On the stability of filament flows and Schrödinger maps

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binormal curvature flow

The equation for binormal curvature flow is:

$$\boxed{\gamma_t = \gamma_s \times \gamma_{ss}} \quad (1)$$

for $\gamma : \mathbb{R} \times (\mathbb{R}/L\mathbb{Z}) \to \mathbb{R}^3$, $t \in \mathbb{R}$ is the time variable, $L =$length, $s \in \mathbb{R}/L\mathbb{Z}$ is assumed to be an arc-length parameter, i.e.

$$|\partial_s \gamma(t, s)|^2 = 1. \quad (2)$$

Equivalently, we can write (1) as

$$\boxed{\partial_t \gamma = \kappa b}$$

where $\kappa =$ curvature, $b =$binormal vector. (see numerical example.) This

- is thought to approximate vortex filament motion in certain fluids. (see *e.g.* simulations of Barenghi, Hanninen, Tsubota, Quantum Fluids Group, University of Newcastle.)
- can be thought of as a “Schrödinger equation for curves”.
- In fact is a Hamiltonian flow for curves, with Hamiltonian = arclength.

History

Starting point: 3d Euler equations for incompressible fluid:

$$\partial_t u + u \cdot \nabla u + \nabla p = 0 \qquad u : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3, \quad \nabla \cdot u = 0.$$

with $p : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$ the pressure.

The vorticity is defined to be

$$\omega = \nabla \times u$$

and satisfies

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u.$$

Conversely, the velocity u can be recovered from ω via

$$u(t, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(y - x) \times \omega}{|y - x|^2} \, dy.$$

- study of motion of vortex filaments in ideal fluids initiated by Helmholz. (1858, Crelle).
  - gave basic definitions and description.
  - noted that vortex filaments are convected by the flow.
  - described translating vortex rings
- stability of vortex rings studied by Kelvin (1967, 1880).
- Kelvin also developed a theory of vortex atoms, a sort of precursor of string theory.
- formal derivation of binormal curvature flow given by Da Rios (1906).
  - work was ignored, except by his advisor, Levi-Civita.
  - Levi-Civita returned to the question in 1932, extended some of da Rios' results. This too was mostly ignored.
  - subsequently rediscovered several times. Recently extended to $\mathbb{R}^n$ (Shashikanth (2011), Khesin (2012).)
  - rigorous justification still completely open.
- binormal curvature flow also believed to describe vortex filaments in superfluids. Fetter (1966) rigorous justification also open.

three cousins: some related equations

The binormal curvature flow is closely related to the following:

- Schrödinger maps from $S^1 \to S^2$:

$$\boxed{u_t = u \times u_{ss}, \qquad u : (0,T) \times S^1 \to S^2.} \tag{3}$$

- cubic nonlinear Schrödinger equation:

$$\boxed{i\partial_t \psi + \partial_{ss}\psi + |\psi|^2\psi = 0, \qquad \psi : (0,T) \times S^1 \to \mathbb{C}} \tag{4}$$

Indeed,

- if $\gamma : (0,T) \times S^1 \to \mathbb{R}^3$ is a binormal curvature flow (parametrized by arclength) then $u := \gamma_s$ solves (3).
- The remarkable Hasimoto transform: (Hasimoto 1970) Given a binormal curvature flow, if we define

$$\psi(t,s) = \kappa(t,s) \exp(i \int_{s_0}^{s} T(t,\sigma) d\sigma)$$

where κ =curvature and T =torsion, then ψ solves (4).

Remarks

- more tools for nonlinear Schrödinger equation than for its two cousins
  - semilinear structure
  - harmonic analysis techniques: multilinear estimates, Strichartz estimates etc....
- BCF, SM often studied by transforming to NLS. this leads to:
  - well-posedness results. (Banica-Vega, Rodnianski-Rubinstein-Staffilani ...)
  - solitons on vortex filaments (Hasimoto 1970)
  - infinite number of conserved quantities: arclength, $\int \kappa^2$, ...
  - more generally, the binormal curvature flow is in some sense integrable.
- NLS, SM never studied by transforming to BCF.

Interpretation:

Chang-Shatah-Uhlenbeck (2000): Hasimoto transform can be seen as choice of gauge on tangent bundle to S^2.

This leads to some generalizations of Hasimoto transform, useful for Schrödinger maps.

Our goal: a geometric measure theory approach to binormal curvature flow.

Motivations:

- allows changes of topology

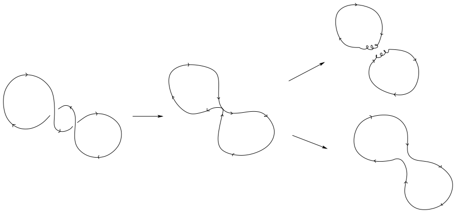

Figure: Non unique evolution through strands recombination and singularity formation.

- thus possibly useful for studying limiting vortex filament dynamics in ideal fluids. (see *e.g.* simulation of R. Tebbs, Quantum Fluids Group, University of Newcastle.)
- turns out to yield strong new stability properties.
- new insight into irregular or oscillatory vortex filaments.

Toward weak solutions

Basic idea:

- view curve evolving by binormal curvature flow as a distribution acting on test functions.
- similar in spirit to geometric measure theory formulations of minimal surfaces (varifolds) and motion by mean curvature.
- basic calculation: If $\gamma : \mathbb{R} \times S^1 \to \mathbb{R}^3$ is a smooth binormal curvature flow parametrized by arclength, then

$$\frac{d}{dt}\int_{S^1} (\phi \circ \gamma) \cdot \gamma_s \, ds \;=\; \int_{S^1} [D(\nabla \times \phi) \circ \gamma] : (\gamma_s \otimes \gamma_s) \, ds \tag{5}$$

  for all $\phi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$.
- we will in effect take this as the definition of a weak binormal curvature flow.
- formally, to give a meaning to (5), we only need to know position and tangent to curve γ.
- no second derivatives needed.

Towards weak solutions

A class of generalized curves that possess (only) position and tangents.

- Given any Lipschitz curve $\gamma : S^1 \to \mathbb{R}^3$, we can define an associated measure V_γ by

$$\int_{\mathbb{R}^3 \times S^2} f(x,\xi) dV_\gamma = \int_{S^1} f(\gamma(s), \frac{\gamma'(s)}{|\gamma'(s)|})|\gamma'(s)| \, ds$$

- For a measure V_γ associated as above to a closed Lipchitz curve,

$$\int_{\mathbb{R}^3 \times S^2} \nabla g(x) \cdot \xi \, V_\gamma(dx, d\xi) = 0 \quad \text{for } g \in C_c^\infty(\mathbb{R}^3) \tag{6}$$

Proof: integration by parts. We interpret this as: "V_γ has no boundary."

- More generally, we can view any measure V on $\mathbb{R}^3 \times S^2$ as a generalized curve. If $A \subset \mathbb{R}^3$, $B \subset S^2$, then we interpret

$$V(A \times B) := \mathcal{H}^1\Big(\{x \in A : \text{ tangent at } x \text{ belongs to } B\}\Big).$$

If (6) holds then we say that V has no boundary.

definition of weak binormal curvature flow

Definition

Let I be an interval of $\mathbb{R}$. A family of measures $(V_t)_{t\in I}$ on $\mathbb{R}^3 \times S^2$ is a *weak binormal curvature flow* of finite mass if

1. V_t has no boundary for a.e. t in the sense of (6)
2. The map $t \mapsto V_t(\mathbb{R}^3 \times S^2)$ is finite and non-increasing on I.
3. For every $\phi \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ the map $t \mapsto \int_{\mathbb{R}^3\times S^2} \phi \cdot \xi \; V_t(dx, d\xi)$ is Lipschitz on I and for a.e. $t \in I$,

$$\boxed{\frac{d}{dt}\int_{\mathbb{R}^3\times S^2} \phi \cdot \xi \; V_t(dx, d\xi) = -\int_{\mathbb{R}^3\times S^2} D(\mathrm{curl}\phi) : (\xi \otimes \xi) \;\; V_t(dx, d\xi).}$$

remarks

Facts stated earlier immediately imply that

Lemma

If $\gamma : I \times S^1 \to \mathbb{R}^3$ *is a smooth binormal curvature flow, then the family of associated measures* $(V_{\gamma(t,\cdot)})_{t\in I}$ *is a weak binormal curvature flow.*

- the definition of weak solutions is *linear* with respect to V
- weak flows may be useful for describing limits of sequences of solutions with rapid oscillatiions.
- it is easy to construct examples of weak solutions exhibiting change of topology.

Existence and (failure of) uniqueness

- Existence: For any lipschitz $\gamma^0 : S^1 \to \mathbb{R}^3$, there exists a weak solution $(V_t)_{t\in\mathbb{R}}$ such that
  - $\mathrm{wklim}_{t\to 0} V_t = V_{\gamma^0}$
  - for $\Gamma_t(\phi) := \int_{S^2} \phi(x) \cdot \xi V_t(dx, d\xi)$,
    - $t \mapsto \Gamma_t$ is continuous in suitable weak norms
    - Γ_t is an integer multiplicity rectifiable 1-current for every t
  - $V_t(\mathbb{R}^3 \times S^2) = |\gamma^0|$ for all t.

- many sources/examples of nonuniqueness:
  - The definition of weak solution only involves $\int_{S^2} \xi V_t(\cdot, d\xi)$ and $\int_{S^2} \xi \otimes \xi V_t(\cdot, d\xi)$.
  - collisions and self-intersections.
  - definition of weak solution does not constrain evolution of second moments.
  - no possibility of uniqueness if mass is allowed to increase.

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results: weak-strong uniqueness

Nonetheless, we have the following

Theorem (J-Smets, 2012)

Let $\gamma \in C([0,T], W^{3,\infty}(S^1, \mathbb{R}^3))$ be a smooth binormal curvature flow without self-intersection.

Let $(V_t)_{t\in[0,T]}$ be a weak binormal curvature flow.

If $V_0 = V_{\gamma(0,\cdot)}$, then $V_t = V_{\gamma(t,\cdot)}$ for all t in $[0,T]$.

1. proof also shows that a WBCF that starts near a smooth flow remains close for some time. This can be seen in numerics if we believe that a simulation of (1) with rough data corresponds to a WBCF.
2. self-intersection should *always* lead to nonuniquenes.
3. uniqueness fails if $t \mapsto V_t(\mathbb{R}^3 \times S^2)$ is allowed to increase.
4. regularity here is much weaker than in Banica-Vega work.

results: orbital continuity for Schrödinger maps

Theorem (J-Smets, 2012)

Let $u \in C(I, H^3(T^1, S^2))$ be a solution of the Schrödinger map equation

$$u_t = u \times u_{ss} \tag{7}$$

on $I = (-T, T)$ for some $T > 0$. Given any other solution $v \in L^\infty(I, H^{1/2}(T^1, S^2))$ of (7), there exists a continuous function $\sigma : I \to T^1$ such that for every $t \in I$,

$$\|v(t, \cdot) - u(t, \cdot + \sigma(t))\|_{L^2(T^1, \mathbb{R}^3)} \leqslant C\|v(0, \cdot) - u(0, \cdot)\|_{L^2(T^1, \mathbb{R}^3)},$$

where the constant $C \equiv C(\|\partial_{sss} u(0, \cdot)\|_{L^2}, T)$, in particular C does not depend on v.

- we do what never works: study SM1 by transforming to BCF.
- Note that here there is no “non-self-intersection” condition.
- the critical Sobolev space is $H^{1/2}$.

sharpness of the above

The above theorem is *not* true without the translation $\sigma(t)$:

Theorem (J-Smets, 2012)

For any given $t \neq 0$*, the flow map for* (7) *at time t is not continuous as a map from* $\mathcal{C}^\infty(T^1, S^2)$*, equipped with the weak topology of* $H^{1/2}$*, to the space of distributions* $(\mathcal{C}^\infty(T^1, \mathbb{R}^3))^*$*. Indeed, for any* $\sigma_0 \in \mathbb{R}$ *there exist a sequence of smooth initial data* $(u_{m,\sigma_0}(0,\cdot))_{m\in\mathbb{N}} \in \mathcal{C}^\infty(T^1, S^2)$ *such that*

$$u_{m,\sigma_0}(0,\cdot) \rightharpoonup u^*(0,\cdot) \qquad \text{in } H^{\frac{1}{2}}_{\text{weak}}(T^1, \mathbb{R}^3),$$

where $u^*(0,s) := (\cos(s), \sin(s), 0)$ *is a stationary solution of* (7)*, and for any* $t \in \mathbb{R}$

$$u_{m,\sigma_0}(t,\cdot) \rightharpoonup u^*(t, \cdot + \sigma_0 t) \qquad \text{in } H^{\frac{1}{2}}_{\text{weak}}(T^1, \mathbb{R}^3).$$

key idea: relative entropy

Given smooth curve $\gamma(s)$, let r_γ :=injectivity radius, and

$$P(x) := \text{nearest point projection onto } \text{Image}(\gamma),$$

when $\text{dist}(x, \gamma) < r_\gamma$;

$$\phi_\gamma(x) := f(\text{dist}(x, \gamma(\cdot)))\, \gamma_s(P(x)).$$

where f is smooth, $0 \leqslant f \leqslant 1$, $f' \leqslant 0$, and

$$f(r) := 1 - r^2 \text{ for } r \leqslant r_\gamma/2, \qquad f(r) = 0 \text{ if } r \geqslant r_\gamma.$$

Finally, define

$$\boxed{E(V;\gamma) := \int_{\mathbb{R}^3 \times S^2} (1 - \phi_\gamma(x) \cdot \xi)\, V(dx, d\xi)}$$

Heuristically,

$$E(V;\gamma) \approx \text{“}H^1 \text{ norm of } V \text{ with respect to } \gamma.\text{"}$$

In fact, near γ

$$\begin{aligned} E(V;\gamma) &= \int (1 - \xi\cdot\tau_\gamma) + \int (1 - f(\text{dist}_\gamma))(\xi\cdot\tau_\gamma) \\ &= \int \frac{1}{2}|\xi - \tau_\gamma|^2 + \int \frac{1}{2}(\text{dist}_\gamma)^2(\xi\cdot\tau_\gamma) \end{aligned}$$

Thus for V supported near γ,

$$\boxed{E(V;\gamma) \leqslant C\int \min\left(1, |\xi - \tau_\gamma|^2 + (\text{dist}_\gamma)^2(\xi\cdot\tau_\gamma)\right) dV.}$$

This leads to

Lemma

*If **V** has no boundary and $E(V;\gamma) = 0$, then $V = cV_\gamma$ for some c.*

main estimate

Key Fact:
Under the hypotheses of Weak-Strong Uniqueness Theorem,

$$\boxed{\frac{d}{dt}E(V_t;\gamma_t) \leqslant KE(V_t;\gamma_t)}$$

for almost every $t \in [0, T]$, where K is a constant depending only on r_γ and $\|\gamma\|_{L^\infty(I, W^{3,\infty}(S^1))}$.

If we know that V is parametrized by a single Lipschitz curve, we can modify the definition of E, exploiting the parametrization to handle self-intersection of γ. A suitably modified version of the key fact still holds and is used for the Schrödinger map “orbital continuity" theorem.

The proof of the key fact is a computation.....
it uses the equation twice:

- the weak form of the equation for V_γ,
- the strong form of the equation to deduce properties of the vector field ϕ_γ

rapidly oscillating data

Given a smooth curve $\gamma_0 : S^1 \to \mathbb{R}^3$, define

$$\gamma_{k,\alpha}(s) = \gamma_0(s) + \frac{\alpha}{k}\Big(\cos(ks)n_0(s) + \sin(ks)b_0(s)\Big)$$

where $n_0(s)$ and $b_0(s)$ are the normal and binormal to γ_0 at s.

Question: behaviour of solution of binormal curvature flow with initial data γ_0?

We do not know.....

- One possibility: extreme instability, due to very large curvature.
- Our framework suggests another possibility:
  - the measures $V_{k,\alpha}$ associated to $\gamma_{k,\alpha}$ converge as $k \to \infty$ to a limiting measure V_{lim} supported on a set of the form

$$\{(x,\xi) \in \mathbb{R}^3 \times S^2 : x \in \text{Image}(\gamma_0), \quad \xi \cdot \partial_s \gamma_0 = (1+\alpha^2)^{-1/2}\}$$

  - one can find an explicit weak binormal curvature (V_t^*) flow with data V_{lim}.
  - V_t^* is supported on

$$\{(x,\xi) \in \mathbb{R}^3 \times S^2 : x \in \text{Image}(\gamma(\beta t,\cdot)), \quad \xi \cdot \partial_s \gamma_0 = (1+\alpha^2)^{-1/2}\}$$

  where γ solves binormal curvature flow with initial data γ_0 and

$$\beta = \frac{2-\alpha^2}{2\sqrt{1+\alpha^2}} \in (-\infty, 1).$$

  - For this solution, oscillations slow (or reverse!) flow of time, compared to smooth solution.
  - This is consistent with numerics. Does it in fact describe sequences of smooth solutions?

Numerics

How would curves with corners evolve via the binormal curvature flow?

or stated differently:

How would maps with jump discontinuities evolve as Schrödinger maps?

or maybe:

How would data with Dirac delta functions evolve via cubic NLS?

All these questions should be harder for closed curves than for non compact curves.

Thank you for your attention!