integrability conditions Initial energy \( e(u_0) := \frac{1}{2}R^2 < +\infty \)

\[
\frac{1}{2} \int |u(x, t)|^2 dx < +\infty \\ \nu \int_0^T \int_D |\nabla u(x, t)|^2 dxdt < +\infty \\ \int\int_{loc} |p|^{5/3} dxdt < +\infty \quad (Sohr \ & \ vonWahl \ (1986))
\]

3. The energy inequality is satisfied

\[
\frac{1}{2} \int_D |u(x, T)|^2 dx + \nu \int_0^T \int_D |\nabla u(x, t)|^2 dxdt \leq \frac{1}{2} \int_D |u_0(x)|^2 dx
\]
Theorem (Leray (1934))

Given $u_0 \in L^2(D)$ divergence free, then there exists at least one weak solution to (1) globally in time. Weak solutions satisfy

$$u \in L^\infty_t(L^2_x) \cap L^2_t(\dot{H}^1_x) \quad p \in L^{5/3}_{loc}(Q)$$

(3)

A lot is known about such solutions, including weak continuity

$$u \in C_t(L^2_x : \text{weak topology})$$

as well as

$$u \in L^s_t(L^p_x), \quad \frac{3}{p} + \frac{2}{s} = \frac{3}{2} \quad 2 \leq p \leq 6$$

Uniqueness and global regularity are unknown
Theorem (Leray (1934))

Given \( u_0 \in L^2(D) \) divergence free, then there exists at least one weak solution to (1) globally in time. Weak solutions satisfy

\[
    u \in L^\infty_t(L^2_x) \cap L^2_t(\dot{H}^1_x) \quad p \in L^{5/3}_{loc}(Q) \tag{3}
\]

A lot is known about such solutions, including weak continuity

\[
    u \in C_t(L^2_x : \text{weak topology})
\]

as well as

\[
    u \in L^s_t(L^p_x), \quad \frac{3}{p} + \frac{2}{s} = \frac{3}{2} \quad 2 \leq p \leq 6
\]

Uniqueness and global regularity are unknown.
Theorem (Leray (1934))

Given $u_0 \in L^2(D)$ divergence free, then there exists at least one weak solution to (1) globally in time. Weak solutions satisfy

$$u \in L^\infty_t(L^2_x) \cap L^2_t(\dot{H}^1_x) \quad p \in L^{5/3}_{loc}(Q)$$  \hspace{1cm} (3)

A lot is known about such solutions, including weak continuity

$$u \in C_t(L^2_x : \text{weak topology})$$

as well as

$$u \in L^s_t(L^p_x), \quad \frac{3}{p} + \frac{2}{s} = \frac{3}{2} \quad 2 \leq p \leq 6$$

Uniqueness and global regularity are unknown.
Theorem (Leray (1934))

Given $u_0 \in L^2(D)$ divergence free, then there exists at least one weak solution to (1) globally in time. Weak solutions satisfy

$$u \in L_t^\infty(L_x^2) \cap L_t^2(\dot{H}_x^1) \quad p \in L_{loc}^{5/3}(Q)$$

A lot is known about such solutions, including weak continuity

$$u \in C_t(L_x^2 : \text{weak topology})$$

as well as

$$u \in L_t^s(L_x^p) , \quad \frac{3}{p} + \frac{2}{s} = \frac{3}{2} \quad 2 \leq p \leq 6$$

Uniqueness and global regularity are unknown
**Definition (Singular set)**

Given a weak solution \((u, p)\) of (1), the **singular set** \(S(u)\) is the set of space-time points at which \(u(x, t)\) is not locally bounded.

That is, \((x_0, t_0) \not\in S(u)\) if there is a neighborhood \(Q_r := Q_r(x_0, t_0)\) such that \(u(x, t)\) is bounded in \(Q_r\)

Hence \(S(u)\) is a closed set

This makes sense due to a theorem of Serrin (1962) which states that if \((x_0, t_0) \not\in S(u)\), then for all \(k\) (and with some \(0 < \alpha < 1\))

\[
\partial_x^k u(x, t) \in C^\alpha(\frac{Q_r}{2}(x_0, t_0)) \tag{4}
\]

Serrin’s condition is actually \(u \in L_t^s(L_x^p)(Q_r(x_0, t_0))\) for \(\frac{3}{p} + \frac{2}{s} < 1\)

Improved by Struwe (1995) to equality, with \(s < \infty\)

And by Escauriaza, Seregin and Sver`ak (2003) to

\(u \in L_t^\infty(L_x^3)(Q_r(x_0, t_0))\)
Definition (Singular set)

Given a weak solution \((u, p)\) of (1), the singular set \(S(u)\) is the set of space-time points at which \(u(x, t)\) is not locally bounded.

That is, \((x_0, t_0) \notin S(u)\) if there is a neighborhood \(Q_r := Q_r(x_0, t_0)\) such that \(u(x, t)\) is bounded in \(Q_r\)

Hence \(S(u)\) is a closed set

This makes sense due to a theorem of Serrin (1962) which states that if \((x_0, t_0) \notin S(u)\), then for all \(k\) (and with some \(0 < \alpha < 1\))

\[
\partial_x^k u(x, t) \in C^\alpha(Q_r/2(x_0, t_0)) \tag{4}
\]

Serrin’s condition is actually \(u \in L^s_t(L^p_x)(Q_r(x_0, t_0))\) for \(\frac{3}{p} + \frac{2}{s} < 1\)

Improved by Struwe (1995) to equality, with \(s < \infty\)

And by Escauriaza, Seregin and Sveràk (2003) to \(u \in L^\infty_t(L^3_x)(Q_r(x_0, t_0))\)
Definition (Singular set)
Given a weak solution \((u, p)\) of (1), the singular set \(S(u)\) is the set of space-time points at which \(u(x, t)\) is not locally bounded.

That is, \((x_0, t_0) \notin S(u)\) if there is a neighborhood \(Q_r := Q_r(x_0, t_0)\) such that \(u(x, t)\) is bounded in \(Q_r\)

Hence \(S(u)\) is a closed set

This makes sense due to a theorem of Serrin (1962) which states that if \((x_0, t_0) \notin S(u)\), then for all \(k\) (and with some \(0 < \alpha < 1\))

\[
\partial^k_x u(x, t) \in C^\alpha(Q_{r/2}(x_0, t_0))
\]  

(S4)

Serrin’s condition is actually \(u \in L^s_t(L^p_x)(Q_r(x_0, t_0))\) for \(\frac{3}{p} + \frac{2}{s} < 1\)

Improved by Struwe (1995) to equality, with \(s < \infty\)

And by Escauriaza, Seregin and Sver`ak (2003) to
\[u \in L^\infty_t(L^3_x)(Q_r(x_0, t_0))\]
Upper bounds on the singular set

- Singular times

**Theorem (Leray, Foiaş & Temam)**

*The set of singular times* $\tau(u) = \pi_t S(u) \in \mathbb{R}^+$ *has zero $1/2$-Hausdorff dimensional measure*

$$\mathcal{H}^{1/2}(\tau(u)) = 0$$  \hspace{1cm} (5)

- Partial regularity

**Theorem (Caffarelli, Kohn & Nirenberg (1982))**

*If $(u, p)$ is a suitable weak solution of (1) then the parabolic one-dimensional Hausdorff measure of $S(u)$ is zero;*

$$\mathcal{P}^1(S(u)) = 0$$  \hspace{1cm} (6)

- The solutions constructed by Leray are suitable.
Upper bounds on the singular set

- Singular times

Theorem (Leray, Foiaș & Temam)

The set of singular times \( \tau(u) = \pi_t S(u) \in \mathbb{R}^+ \) has zero \( 1/2 \)-Hausdorff dimensional measure

\[ H^{1/2}(\tau(u)) = 0 \] (5)

- Partial regularity

Theorem (Caffarelli, Kohn & Nirenberg (1982))

If \((u, p)\) is a suitable weak solution of (1) then the parabolic one-dimensional Hausdorff measure of \( S(u) \) is zero;

\[ P^1(S(u)) = 0 \] (6)

- The solutions constructed by Leray are suitable.
Upper bounds on the singular set

- Singular times

Theorem (Leray, Foiaș & Temam)

The set of singular times \( \tau(u) = \pi_t S(u) \in \mathbb{R}^+ \) has zero \( \frac{1}{2} \)-Hausdorff dimensional measure

\[
\mathcal{H}^{1/2}(\tau(u)) = 0
\]  

- Partial regularity

Theorem (Caffarelli, Kohn & Nirenberg (1982))

If \((u, p)\) is a suitable weak solution of (1) then the parabolic one-dimensional Hausdorff measure of \( S(u) \) is zero;

\[
\mathcal{P}^1(S(u)) = 0
\]

- The solutions constructed by Leray are suitable.
Hausdorff dimension

Definition (Hausdorff dimension)

Cover a set \( S \) with balls \( B_{r_j} \) of radii \( r_j < \delta \). The \( \beta \)-dimensional Hausdorff measure of \( S \) is

\[
\mathcal{H}^\beta(S) := \liminf_{\delta \to 0} \sum_{j} r_j^\beta
\]

The Hausdorff dimension of \( S \) is the infimum of \( \beta \) such that \( \mathcal{H}^\beta(S) = 0 \)

The parabolic Hausdorff dimension is the same, however using parabolic cylinders \( Q_r \) for space-time

\[
Q_r(x_0, t_0) := \{(x, t) : |x_0 - x| < r, 0 < t_0 - t < r^2\}
\]
Hausdorff dimension

Definition (Hausdorff dimension)

Cover a set $S$ with balls $B_{r_j}$ of radii $r_j < \delta$. The $\beta$-dimensional Hausdorff measure of $S$ is

$$\mathcal{H}^\beta(S) := \liminf_{\delta \to 0} \sum_j r_j^\beta$$

The Hausdorff dimension of $S$ is the infimum of $\beta$ such that $\mathcal{H}^\beta(S) = 0$

The parabolic Hausdorff dimension is the same, however using parabolic cylinders $Q_r$ for space-time

$$Q_r(x_0, t_0) := \{(x, t) : |x_0 - x| < r, 0 < t_0 - t < r^2\}$$
Homogeneous (or box counting) dimension

- **Definition (homogeneous dimension)**
  Given a closed set $S$, consider $C_0^\infty$ cutoff functions $0 \leq \varphi_\varepsilon \leq 1$ such that on an $\varepsilon$-tubular neighborhood $o_\varepsilon(S)$ of $S$

  $$\varphi_\varepsilon(x) = 1$$

  Then the **homogeneous dimension** of $S$ is

  $$D(S) := d - \liminf_{\varepsilon \to 0} \frac{\log(\int \varphi_\varepsilon)}{\log(\varepsilon)}$$

  - This is to say that
    $$\int \varphi_\varepsilon \, dx \sim \varepsilon^{d - D(S)}$$

  - The main lemma of the Caffarelli Kohn Nirenberg theorem also implies that $D(S) \leq 1$
Homogeneous (or box counting) dimension

**Definition (homogeneous dimension)**

Given a closed set $S$, consider $C^\infty_0$ cutoff functions $0 \leq \varphi_\varepsilon \leq 1$ such that on an $\varepsilon$-tubular neighborhood $o_\varepsilon(S)$ of $S$

$$\varphi_\varepsilon(x) = 1$$

Then the **homogeneous dimension** of $S$ is

$$D(S) := d - \liminf_{\varepsilon \to 0} \frac{\log(\int \varphi_\varepsilon)}{\log(\varepsilon)}$$

**This is to say that**

$$\int \varphi_\varepsilon \, dx \sim \varepsilon^{d-D(S)}$$

**The main lemma of the Caffarelli Kohn Nirenberg theorem also implies that** $D(S) \leq 1$
Restrict to a time slice

- For $t_0$ fixed, the singular set $S_{t_0} := S(u) \cap \{t = t_0\}$ in each time slice is at most one-dimensional, and

$$\mathcal{H}^1(S_{t_0}) = 0$$ (7)

- Suitable weak solutions are those satisfying a local energy inequality

$$\int_D \frac{1}{2}|u(\cdot, t)|^2 \varphi \, dx \bigg|_{t=0}^T + \nu \int_0^T \int_D |\nabla u(\cdot, t)|^2 \varphi \, dx \, dt$$

$$\leq \frac{1}{2} \int_0^T \int_D |u(\cdot, t)|^2 \left( \partial_t \varphi + \nu \Delta \varphi \right) \, dx \, dt$$

$$+ \int_0^T \int_D \left( p + \frac{1}{2}|u(\cdot, t)|^2 \right) u \cdot \nabla \varphi \, dx \, dt$$ (8)
Restrict to a time slice

- For $t_0$ fixed, the singular set $S_{t_0} := S(u) \cap \{ t = t_0 \}$ in each time slice is at most one-dimensional, and

$$H^1(S_{t_0}) = 0 \quad (7)$$

- Suitable weak solutions are those satisfying a local energy inequality

$$\int_D \frac{1}{2} |u(\cdot, t)|^2 \varphi \, dx \bigg|_{t=0} + \nu \int_0^T \int_D |\nabla u(\cdot, t)|^2 \varphi \, dxdt \leq \frac{1}{2} \int_0^T \int_D |u(\cdot, t)|^2 \left( \partial_t \varphi + \nu \Delta \varphi \right) \, dxdt$$

$$+ \int_0^T \int_D \left( p + \frac{1}{2} |u(\cdot, t)|^2 \right) u \cdot \nabla \varphi \, dxdt$$

(8)
Theorem 1: lower bounds in phase space

Theorem (2011)

If \( t_0 \in \tau(u) \) is a singular time for \( u \) then

\[
\dim (WF(u)) \geq \frac{1}{2} \quad \text{(9)}
\]

- **dimension comparison**: \( S(u) \cap \{ t = t_0 \} := S_{t_0}(u) \) is a subset of \( \mathbb{R}^3 \) while the wave front set \( WF(u) \subseteq T^*(\mathbb{R}^3) \) can be considered as a subset of \( S^*(\mathbb{R}^3) \), which is 5 dimensional.

- This lower bound is essentially valid fiber-wise \( WF_{x_0} \subseteq T^*_{x_0}(D) \), for each fiber for which \( (x_0, t_0) \in S_{t_0}(u) \).
Phase space dimension of $WF(u)$

- Consider cutoff symbols $0 \leq a(x, \xi) \leq 1$ in $S^0_{\rho\delta}$ such that and $a(x, \xi) = 1$ on $WF(u) \cap B_r(x_0)$ and

$$ (1 - a(x, D))u \in C_t(C^\infty_x)(Q_r(x_0, t0)) $$

- The volume growth of $\text{supp}(a)$ gives an upper bound on the phase space neighborhood of $WF(u)_{x_0} \subseteq T^*_x(D)$ supporting the singularity

$$ \text{vol} (\pi_\xi \text{supp}(a) \cap B_R(0)) \sim R^{1+\beta} $$

Definition (Dim$WF(u)_{x_0}$)

$$ \bar{\beta}_{x_0}(u) := \liminf_{r,a} (\beta) $$ (10)
Three inequalities

1. The energy inequality

\[ \frac{1}{2} \int_D |u(x, T)|^2 \, dx + \nu \int_0^T \int_D |\nabla u(x, t)|^2 \, dx \, dt \leq \frac{1}{2} \int_D |u_0(x)|^2 \, dx \]

and its consequences under interpolation


\[ \int_0^T \|u(\cdot, t)\|_{L^\infty} \, dt < +\infty \ , \quad \forall T \in \mathbb{R}^+ \quad (11) \]

3. Theorem (Biryuk & C. (2009))

Let \( B_R(0) \subseteq L^2(D) \) and define

\[ A_{R_1} := \left\{ (\hat{u}(\xi))_{\xi \in \mathbb{R}^d} : |\xi||\hat{u}(\xi)| < R_1 \right\} \quad (12) \]

If \( R^2/\sqrt{2\pi}^3 < \nu R_1 \) then \( A_{R_1} \cap B_R(0) \) is a (future) invariant set for Navier – Stokes flow
Three inequalities

1. The energy inequality

$$\frac{1}{2} \int_{D} |u(x, T)|^2 \, dx + \nu \int_{0}^{T} \int_{D} |\nabla u(x, t)|^2 \, dx \, dt \leq \frac{1}{2} \int_{D} |u_0(x)|^2 \, dx$$

and its consequences under interpolation


$$\int_{0}^{T} \|u(\cdot, t)\|_{L^\infty} \, dt < +\infty , \quad \forall T \in \mathbb{R}^+ \quad (11)$$

3. Theorem (Biryuk & C. (2009))

Let $B_R(0) \subseteq L^2(D)$ and define

$$A_{R_1} := \left\{ (\hat{u}(\xi))_{\xi \in \mathbb{R}^d} : |\xi| |\hat{u}(\xi)| < R_1 \right\} \quad (12)$$

If $R^2 / \sqrt{2\pi^3} < \nu R_1$ then $A_{R_1} \cap B_R(0)$ is a (future) invariant set for Navier – Stokes flow
Three inequalities

1. The energy inequality

\[ \frac{1}{2} \int_D |u(x, T)|^2 \, dx + \nu \int_0^T \int_D |\nabla u(x, t)|^2 \, dx \, dt \leq \frac{1}{2} \int_D |u_0(x)|^2 \, dx \]

and its consequences under interpolation


\[ \int_0^T \|u(\cdot, t)\|_{L^\infty} \, dt < +\infty, \quad \forall T \in \mathbb{R}^+ \tag{11} \]

3. Theorem (Biryuk & C. (2009))

Let \( B_R(0) \subseteq L^2(D) \) and define

\[ A_{R_1} := \{ (\hat{u}(\xi))_{\xi \in \mathbb{R}^d} : |\xi| |\hat{u}(\xi)| < R_1 \} \tag{12} \]

If \( R^2 / \sqrt{2\pi^3} < \nu R_1 \) then \( A_{R_1} \cap B_R(0) \) is a (future) invariant set for Navier – Stokes flow
That is to say, if initial data satisfies \(|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}\) then for all time \(t > 0\)

\[|\hat{u}(\xi, t)| < \frac{R_1}{|\xi|}, \quad \forall \xi\] (13)

Corollary (Biryuk & C (2009))

If the initial data satisfies \(|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}\), then additionally for all \(T \geq 0\)

\[\nu \int_0^T |\hat{u}(\xi, t)|^2 dt \leq \frac{R_2^2}{|\xi|} \] (14)

Proof of theorem and corollary given at end of talk (if there is time)

The quantity \(\sup_t \|\xi|\hat{u}(\xi, t)\|_{L^\infty}\) scales like the \(BV\) norm \(\sup_t \|\partial_x u(\cdot, t)\|_{L^1}\) (for which there are no known bounds).
That is to say, if initial data satisfies $|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}$ then for all time $t > 0$

$$|\hat{u}(\xi, t)| < \frac{R_1}{|\xi|}, \quad \forall \xi \quad (13)$$

**Corollary (Biryuk & C (2009))**

*If the initial data satisfies $|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}$, then additionally for all $T \geq 0*$

$$\nu \int_0^T |\hat{u}(\xi, t)|^2 \, dt \leq \frac{R_2^2}{|\xi|^4} \quad (14)$$

**Proof of theorem and corollary given at end of talk**

(if there is time)

The quantity $\sup_t \|\xi|\hat{u}(\xi, t)\|_{L^\infty}$ scales like the $BV$ norm $\sup_t \|\partial_x u(\cdot, t)\|_{L^1}$ (for which there are no known bounds).
That is to say, if initial data satisfies $|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}$ then for all time $t > 0$

$$|\hat{u}(\xi, t)| < \frac{R_1}{|\xi|}, \quad \forall \xi \quad (13)$$

**Corollary (Biryuk & C (2009))**

*If the initial data satisfies $|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}$, then additionally for all $T \geq 0$*

$$\nu \int_0^T |\hat{u}(\xi, t)|^2 \, dt \leq \frac{R_2^2}{|\xi|^4} \quad (14)$$

**Proof of theorem and corollary given at end of talk (if there is time)**

The quantity $\sup_t \|\xi|\hat{u}(\xi, t)\|_{L^\infty}$ scales like the $BV$ norm $\sup_t \|\partial_x u(\cdot, t)\|_{L^1}$ (for which there are no known bounds).
Energy discontinuities

- The energy $e(u(\cdot, t)) = \frac{1}{2} \|u(\cdot, t)\|_{L^2}^2$ could be discontinuous at $t_0 \in \tau(u)$.

  Or else it may be continuous (but nonetheless $\|\nabla u(\cdot, t)\|_{L^2}^2$ is necessarily unbounded on $[t_0 - \delta, t_0]$ for any $\delta$)

- Decompose the set of singular times $\tau(u)$ into

$$\tau(u) := \tau_1 \cup \tau_2$$

where $\tau_1$ are the energy discontinuities

$$\tau_1 := \left\{ t_0 : \lim_{t \to t_0^-} \sup \|u(\cdot, t)\|_{L^2}^2 > \|u(\cdot, t_0)\|_{L^2}^2 \right\}$$

and $\tau_2 = \tau(u) \setminus \tau_1$ is the remainder
Energy discontinuities

The energy \( e(u(\cdot, t)) = \frac{1}{2} \|u(\cdot, t)\|_{L^2}^2 \) could be discontinuous at \( t_0 \in \tau(u) \).

Or else it may be continuous (but nonetheless \( \|\nabla u(\cdot, t)\|_{L^2}^2 \) is necessarily unbounded on \([t_0 - \delta, t_0]\) for any \( \delta \))

Decompose the set of singular times \( \tau(u) \) into

\[ \tau(u) := \tau_1 \cup \tau_2 \]

where \( \tau_1 \) are the energy discontinuities

\[ \tau_1 := \{ t_0 : \limsup_{t \to t_0^-} \|u(\cdot, t)\|_{L^2}^2 > \|u(\cdot, t_0)\|_{L^2}^2 \} \]

and \( \tau_2 = \tau(u) \setminus \tau_1 \) is the remainder
The energy concentration set $S^{L^2}$

Suppose that $t_0 \in \tau_1$; that is, $u(x, t)$ is not strong $L^2$ continuous for $t \mapsto t_0^-$. 

**Definition ($L^2$ concentration set)**

The point $x_0 \notin S^{L^2}_{t_0}$ if there exists $r > 0$ such that

$$
\lim_{t \to t_0^-} \|u(\cdot, t)\|^2_{L^2(B_r(x_0))} = \|u(\cdot, t_0)\|^2_{L^2(B_r(x_0))}
$$

(15)

Thus $L^2$ concentration is associated with a point set $S^{L^2}_{t_0} \in \mathbb{R}^3$

The set $S^{L^2}_{t_0}$ is closed 

Indeed norm convergence plus weak convergence implies strong convergence, and any $B_{r_1}(x_1) \subseteq B_r(x_0)$ shares this strong convergence.

Since $u(\cdot, t)$ is smooth outside the singular set $S_{t_0}(u)$,

$$
S^{L^2}_{t_0} \subseteq S_{t_0}(u)
$$

(16)
The energy concentration set $S^{L^2}$

- Suppose that $t_0 \in \tau_1$; that is, $u(x, t)$ is not strong $L^2$ continuous for $t \mapsto t_0^-$. 

Definition ($L^2$ concentration set)

The point $x_0 \not\in S^{L^2}_{t_0}$ if there exists $r > 0$ such that

$$\lim_{t \to t_0^-} \|u(\cdot, t)\|^2_{L^2(B_r(x_0))} = \|u(\cdot, t_0)\|^2_{L^2(B_r(x_0))} \quad (15)$$

Thus $L^2$ concentration is associated with a point set $S^{L^2}_{t_0} \in \mathbb{R}^3$

- The set $S^{L^2}_{t_0}$ is closed

Indeed norm convergence plus weak convergence implies strong convergence, and any $B_{r_1}(x_1) \subseteq B_r(x_0)$ shares this strong convergence

- Since $u(\cdot, t)$ is smooth outside the singular set $S_{t_0}(u)$,

$$S^{L^2}_{t_0} \subseteq S_{t_0}(u) \quad (16)$$
The energy concentration set $S^{L^2}$

- Suppose that $t_0 \in \tau_1$; that is, $u(x, t)$ is not strong $L^2$ continuous for $t \mapsto t_0^-$. 

Definition ($L^2$ concentration set)

The point $x_0 \not\in S_{t_0}^{L^2}$ if there exists $r > 0$ such that

$$\lim_{t \to t_0^-} \|u(\cdot, t)\|_{L^2(B_r(x_0))}^2 = \|u(\cdot, t_0)\|_{L^2(B_r(x_0))}^2$$  \hspace{1cm} \text{(15)}

Thus $L^2$ concentration is associated with a point set $S_{t_0}^{L^2} \in \mathbb{R}^3$

- The set $S_{t_0}^{L^2}$ is closed

Indeed norm convergence plus weak convergence implies strong convergence, and any $B_{r_1}(x_1) \subseteq B_r(x_0)$ shares this strong convergence

- Since $u(\cdot, t)$ is smooth outside the singular set $S(t_0(u)$,

$$S_{t_0}^{L^2} \subseteq S_{t_0}(u)$$  \hspace{1cm} \text{(16)}
Theorem 2: lower bounds on the energy concentration set

- A lower bound on the size of the energy concentration wave front set $WF^{L^2}(u)$ when $t_0 \in \tau_1$

Theorem (Arnold & C. (2010))

If $t_0 \in \tau_1$ is an energy discontinuity for $u$ then

$$\text{Dim} \left( WF^{L^2}(u) \right) \geq 1$$  \hspace{1cm} (17)

- Remark: $S_{t_0}^{L^2} \subseteq S_{t_0}$, and as well $WF^{L^2} \subseteq WF$, so that whenever $x_0 \in S_{t_0}^{L^2}$ then Theorem 2 implies Theorem 1 with

$$\text{Dim} \left( WF(u) \right) \geq 1$$

However Theorem 1 applies in the case that $x_0 \in S_{t_0} \setminus S_{t_0}^{L^2}$
Theorem 2: lower bounds on the energy concentration set

- A lower bound on the size of the energy concentration wave front set $WF^{L^2}(u)$ when $t_0 \in \tau_1$

Theorem (Arnold & C. (2010))

If $t_0 \in \tau_1$ is an energy discontinuity for $u$ then

$$\text{Dim} \ (WF^{L^2}(u)) \geq 1 \quad (17)$$

- Remark: $S^{L^2}_{t_0} \subseteq S_{t_0}$, and as well $WF^{L^2} \subseteq WF$, so that whenever $x_0 \in S^{L^2}_{t_0}$ then Theorem 2 implies Theorem 1 with

$$\text{Dim} \ (WF(u)) \geq 1$$

However Theorem 1 applies in the case that $x_0 \in S_{t_0} \setminus S^{L^2}_{t_0}$
Outline of the proof of Theorem 2

- Probe the solution with Weyl calculus pseudodifferential operators

- Identify the $L^2$ discontinuities with defect measures, and $WF^{L^2}(u) \subseteq WF(u)$ with their support. This identifies the microlocal $L^2$ concentration set $WF^{L^2}(u)$ as a geometric subset of $T^*(\mathbb{R}^3)$

- Define the dimension of the sets $WF(u)$ and $WF^{L^2}(u)$ using symbol classes $S^0_{\rho\delta}$, for $0 < \rho \leq 1$

- If the solution concentrates onto $WF^{L^2}(u)$ on a set which is too small, argue by contradiction, using the third estimate (13) and the Lebesgue dominated convergence theorem
Thank you
proof of the theorem that $A \cap B_R(0)$ is invariant

- For fixed $\xi$ the field $\hat{u}(\xi) \in \mathbb{C}_\xi^2 \subseteq \mathbb{C}^3$
  Because of incompressibility $\xi \cdot \hat{u}(\xi) = 0$

- Proposition

  The function $\hat{u}(\xi, t)$ is Lipschitz continuous as a function of $t$ for every $\xi$

  - The Fourier transform satisfies

    \[
    \partial_t \hat{u}(\xi) = -\nu |\xi|^2 \hat{u}(\xi) - \frac{i\xi}{\sqrt{(2\pi)^3}} \Pi_\xi \int \hat{u}(\xi - \xi_1) \cdot \hat{u}(\xi_1) d\xi_1
    \]

    \[
    := X(u)_\xi
    \]  

    (18)
proof of the theorem that $A \cap B_R(0)$ is invariant

- For fixed $\xi$ the field $\hat{u}(\xi) \in \mathbb{C}_\xi^2 \subset \mathbb{C}^3$
  Because of incompressibility $\xi \cdot \hat{u}(\xi) = 0$

- Proposition
  *The function $\hat{u}(\xi, t)$ is Lipschitz continuous as a function of $t$ for every $\xi$*

- The Fourier transform satisfies

$$
\partial_t \hat{u}(\xi) = -\nu |\xi|^2 \hat{u}(\xi) - \frac{i\xi}{\sqrt{(2\pi)^3}} \Pi_\xi \int \hat{u}(\xi - \xi_1) \cdot \hat{u}(\xi_1) d\xi_1 \\
:= X(u)_\xi
$$

(18)
proof of the theorem that $A \cap B_R(0)$ is invariant

- For fixed $\xi$ the field $\hat{u}(\xi) \in C_\xi^2 \subseteq C^3$
  
  Because of incompressibility $\xi \cdot \hat{u}(\xi) = 0$

- Proposition

  *The function $\hat{u}(\xi, t)$ is Lipschitz continuous as a function of $t$ for every $\xi$*

- The Fourier transform satisfies

  $$
  \partial_t \hat{u}(\xi) = -\nu |\xi|^2 \hat{u}(\xi) - \frac{i\xi}{\sqrt{(2\pi)^3}} \Pi_\xi \int \hat{u}(\xi - \xi_1) \cdot \hat{u}(\xi_1) d\xi_1
  $$

  $$
  := X(u)_{\xi}
  $$

  (18)
Suppose that $\|u(\cdot)\|_{L^2} \leq R$ and consider the vector field $X(u)\xi$ when $|\hat{u}(\xi)| = R_1/|\xi|$. The radial component is

$$\text{re}(\hat{u}(\xi) \cdot X(u)\xi) < -\nu |\xi|^2 \left(\frac{R_1}{|\xi|}\right)^2 + \left(\frac{R_1}{|\xi|}\right)|\xi| \frac{R^2}{\sqrt{(2\pi)^3}}$$

and therefore the LHS is negative when $R^2/\sqrt{(2\pi)^3} < \nu R_1$

A similar argument holds when a forcing $f(x, t)$ is present.
Suppose that $\|u(\cdot)\|_{L^2} \leq R$ and consider the vector field $X(u)\xi$ when $|\hat{u}(\xi)| = R_1/|\xi|$. The radial component is

$$\text{re}(\hat{u}(\xi) \cdot X(u)\xi) < -\nu |\xi|^2 (R_1/|\xi|)^2 + (R_1/|\xi|)|\xi| \frac{R^2}{\sqrt{(2\pi)^3}}$$

(19)

and therefore the LHS is negative when $R^2/\sqrt{(2\pi)^3} < \nu R_1$

A similar argument holds when a forcing $f(x, t)$ is present
A fact about the vector field $X(\hat{u})$ is that solutions obey

$$|\hat{u}(\xi, T)|^2 - |\hat{u}_0(\xi)|^2 + 2\nu \int_0^T |\xi|^2 |\hat{u}(\xi, t)|^2 \, dt$$

$$= \frac{2}{\sqrt{(2\pi)^3}} \text{Im} \left[ \int_0^T \hat{u}(\xi) \cdot \int \hat{u}(\xi - \xi_1) \cdot \xi_1 \hat{u}(\xi_1) \, d\xi_1 \, dt \right] \quad (20)$$

(again setting $f = 0$ for simplicity)

Writing $I^2(\xi) = \nu \int_0^T |\xi|^4 |\hat{u}(\xi, t)|^2 \, dt$

this gives an inequality

$$I^2(\xi) - \frac{R^2}{2\nu \sqrt{(2\pi)^3}} I(\xi) - \frac{R_1^2}{2} \leq 0 \quad (21)$$
proof of corollary

- A fact about the vector field $X(\hat{u})$ is that solutions obey

$$|\hat{u}(\xi, T)|^2 - |\hat{u}_0(\xi)|^2 + 2\nu \int_0^T |\xi|^2 |\hat{u}(\xi, t)|^2 \, dt$$

$$= \frac{2}{\sqrt{(2\pi)^3}} \text{im}\left[ \int_0^T \hat{u}(\xi) \cdot \int \hat{u}(\xi - \xi_1) \cdot \xi_1 \hat{u}(\xi_1) \, d\xi_1 \, dt \right]$$

(again setting $f = 0$ for simplicity)

- Writing $I^2(\xi) = \nu \int_0^T |\xi|^4 |\hat{u}(\xi, t)|^2 \, dt$

this gives an inequality

$$I^2(\xi) - \frac{R^2}{2\nu \sqrt{(2\pi)^3}} I(\xi) - \frac{R_1^2}{2} \leq 0 \quad (21)$$
Microlocal behavior in $x$ and $\xi$

- The Fourier transform $\hat{u}(\xi, t) \in Lip$ as a function of time
- Take $x_0 \in S_T^{L^2}$, and localize the convergence question

$$v(x, t) := (u(x, t) - u(x, T))\varphi(x), \quad 0 \leq \varphi \in C_0^\infty(B_r(x_0))$$

**Proposition**

As $t \to T^-$ then $\hat{v}(\xi, t) \to 0$ pointwise in $\xi \in \mathbb{R}^3$

- What causes a lack of strong convergence is loss of $L^2$ mass at $|\xi| \to \infty$
Weyl calculus

- Given a point $x_0 \in S^L_T$ we test for energy concentration using the Weyl pseudodifferential calculus

$$a^w(x, D)v(x, t) = \int\int e^{i\xi \cdot (x-y)} a\left(\frac{x+y}{2}, \xi\right)v(y, t) \, dyd\xi$$  \hspace{1cm} (22)

for $a(x, \xi) \in S^0_{\rho\delta}$.

- A microlocal test of the energy is

$$\langle v | a^w(x, D)v \rangle = \int\int a(x, \xi)W[v](x, \xi) \, dxd\xi$$  \hspace{1cm} (23)

where $W[v]$ is the Wigner transform of $v$

$$W[v](x, \xi) := \frac{1}{(2\pi)^d} \int e^{i\xi \cdot y} v(x + y/2)\overline{v}(x - y/2) \, dy$$  \hspace{1cm} (24)
Given a point $x_0 \in S^L_T$ we test for energy concentration using the Weyl pseudodifferential calculus

$$a^w(x, D)v(x, t) = \int\int e^{i\xi \cdot (x-y)} a\left(\frac{x+y}{2}, \xi\right)v(y, t) \, dy \, d\xi$$

(22)

for $a(x, \xi) \in S^{0}_{\rho_\delta}$.

A microlocal test of the energy is

$$\langle v | a^w(x, D)v \rangle = \int\int a(x, \xi)W[v](x, \xi) \, dx \, d\xi$$

(23)

where $W[v]$ is the Wigner transform of $v$

$$W[v](x, \xi) := \frac{1}{(2\pi)^d} \int e^{i\xi \cdot y} v(x + y/2)\overline{v}(x - y/2) \, dy$$

(24)
For $a(x, \xi) \in S^{0}_{\rho\delta}$ the operators $a^w(x, D)$ are continuous on $L^2$. Whenever $u(\cdot, t)$ converges strongly to $u(\cdot, T)$ on $B_r(x_0)$, for $v(x, t) := (u(x, t) - u(x, T))\varphi(x)$ then

$$\lim_{t \to T^-} \langle a \mid W[v(t)] \rangle = 0 \quad (25)$$

However if $u(\cdot, t)$ converges weakly but not strongly to $u(\cdot, T)$, it is detected by a microlocal defect measure $\mu \in \mathcal{M}(S^*(\mathbb{R}^3))$.

**Theorem (L. Tartar (1990), P. Gérard (1991))**

Let $\mu_{x_0}$ be a microlocal defect measure of $\lim_{t_j \to T^-} v(x, t_j)$. Suppose that for all symbols $a \in S^{0}_{10}$ homogeneous degree zero

$$\lim_{t_j \to T^-} \langle a \mid W[v(t_j)] \rangle := \iint_{S^*(\mathbb{R}^3)} a(x, \xi)\mu_{x_0}(dx dS_{\xi}) = 0 \quad (26)$$

then $\mu_{x_0} = 0$ and $u(\cdot, t_j)$ converges strongly in $L^2(B_r(x_0))$ to $u(\cdot, T)$. 
microlocal $L^2$ defects

For $a(x, \xi) \in S^0_{\rho \delta}$ the operators $a^w(x, D)$ are continuous on $L^2$
Whenever $u(\cdot, t)$ converges strongly to $u(\cdot, T)$ on $B_r(x_0)$, for $v(x, t) := (u(x, t) - u(x, T))\varphi(x)$ then

$$\lim_{t \to T^{-}} \langle a \mid W[v(t)] \rangle = 0 \quad (25)$$

However if $u(\cdot, t)$ converges weakly but not strongly to $u(\cdot, T)$, it is detected by a microlocal defect measure $\mu \in \mathcal{M}(S^*_{\mathbb{R}^3})$.

Theorem (L. Tartar (1990), P. Gérard (1991))

Let $\mu_{x_0}$ be a microlocal defect measure of $\lim_{t_j \to T^{-}} v(x, t_j)$. Suppose that for all symbols $a \in S^0_{10}$ homogeneous degree zero

$$\lim_{t_j \to T^{-}} \langle a \mid W[v(t_j)] \rangle := \iint_{S^*_{\mathbb{R}^3}} a(x, \xi) \mu_{x_0} (dxdS_\xi) = 0 \quad (26)$$

then $\mu_{x_0} = 0$ and $u(\cdot, t_j)$ converges strongly in $L^2(B_r(x_0))$ to $u(\cdot, T)$. 


the $L^2$ wave front set $WF^{L^2}$

**Definition**

Let $A_r$ be the set of all microlocal defect measures $\mu_{x_0,r}$ as $t \to T^-$ of $\varphi_r(\cdot)(u(\cdot, t) - u(\cdot, T)) := v_r$, with $\text{supp } \varphi_r \subseteq B_r(x_0)$. The $L^2$ wave front set is

$$WF^{L^2}_{x_0 T} := \bigcap_{r > 0} \bigcup A_r \text{supp } (\mu_{x_0,r}) \tag{27}$$

- The sets $\text{supp } \mu_{x_0,r}$ are monotone decreasing in $r \to 0$
- Our definition of “dimension” gives an upper bound on the Hausdorff dimension of $WF^{L^2}_{T} \cap S^*(D)$, which is geometric. But it is not necessarily an equality.
- Our resulting lower bounds contain more analytic information, in non-conic neighborhoods of $WF^{L^2}_{T} \subseteq T^*(D)$. 

the $L^2$ wave front set $WF^{L^2}$

**Definition**
Let $A_r$ be the set of all microlocal defect measures $\mu_{x_0,r}$ as $t \to T^-$ of $
(\varphi_r(\cdot)(u(\cdot, t) - u(\cdot, T)) := v_r, \text{ with supp } \varphi_r \subseteq B_r(x_0)$. The $L^2$ wave front set is

$$WF^{L^2}_{x_0T} := \bigcap_{r > 0} \bigcup_{A_r} \text{supp } (\mu_{x_0,r})$$ (27)

- The sets $\text{supp } \mu_{x_0,r}$ are monotone decreasing in $r \to 0$
- Our definition of “dimension” gives an upper bound on the Hausdorff dimension of $WF^{L^2}_T \cap S^*(D)$, which is geometric. But it is not necessarily an equality.
- Our resulting lower bounds contain more analytic information, in non-conic neighborhoods of $WF^{L^2}_T \subseteq T^*(D)$. 

Hörmander symbol classes $S^{0}_{\rho\delta}$ and $WF^{L^2}$

- One probes $WF^{L^2}_{x_0T}$ more finely with symbols $a \in S^{0}_{\rho\delta}$, with $0 \leq \delta < \rho \leq 1$

For example $a(x, \xi) = \varphi(x) \chi(\frac{\xi'}{(\xi_1)^{\rho}}) \in S^{0}_{\rho_0}$ for $0 < \rho \leq 1$

By standard theory, the operators $a^w(x, D)$ are bounded on $L^2$.

These symbol classes include ones of quasi-homogeneous type


- Suppose that $0 \leq a(x, \xi) \leq 1$ is such that

$$\lim_{t \to T^-} \langle v | (1 - a)^w(x, D)v \rangle = 0$$

then

$$WF^{L^2}_{x_0T} \subseteq \text{supp} \ (a)$$

and $L^2$ mass is transported to infinity in $\text{supp} \ (a)$, microlocally near $x_0$
Hörmander symbol classes $S_{\rho \delta}^0$ and $WF^{L^2}$

- One probes $WF^{L^2}_{x_0 T}$ more finely with symbols $a \in S_{\rho \delta}^0$, with $0 \leq \delta < \rho \leq 1$

  For example $a(x, \xi) = \varphi(x) \chi\left(\frac{\xi'}{\langle \xi_1 \rangle^\rho}\right) \in S_{\rho_0}^0$ for $0 < \rho \leq 1$

  By standard theory, the operators $a^w(x, D)$ are bounded on $L^2$.

  These symbol classes include ones of quasi-homogeneous type


- Suppose that $0 \leq a(x, \xi) \leq 1$ is such that

  \[
  \lim_{t \to T^-} \langle v \vert (1 - a)^w(x, D)v \rangle = 0 \tag{28}
  \]

  then

  \[
  WF^{L^2}_{x_0 T} \subseteq \text{supp} (a)
  \]

  and $L^2$ mass is transported to infinity in $\text{supp} (a)$, microlocally near $x_0$
Given test symbols $a(x, \xi)$, the volume growth of $\text{supp} (a)$ gives an upper bound on the neighborhood of $WF_{x_0T}^{L^2} \subseteq T^*(D)$ with $L^2$ mass concentration.

Consider $a(x, \xi) \in S^{0}_{\rho \delta}$ such that $0 \leq a \leq 1$,

\[
\lim_{t \to T^-} \langle (1 - a) | W[v(t)] \rangle = 0 \quad (29)
\]

and

\[
\text{vol} \left( \pi_{\xi} \text{supp} (a) \cap B_R(0) \right) \sim R^{1+\beta} \quad (30)
\]

Definition (size of $WF_{x_0T}^{L^2}$)

\[
\bar{\beta}_{x_0}(v) := \inf(\beta) \quad (31)
\]
Lower bounds for the $L^2$ wave front set

**Theorem (Arnold & C. (2010))**

The set $WF_{x_0}^{L^2} \subseteq WF_{x_0}^T(u)$ is not too small (if it is nonempty), in that

$$\bar{\beta}_{x_0}(v) \geq 1$$

(32)

This is essentially a lower bound in each fiber of $T^*_{x_0}(\mathbb{R}^3)$.
Lower bounds for the $L^2$ wave front set

Theorem (Arnold & C. (2010))

The set $WF_{x_0T}^{L^2} \subseteq WF_{x_0T}(u)$ is not too small (if it is nonempty), in that

$$\bar{\beta}_{x_0}(v) \geq 1$$ \hspace{1cm} (32)

This is essentially a lower bound in each fiber of $T_{x_0}^*(\mathbb{R}^3)$.
Thank you
Proof of Theorem 2

Taking into account the local character of the problem, the real result states.

Conjecture (A & C)(work in progress)

Suppose that $\bar{\alpha}$ is the Hausdorff dimension of the set $S^{L^2}_T$, then

$$\bar{\beta}_{x_0}(v) \geq \bar{\alpha} + 1$$

(33)
Assume that $\overline{\beta}_{x_0}(v) < 1$. Choose $a \in S^0_{\rho\delta}$ with above support properties and volume growth $\overline{\beta}_{x_0}(v) < \beta < 1$, and test $v(\cdot, t)$

$$
\lim_{t \to T^-} \langle a | W[v] \rangle = \lim_{t \to T^-} \int va^w(x, D)v \, dx
$$

$$
= \lim_{t \to T^-} \int d\eta \left[ \int \tilde{a}(\eta, \xi) \hat{v}(\xi + \eta/2, t)\hat{v}(\xi - \eta/2, t) \, d\xi \right]
$$

where $\tilde{a}(\eta, \xi)$ is the Fourier transform of $a$ with respect to $x$.

For each $\eta$, the volume growth of $\text{supp} \tilde{a}(\eta, \xi)$ is bounded by $R^{1+\beta}$.

There is now a majorant: for each $\eta$

$$
|\tilde{a}(\eta, \xi)||\hat{v}(\xi+\eta/2, t)||\hat{v}(\xi-\eta/2, t)| \leq |\tilde{a}(\eta, \xi)||\langle \xi+\eta/2 \rangle^{-1}\langle \xi-\eta/2 \rangle^{-1}
$$

This is integrable over $\xi$. Indeed

$$
\int_{\xi} (\ast) \, d\xi \leq \int \langle r \rangle^{-2} r^\beta \, dr < +\infty
$$

(35)

The Lebesgue dominated convergence theorem implies that as $t \to T^-$ the limit vanishes.
Assume that $\bar{\beta}_{x_0}(v) < 1$. Choose $a \in S^0_{\rho\delta}$ with above support properties and volume growth $\bar{\beta}_{x_0}(v) < \beta < 1$, and test $v(\cdot, t)$

$$\lim_{t \to T^-} \langle a| W[v] \rangle = \lim_{t \to T^-} \int v a^w(x, D)v \, dx$$

$$= \lim_{t \to T^-} \int d\eta \left[ \int \tilde{a}(\eta, \xi) \hat{v}(\xi + \eta/2, t)\hat{v}(\xi - \eta/2, t) \, d\xi \right]$$

where $\tilde{a}(\eta, \xi)$ is the Fourier transform of $a$ with respect to $x$.

For each $\eta$, the volume growth of $\text{supp} \tilde{a}(\eta, \xi)$ is bounded by $R^{1+\beta}$.

There is now a majorant: for each $\eta$

$$|\tilde{a}(\eta, \xi)||\hat{v}(\xi+\eta/2, t)||\hat{v}(\xi-\eta/2, t)| \leq |\tilde{a}(\eta, \xi)||\langle \xi+\eta/2 \rangle^{-1} \langle \xi-\eta/2 \rangle^{-1}$$

This is integrable over $\xi$. Indeed

$$\int_{\xi} (*) \, d\xi \leq \int \langle r \rangle^{-2} r^{\beta} \, dr < +\infty$$

The Lebesgue dominated convergence theorem implies that as $t \to T^-$ the limit vanishes.
Assume that $\bar{\beta}_{x_0}(v) < 1$. Choose $a \in S^0_{\rho \delta}$ with above support properties and volume growth $\bar{\beta}_{x_0}(v) < \beta < 1$, and test $v(\cdot, t)$

\[
\lim_{t \to T^-} \langle a | W[v] \rangle = \lim_{t \to T^-} \int v a^w(x, D)v \, dx
\]

\[
= \lim_{t \to T^-} \int d\eta \left[ \int \tilde{a}(\eta, \xi) \hat{v}(\xi + \eta/2, t) \hat{v}(\xi - \eta/2, t) \, d\xi \right]
\]

where $\tilde{a}(\eta, \xi)$ is the Fourier transform of $a$ with respect to $x$.

For each $\eta$, the volume growth of $\text{supp} \tilde{a}(\eta, \xi)$ is bounded by $R^{1+\beta}$

There is now a majorant: for each $\eta$

\[
|\tilde{a}(\eta, \xi)||\hat{v}(\xi+\eta/2, t)||\hat{v}(\xi-\eta/2, t)| \leq |\tilde{a}(\eta, \xi)| \langle \xi + \eta/2 \rangle^{-1} \langle \xi - \eta/2 \rangle^{-1}
\]

This is integrable over $\xi$. Indeed

\[
\int_{\xi}(\ast) \, d\xi \leq \int \langle r \rangle^{-2} r^\beta \, dr < +\infty \tag{35}
\]

The Lebesgue dominated convergence theorem implies that as $t \to T^-$ the limit vanishes.
Assume that $\bar{\beta}_{x_0}(v) < 1$. Choose $a \in S^0_{\rho \delta}$ with above support properties and volume growth $\bar{\beta}_{x_0}(v) < \beta < 1$, and test $v(\cdot, t)$

$$
\lim_{t \to T^-} \langle a | W[v] \rangle = \lim_{t \to T^-} \int v a^w(x, D)v \, dx
$$

$$
= \lim_{t \to T^-} \int d\eta \left[ \int \tilde{a}(\eta, \xi) \hat{v}(\xi + \eta/2, t) \hat{v}(\xi - \eta/2, t) \, d\xi \right]
$$

where $\tilde{a}(\eta, \xi)$ is the Fourier transform of $a$ with respect to $x$.

For each $\eta$, the volume growth of $\text{supp} \, \tilde{a}(\eta, \xi)$ is bounded by $R^{1+\beta}$.

There is now a majorant: for each $\eta$

$$
|\tilde{a}(\eta, \xi)||\hat{v}(\xi + \eta/2, t)||\hat{v}(\xi - \eta/2, t)| \leq |\tilde{a}(\eta, \xi)|\langle \xi + \eta/2 \rangle^{-1}\langle \xi - \eta/2 \rangle^{-1}
$$

This is integrable over $\xi$. Indeed

$$
\int_\xi (\star) \, d\xi \leq \int \langle r \rangle^{-2} r^\beta \, dr < +\infty
$$

The Lebesgue dominated convergence theorem implies that as $t \to T^-$ the limit vanishes.
Thank you again