Local invertibility in Sobolev spaces

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(joint work with Marco Barchiesi and Duvan Henao)

Nonlinear Elasticity – Calculus of Variations approach

A deformation of a body $\Omega \subset \mathbb{R}^n$ is described by a map $\mathbf{u} : \Omega \to \mathbb{R}^n$.

$$\text{Elastic energy} = \int_\Omega W(D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

where $W : \mathbb{R}^{n\times n} \to [0, \infty]$ is the stored energy function of the material.

An equilibrium solution $\mathbf{u}$ (Statics) is a solution of

$$\min_{\substack{\mathbf{u}\in W^{1,p} \\ \text{b.c.}}} \int_\Omega W(D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \;+\; \text{forces}.$$

Apart from solving a minimization problem, every physically realistic solution $\mathbf{u}$ must:

- preserve the orientation: $\det D\mathbf{u} > 0$,
- be invertible (no interpenetration of matter).

Existence in Nonlinear Elasticity

Minimizers of

$$\int_\Omega W(D\mathbf{u})\,\mathrm{d}\mathbf{x}$$

exist as long as W is polyconvex ($=$ convex in the minors of $D\mathbf{u}$) and satisfies

$$W(\mathbf{F}) \geq c\,|\mathbf{F}|^p + c\,|\mathrm{cof}\,\mathbf{F}|^q + h(\det D\mathbf{u})$$

for certain p, q and h.

Key of the proof: if $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,p}$ then $\det D\mathbf{u}_j \rightharpoonup \det D\mathbf{u}$ in L^1.

Finding the optimal p, q for which $\det D\mathbf{u}$ is continuous has been an active field of research.

There is a remarkable parallelism between the optimal p, q for which $\det D\mathbf{u}$ is continuous and other nice properties that a Sobolev $W^{1,p}$ deformation may have.

	$W^{1,p}$, $p > n$	$W^{1,n}$, $\det D\mathbf{u} > 0$	$\mathcal{A}_{p,q}$, $p > n-1,\ q \geq \frac{n}{n-1}$
continuous	✓	✓	$\mathcal{H}^{n-p}$-a.e.
diff. a.e.	✓	✓	✓
$\det D\mathbf{u}$ cont.	✓	✓	✓
Lusin's N	✓	✓	✓
loc. invertible	$\det D\mathbf{u} > 0$	✓	

Cesari 41, Calderón 51, Morrey 66, Resetnyak 67, Marcus, Mizel 73, Vodopyanov, Goldshtein 76, Ball 77, Šverák 88, Müller, Qi, Yan 94, Manfredi 94, Fonseca, Gangbo 95, Hajlasz, Malý 02.

New class (to be defined): $\mathcal{A}_p$, $p > n-1$.

	$W^{1,p}$, $p > n$	$W^{1,n}$, $\det D\mathbf{u} > 0$	$\mathcal{A}_{p,q}$, $p > n-1$, $q \geq \frac{n}{n-1}$	$\mathcal{A}_p$, $p > n-1$
continuous	✓	✓	$\mathcal{H}^{n-p}$-a.e.	
diff. a.e.	✓	✓	✓	
$\det D\mathbf{u}$ cont.	✓	✓	✓	
Lusin's N	✓	✓	✓	
loc. invertible	✓	✓		

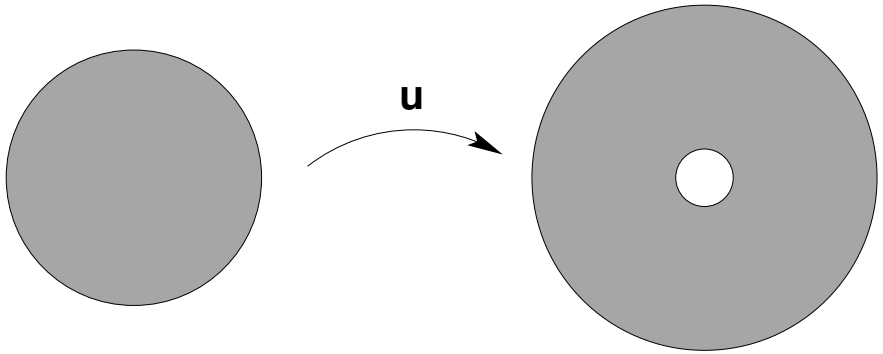
Aim of this talk: fill the table.

Applications to model nonlinear elasticity + other phenomena.

Energies defined in reference and deformed configurations.

Standard counterexample: $\frac{\mathbf{x}}{|\mathbf{x}|}$ or, rather, a variant of

$$\mathbf{u}(\mathbf{x}) = r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}.$$



This deformation opens a hole at $\mathbf{0}$ of radius $r(0)$.

This is the phenomenom of **cavitation**.

It satisfies $\mathbf{u} \in W^{1,p}$ for all $p < n$, and $\operatorname{cof} D\mathbf{u} \in L^q$ for all $q < \frac{n}{n-1}$.

Cavitation

Cavitation is the phenomenon of sudden formation of voids in solids subject to sufficiently large tension.

Cavitation in rubber (A. Gent & P. Lindley 59)

Global invertibility

- J. Ball 81
- P. Ciarlet & J. Nečas 87
- V. Šverák 88
- Tang Qi 88
- S. Müller, T. Qi & B. Yan 94
- S. Müller & S. Spector 95
- S. Conti & C. De Lellis 03
- D. Henao & C.M-C. 10–14

Local invertibility

- I. Fonseca & W. Gangbo 95
- Geometric function theory: Y. Resetnyak 67, T. Iwaniec & V. Šverák 93... All for $W^{1,n}$.

Condition INV of S. Müller & S. Spector 95

Let $p > n-1$. Let $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ and $\mathbf{x}_0 \in \Omega$. By Fubini,

$$\mathbf{u}|_{\partial B(\mathbf{x}_0,r)} \in W^{1,p}(\partial B(\mathbf{x}_0, r), \mathbb{R}^n) \qquad \text{for a.e. } r > 0.$$

Hence $\mathbf{u}|_{\partial B(\mathbf{x}_0,r)}$ is continuous and has a degree

$$\deg(\mathbf{u}, B(\mathbf{x}_0, r), \mathbf{y}) \qquad \text{for } \mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial B(\mathbf{x}_0, r)).$$

We say that $\mathbf{u}$ satisfies INV if

$$\deg(\mathbf{u}, B(\mathbf{x}_0, r), \mathbf{u}(\mathbf{x})) \neq 0 \quad \text{a.e. } \mathbf{x} \in B(\mathbf{x}_0, r),$$

$$\deg(\mathbf{u}, B(\mathbf{x}_0, r), \mathbf{u}(\mathbf{x})) = 0 \quad \text{a.e. } \mathbf{x} \in \Omega \setminus B(\mathbf{x}_0, r).$$

Almost every sphere is impenetrable: inside goes inside, and outside goes outside.

Müller & Spector proved: INV is closed under weak $W^{1,p}$ limit.

INV and $\det D\mathbf{u} > 0 \implies \mathbf{u}$ one-to-one a.e.

Well-posed models for cavitation

- S. Müller & S. Spector 95
- J. Sivaloganathan & S. Spector 00
- S. Conti & C. De Lellis 03
- D. Henao & C.M.-C. 10–12

In the good spaces above, we have the key equality

$\mathrm{Det}\, D\mathbf{u} = \det D\mathbf{u}$, where

$\det D\mathbf{u} =$ determinant of $D\mathbf{u}$.

$\mathrm{Det}\, D\mathbf{u} =$ distributional determinant.

$$\langle \mathrm{Det}\, D\mathbf{u}, \phi \rangle := -\frac{1}{n} \int_\Omega \mathbf{u}(\mathbf{x}) \cdot (\mathrm{cof}\, D\mathbf{u}(\mathbf{x})\, D\phi(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \qquad \phi \in C_c^\infty(\Omega)$$

A variant of condition $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$

$\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$ is equivalent to

$$\det D\mathbf{u}(\mathbf{x}) = \frac{1}{n} \operatorname{Div}\left[\operatorname{adj} D\mathbf{u}(\mathbf{x})\, \mathbf{u}(\mathbf{x})\right].$$

S. Müller 88 introduced

$$\operatorname{div} \mathbf{g}(\mathbf{u}(\mathbf{x})) \det D\mathbf{u}(\mathbf{x}) = \operatorname{Div}\left[\operatorname{adj} D\mathbf{u}(\mathbf{x})\, \mathbf{g}(\mathbf{u}(\mathbf{x}))\right] \quad \forall \mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n) \tag{*}$$

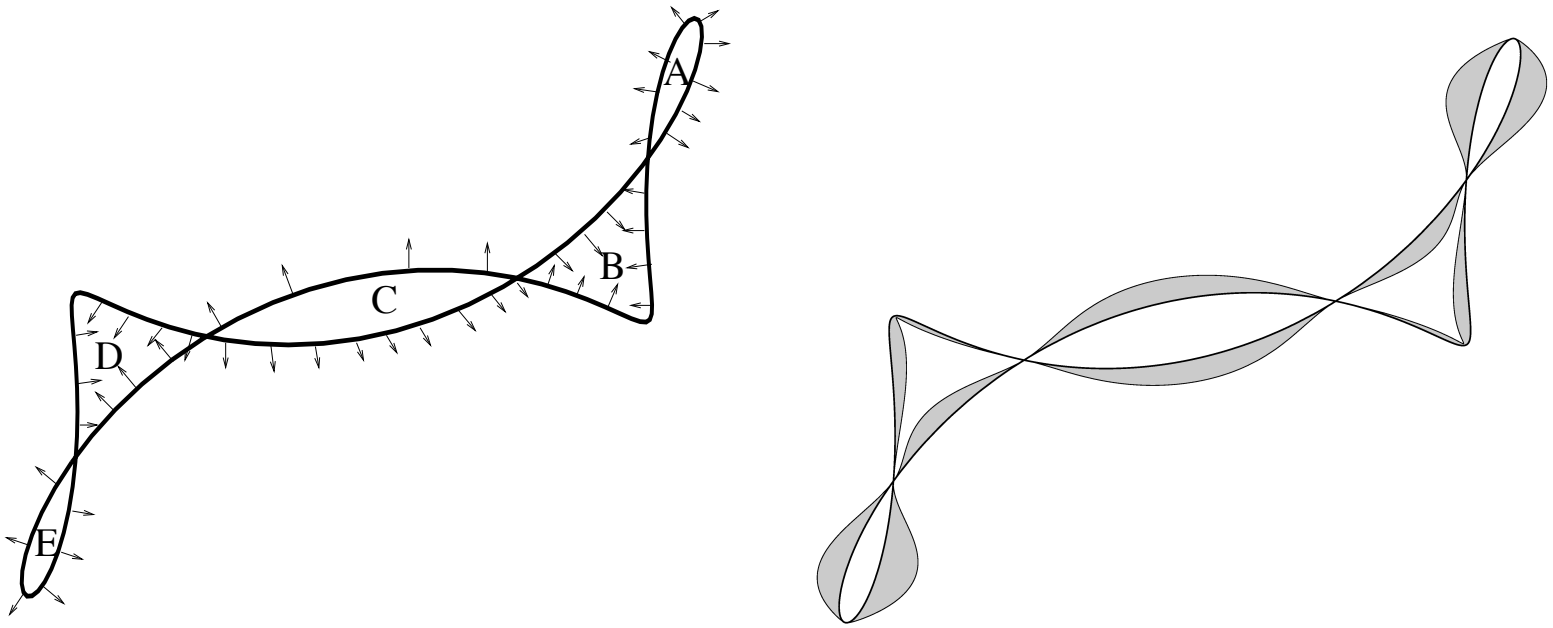
(also Giaquinta, Modica, Souček 98).

D. Henao & C.M.-C. 10 introduced $\mathcal{E}(\mathbf{u})$ to quantify the failure of (*). $\mathcal{E}(\mathbf{u})$ measures the created surface in the deformed configuration (area of the cavities).

For this talk, we just need that

$$\mathcal{E}(\mathbf{u}) = 0 \iff (*) \text{ holds.}$$

Example D. Henao & C.M.-C. 12 $\exists\, \mathbf{u}$ one-to-one a.e., $\det D\mathbf{u} > 0$, $\mathbf{u} \in W^{1,p}$, $p < n$, $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$, but creates a cavity.



$\operatorname{Det} D\mathbf{u} = \det D\mathbf{u} + c\,\delta_0$ with $c = \mathcal{L}^2(A \cup C \cup E) - \mathcal{L}^2(B \cup D) = 0$.

$\mathbf{u}$ does not satisfy INV, $\deg(\mathbf{u}, B, \cdot)$ is sometimes negative.

Orientation reversing and sort of interpenetration.

Without INV, $\operatorname{Det} D\mathbf{u}$ is unable to detect the creation of cavities.

What is orientation preserving?

Analytical definition: $\det D\mathbf{u} > 0$ a.e. (or $\det D\mathbf{u} \geq 0$ a.e.)

$\rightsquigarrow$ But recall counterexample above

Topological definition: $\deg(\mathbf{u}, B, \cdot) \geq 0$ for a.e. balls $B \subset \Omega$.

Characterization of our functional space

$\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n-1$, $\det D\mathbf{u} \in L^1_{\text{loc}}(\Omega)$. The following conditions are equivalent:

a) $\mathcal{E}(\mathbf{u}) = 0$ and $\det D\mathbf{u} > 0$ a.e.

b) $(\operatorname{adj} D\mathbf{u})\,\mathbf{u} \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$, $\det D\mathbf{u}(\mathbf{x}) \neq 0$ for a.e. $\mathbf{x} \in \Omega$, $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}$, and $\deg(\mathbf{u}, B, \cdot) \geq 0$ for a.e. ball B.

We call this class $\mathcal{A}_p$.

Can prove existence in Nonlinear elasticity in the class $\mathcal{A}_p$.

Regularity in $\mathcal{A}_p$, $p > n-1$.

$\mathcal{A}_p = \left\{ \mathbf{u} \in W^{1,p} : \det D\mathbf{u} \in L^1_{\text{loc}},\ \mathcal{E}(\mathbf{u}) = 0,\ \det D\mathbf{u} > 0 \right\}.$

- Weakly monotone (J. Manfredi 94): "maximum and minimum principles"

$$\inf_{\partial B} u^i \leq \inf_{B} u^i \leq \sup_{B} u^i \leq \sup_{\partial B} u^i.$$

- $\mathbf{u} \in L^\infty_{\text{loc}}$.
- $\mathbf{u}$ is continuous $\mathcal{H}^{n-p}$-a.e.
- $\mathbf{u}$ is differentiable a.e.
- $\mathbf{u}$ satisfies Lusin's (N) condition.

Local invertibility

Definition: $\mathbf{u} \in \mathcal{A}_p$, $p > n-1$. For each ball $B \subset \Omega$ for which $\mathbf{u}|_{\partial B}$ is continuous, define

$$\mathrm{im}_{\mathrm{l}}(\mathbf{u}, B) := \{\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{u}(\partial B) : \deg(\mathbf{u}, B, \mathbf{y}) = 1\},$$

$$\mathrm{im}_{\mathrm{l}}(\mathbf{u}, \Omega) := \bigcup_B \mathrm{im}_{\mathrm{l}}(\mathbf{u}, B) \qquad \text{(open sets).}$$

Theorem: Let $\mathbf{u} \in \mathcal{A}_p$, $p > n-1$. Then for a.e. $\mathbf{x} \in \Omega$ there exists $r_{\mathbf{x}} > 0$ such that $B := B(\mathbf{x}, r_{\mathbf{x}})$ satisfies

a) $\mathbf{u}|_B$ is injective a.e. and $\mathbf{u}(B) = \mathrm{im}_{\mathrm{l}}(\mathbf{u}, B)$ a.e.

b) $(\mathbf{u}|_B)^{-1} \in W^{1,1}(\mathrm{im}_{\mathrm{l}}(\mathbf{u}, B), \mathbb{R}^n)$ and
$D(\mathbf{u}|_B)^{-1} = \left(D\mathbf{u} \circ (\mathbf{u}|_B)^{-1}\right)^{-1}$.

c) $\mathbf{u}(\Omega) = \mathrm{im}_{\mathrm{l}}(\mathbf{u}, \Omega)$ a.e.

Thus, the class of

$$\det D\mathbf{u} \in L^1_{\mathrm{loc}}(\Omega), \quad \det D\mathbf{u} > 0 \ \text{ and } \ \mathcal{E}(\mathbf{u}) = 0$$

is (probably) the right analogue of 'orientation-preserving local homeomorphism', in $W^{1,p}$, $p > n-1$.

Stability of the inverses

Theorem. Let $\mathbf{u}_j, \mathbf{u} \in \mathcal{A}_p$ satisfy $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,p}$. Let $B \subset \Omega$ be such that $\mathbf{u}|_B$ satisfies INV and take $V \subset\subset \mathrm{im}_{\mathrm{l}}(\mathbf{u}, B)$. Then for $j \geq j_0$, $\mathbf{u}_j|_B$ satisfies INV, $V \subset \mathrm{im}_{\mathrm{l}}(\mathbf{u}_j, B)$ and

$$
\begin{aligned}
&(\mathbf{u}_j|_B)^{-1} \rightharpoonup (\mathbf{u}|_B)^{-1} \quad \text{in } W^{1,1}(V, \mathbb{R}^n),\\
&\operatorname{adj} D(\mathbf{u}_j|_B)^{-1} \rightharpoonup \operatorname{adj} D(\mathbf{u}|_B)^{-1} \quad \text{in } L^1(V, \mathbb{R}^{n\times n}),\\
&\det D(\mathbf{u}_j|_B)^{-1} \rightharpoonup \det D(\mathbf{u}|_B)^{-1} \quad \text{in } L^1(V),\\
&\chi_{\mathbf{u}_j(\Omega)} \to \chi_{\mathbf{u}(\Omega)} \quad \text{in } L^1(\mathbb{R}^n).
\end{aligned}
$$

Applications

Existence of minimizers for models that

- involve reference Ω and deformed $\mathbf{u}(\Omega)$ configurations.
- involve a composition of maps $\mathbf{u} \circ \mathbf{n}$.

Examples: nematic elastomers and magnetoelasticity.

Nematic elastomers

(cf. M. Barchiesi & A. DeSimone 15 and C. Calderer, C. Garavito & B. Yan 15)

$W : \mathbb{R}^{n\times n}_{+} \to [0,\infty)$ polyconvex, $p > n-1$

$$W(\mathbf{F}) \geq |\mathbf{F}|^p + h(\det \mathbf{F}) \quad \text{with } \lim_{t\searrow 0} h(t) = \lim_{t\to\infty} \frac{h(t)}{t} = \infty.$$

$\mathbf{V_n} := \alpha\, \mathbf{n}\otimes\mathbf{n} + \alpha^{\frac{1}{1-n}}\,(\mathbf{1} - \mathbf{n}\otimes\mathbf{n})$ stretch in direction $\mathbf{n} \in \mathbb{S}^{n-1}$.

Mechanical response $W_{\text{mec}}(\mathbf{F},\mathbf{n}) := W(\mathbf{V_n}^{-1}\,\mathbf{F})$.

Energy

$$I(\mathbf{u},\mathbf{n}) := \int_\Omega W_{\text{mec}}(D\mathbf{u}(\mathbf{x}), \mathbf{n}(\mathbf{u}(\mathbf{x}))) \,\mathrm{d}\mathbf{x} + \int_{\mathbf{u}(\Omega)} |D\mathbf{n}(\mathbf{y})|^2 \,\mathrm{d}\mathbf{y}.$$

Then there exists a minimizer $\mathbf{u} \in \mathcal{A}_p$, $\mathbf{n} \in W^{1,2}(\mathbf{u}(\Omega), \mathbb{S}^{n-1})$ of I.

Proof, part 1: Deal with $\int_{\mathbf{u}(\Omega)} |D\mathbf{n}(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y}$.

Take $B \subset \Omega$ where $\mathbf{u}, \mathbf{u}_j$ are invertible. Take $V \subset\subset \mathrm{im}_{\mathrm{I}}(\mathbf{u}, B)$.

Then $V \subset \mathrm{im}_{\mathrm{I}}(\mathbf{u}_j, B)$ and $\mathbf{n}_j \rightharpoonup \mathbf{n}$ in $W^{1,2}(V)$. Covering argument

$$\chi_{\mathrm{im}_{\mathrm{I}}(\mathbf{u}_j,\Omega)} D\mathbf{n}_j \rightharpoonup \chi_{\mathrm{im}_{\mathrm{I}}(\mathbf{u},\Omega)} D\mathbf{n} \quad \text{in } L^2(\mathbb{R}^n),$$

so

$$\int_{\mathbf{u}(\Omega)} |D\mathbf{n}(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y} \leq \liminf_{j\to\infty} \int_{\mathbf{u}_j(\Omega)} |D\mathbf{n}_j(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y}.$$

Proof, part 2: Deal with $\int_\Omega W(D\mathbf{u}(\mathbf{x}), \mathbf{n}(\mathbf{u}(\mathbf{x}))) \, \mathrm{d}\mathbf{x}$.

Minimizing sequence, coercivity: $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,p}(\Omega, \mathbb{R}^n)$,

$\mathbf{n}_j$ bdd in $W^{1,2}(\mathbf{u}_j(\Omega), \mathbb{R}^n)$. We saw $\mathbf{n}_j \rightharpoonup \mathbf{n}$, but $\mathbf{n}_j \circ \mathbf{u}_j \rightharpoonup \mathbf{n} \circ \mathbf{u}$?

Solution: Take $B \subset \Omega$ where $\mathbf{u}, \mathbf{u}_j$ are invertible. Change variables

$$\int_B W(D\mathbf{u}(\mathbf{x}), \mathbf{n}(\mathbf{u}(\mathbf{x}))) \, \mathrm{d}\mathbf{x}$$
$$= \int_{\mathbf{u}(B)} W(D(\mathbf{u}|_B)^{-1}(\mathbf{y})^{-1}, \mathbf{n}(\mathbf{y})) \det D(\mathbf{u}|_B)^{-1}(\mathbf{y}) \, \mathrm{d}\mathbf{y}$$

(J. Ball 77): $W(\mathbf{F})$ polyconvex $\Rightarrow$ $W(\mathbf{F}^{-1}) \det \mathbf{F}$ polyconvex, hence

$$\int_B W(D\mathbf{u}(\mathbf{x}), \mathbf{n}(\mathbf{u}(\mathbf{x}))) \, \mathrm{d}\mathbf{x} \leq \liminf_{j \to \infty} \int_B W(D\mathbf{u}_j(\mathbf{x}), \mathbf{n}_j(\mathbf{u}_j(\mathbf{x}))) \, \mathrm{d}\mathbf{x}$$

A covering argument $\Omega = \bigcup_B B$ concludes.