New regularity estimates for transport equations

P-E Jabin

Compressible Fluid dynamics

The compressible Navier-Stokes system reads

$$\partial_t \rho + \mathrm{div}\,(u\,\rho) = 0,$$
$$\partial_t(\rho\, u) + \mathrm{div}\,(\rho\, u \otimes u) - \Delta u = -\nabla p(\rho)$$

for some barotropic pressure term $p(\rho) \sim \rho^\gamma$, in a bounded domain $\Omega \subset \mathbb{R}^d$ with for instance Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0.$$

Compressible Fluid dynamics

The compressible Navier-Stokes system reads

$$\partial_t \rho + \mathrm{div}\,(u\,\rho) = 0,$$
$$\partial_t(\rho\,u) + \mathrm{div}\,(\rho\,u \otimes u) - \mu\Delta u + \lambda\nabla \mathrm{div}\,u = -\nabla p(\rho)$$

for some barotropic pressure term $p(\rho) \sim \rho^\gamma$

Variants of course exist. Some mostly fit within the same theory:

- With more physical diffusion

Compressible Fluid dynamics

The compressible Navier-Stokes system reads

$$\partial_t \rho + \operatorname{div}(u\,\rho) = 0,$$
$$\partial_t(\rho\, u) + \operatorname{div}(\rho\, u \otimes u) - \operatorname{div}(\mu(\theta)\, D\, u) + \nabla(\lambda(\theta) \operatorname{div} u) = -\nabla p(\rho, \theta)$$
$$\partial_t(\rho\theta) + \operatorname{div}(\rho\, u\, \theta) + R\, \rho\theta \operatorname{div} u = \mu\, |D\, u|^2 + \lambda |\operatorname{div} u|^2 + \operatorname{div}(\kappa(\theta)\, \nabla\theta)$$

for some barotropic pressure term $p(\rho) \sim \rho^\gamma$

Variants of course exist. Some mostly fit within the same theory:

- With more physical diffusion
- With temperature

Compressible Fluid dynamics

The compressible Navier-Stokes system reads

$$\partial_t \rho + \mathrm{div}\,(u\,\rho) = 0,$$
$$\partial_t(\rho\,u) + \mathrm{div}\,(\rho\,u \otimes u) - \mathrm{div}\,(\mu(\rho)\,D\,u) + \nabla(\lambda(\rho)\,\mathrm{div}\,u) = -\nabla p(\rho)$$

for some barotropic pressure term $p(\rho) \sim \rho^\gamma$

Variants of course exist. Some mostly fit within the same theory:

- With more physical diffusion
- With temperature

or are inaccessible

- With density dependent viscosity (see nevertheless Bresch-Desjardins)

Compressible Fluid dynamics

The compressible Navier-Stokes system reads

$$\partial_t \rho + \mathrm{div}\,(u\,\rho) = 0,$$
$$\partial_t(\rho\,u) + \mathrm{div}\,(\rho\,u\otimes u) - \mathrm{div}\,(A(x)\,D\,u) + \nabla(A(x)\,\mathrm{div}\,u) = -\nabla p(\rho)$$

for some barotropic pressure term $p(\rho)\sim\rho^\gamma$

Variants of course exist. Some mostly fit within the same theory:

- With more physical diffusion
- With temperature

or are inaccessible

- With density dependent viscosity (see nevertheless Bresch-Desjardins)
- With non homogeneous viscosity given by a matrix A, or non local $p(\rho)$.

Compressible Fluid dynamics

The compressible Navier-Stokes system reads

$$\begin{aligned} &\partial_t \rho + \operatorname{div}(u\,\rho) = 0, \\ &\partial_t(\rho\, u) + \operatorname{div}(\rho\, u \otimes u) - \Delta u = -\nabla p(\rho) \end{aligned}$$

for some barotropic pressure term $p(\rho) \sim \rho^\gamma$
For simplicity this talk deals only with the first system.

Compressible Fluid dynamics

The compressible Navier-Stokes system reads

$$\partial_t \rho + \mathrm{div}\,(u\,\rho) = 0,$$
$$\partial_t(\rho\, u) + \mathrm{div}\,(\rho\, u \otimes u) - \Delta u = \; - \nabla p(\rho)$$

for some barotropic pressure term $p(\rho) \sim \rho^\gamma$

Goal: Revisit the classical compactness theory by obtaining quantitative regularity estimates.

Other similar models

The same theory applies to many other models, for instance

$$\partial_t \rho + \mathrm{div}\,(u\,\rho) = 0,$$
$$-\Delta u = -\nabla p(\rho) + S.$$

In some applications to biology, $u = \nabla c$ with c the concentration of some chemical (or a sum of chemicals) used by the biological agents to interact.
Therefore, in some cases, $p(\rho)$ should include repulsive and attractive interactions.

State of the art: A priori estimates

Let us describe the available theory as developed first by P.L. Lions and extended by E. Feireisl. Start with the a priori estimates
Conservation of mass

$$\int \rho(t,x)\,dx = const.$$

Energy estimate For $P(\rho)$ s.t. $P'\,\rho - P = p(\rho)$, *i.e.* $P(\rho) \sim \rho^\gamma$

$$\int \left(P(\rho(t,x)) + \frac{1}{2}\,\rho\,u^2\right)\,dx + \int_0^t \int |\nabla u|^2 = const.$$

State of the art: A priori estimates

Let us describe the available theory as developed first by P.L. Lions and extended by E. Feireisl. Start with the a priori estimates

Conservation of mass

$$\int \rho(t,x)\,dx = const.$$

Energy estimate For $P(\rho)$ s.t. $P'\,\rho - P = p(\rho)$, *i.e.* $P(\rho) \sim \rho^\gamma$

$$\int \left(P(\rho(t,x)) + \frac{1}{2}\,\rho\,u^2\right)\,dx + \int_0^t \int |\nabla u|^2 = const.$$

Pressure estimates

$$\int_0^t \int \rho^\theta\,p(\rho)\,dx\,dt \leq C, \qquad \theta < \frac{2}{d}\,\gamma.$$

Compactness of ρ

Take a sequence ρ_k, u_k of (approximate) solutions. u_k is compact and the only difficulty is the Compactness of ρ_k.

- P.L. Lions: Show that $w - \lim \rho_k^2 = A = (w - lim \rho_k)^2 = \rho^2$.

$$\partial_t \rho_k^2 + \mathrm{div}\,(u_k\,\rho_k^2) = -\mathrm{div}\,u_k\,\rho_k^2.$$

But

$$\mathrm{div}\,u_k = p(\rho_k) + \Delta^{-1}(\partial_t(\rho_k u_k) + \mathrm{div}\,(\rho_k\,u_k \otimes u_k)).$$

So

$$w - \lim \rho_k^2\,\mathrm{div}\,u_k \geq \rho B + A\,\Delta^{-1}(\partial_t(\rho u) + \mathrm{div}\,(\rho\,u \otimes u)),$$

where $B = w - lim \rho\, p(\rho)$.

Compactness of ρ

Take a sequence ρ_k, u_k of (approximate) solutions. u_k is compact and the only difficulty is the Compactness of ρ_k.

• P.L. Lions: Show that $w-\lim \rho_k^2 = A = (w-lim\rho_k)^2 = \rho^2$.

Hence

$$\partial_t A + \mathrm{div}\,(u\,A) \leq -\rho\,B - A\,\Delta^{-1}(\partial_t(\rho u) + \mathrm{div}\,(\rho\,u \otimes u)).$$

while recalling $B = w - lim\rho\,p(\rho)$

$$\partial_t \rho^2 + \mathrm{div}\,(u\,\rho^2) = -\rho\,B - \rho^2\,\Delta^{-1}(\partial_t(\rho u) + \mathrm{div}\,(\rho\,u \otimes u)).$$

Thus $A \leq \rho^2$ and then $A = \rho^2$.

Compactness of ρ

Take a sequence ρ_k, u_k of (approximate) solutions. u_k is compact and the only difficulty is the Compactness of ρ_k.

• P.L. Lions: Show that $w - \lim \rho_k^2 = A = (w - lim \rho_k)^2 = \rho^2$.

• Of course show instead $w - \lim \beta(\rho_k) = \beta(w - \lim \rho_k)$ for some convex β.

Compactness of ρ

Take a sequence ρ_k, u_k of (approximate) solutions. u_k is compact and the only difficulty is the Compactness of ρ_k.

- P.L. Lions: Show that $w - \lim \rho_k^2 = A = (w - lim \rho_k)^2 = \rho^2$.
- Of course show instead $w - \lim \beta(\rho_k) = \beta(w - \lim \rho_k)$ for some convex β.
- Only gives compactness at the limit: No regularity estimates on ρ_k.

Compactness of ρ

Take a sequence ρ_k, u_k of (approximate) solutions. u_k is compact and the only difficulty is the Compactness of ρ_k.

- P.L. Lions: Show that $w - \lim \rho_k^2 = A = (w - lim \rho_k)^2 = \rho^2$.
- Of course show instead $w - \lim \beta(\rho_k) = \beta(w - \lim \rho_k)$ for some convex β.
- Only gives compactness at the limit: No regularity estimates on ρ_k.
- The critical step is

$$w - \lim\ \rho_k^2\, p(\rho_k) \geq \rho\, w - \lim \rho_k\, p(\rho_k),$$

which requires p increasing.

Compactness of ρ

Take a sequence ρ_k, u_k of (approximate) solutions. u_k is compact and the only difficulty is the Compactness of ρ_k.

- P.L. Lions: Show that $w - \lim \rho_k^2 = A = (w - lim \rho_k)^2 = \rho^2$.
- Of course show instead $w - \lim \beta(\rho_k) = \beta(w - \lim \rho_k)$ for some convex β.
- Only gives compactness at the limit: No regularity estimates on ρ_k.
- The critical step is

$$w - \lim \; \rho_k^2 \, p(\rho_k) \geq \rho \, w - \lim \rho_k \, p(\rho_k),$$

which requires p increasing.

- Feireisl uses Riesz operators and does not need $\rho \operatorname{div} u$.

Existence of weak solutions

The previous method yields

Theorem *P.L. Lions*
Assume $p(\rho) \sim \rho^\gamma$ *with* $\gamma > 9/5$ *and p monotone.*
Then there exists a weak solution to compressible Navier-Stokes.

While with refined techniques

Theorem *E. Feireisl*
Assume $p(\rho) \sim \rho^\gamma$ *with* $\gamma > 3/2$ *and p monotone.*
Then there exists a weak solution to compressible Navier-Stokes.

The idea

Propagate some explicit regularity on ρ by computing

$$\int \frac{|\rho(x)-\rho(y)|^a}{(|x-y|+h)^k}\,dx\,dy,$$

for some $k>d$.

However this corresponds to a Sobolev like regularity on ρ which cannot work. So instead...

The idea

Propagate some explicit regularity on ρ by computing

$$\int \frac{|\rho(x)-\rho(y)|^a}{(|x-y|+h)^k} W(x,y)\, dx\, dy,$$

for some $k > d$.

Where the weight W solves the same transport equation

$$\partial_t W + u(t,x)\cdot\nabla_x W + u(t,y)\cdot\nabla_y W = -P,$$

for a well chosen penalization P.

The idea

Propagate some explicit regularity on ρ by computing

$$\int \frac{|\rho(x)-\rho(y)|^a}{(|x-y|+h)^k}\, W(x,y)\, dx\, dy,$$

for some $k > d$.

Where the weight W solves the same transport equation

$$\partial_t W + u(t,x)\cdot\nabla_x W + u(t,y)\cdot\nabla_y W = -P,$$

for a well chosen penalization P.

Then explain that W cannot be too small, too often to bound

$$\int \frac{|\rho(x)-\rho(y)|^a}{(|x-y|+h)^k}\, dx\, dy,$$

in terms of h.

A new result

The main improvements are: No monotonicity assumption on p, explicit regularity.

Theorem

Assume $p(\rho) \sim \rho^\gamma$ with $\gamma > 9/5$.
Then there exists a weak solution to compressible Navier-Stokes.
Moreover the solution satisfies for any $k > d$

$$\int_{\rho(x),\ \rho(y)>\eta} \frac{|\rho(x)-\rho(y)|^a}{(|x-y|+h)^k}\,dx\,dy \leq C_\eta\,\frac{h^{-(k-d)}}{|\log h|^\mu},$$

with $\mu = 1$ if $\gamma > 3$.

Extensions

The new theory should also be applied to

- Non homogeneous viscosity for some given matrix $A(t,x)$

$$\partial_t \rho + \mathrm{div}\,(u\,\rho) = 0,$$
$$-\,\mathrm{div}\,(A(t,x)\,\nabla u) = -\nabla p(\rho) + S.$$

- Well posedness for the linear kinetic equations

$$\partial_t f + v \cdot \nabla_x f + E(t,x) \cdot \nabla_v f = 0,$$

with only $E \in L^2_t H^{3/4}_x$.