CMU-Talk: Strong Maximum Principle for Reaction-Diffusion Systems

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<u>Setting</u>: $X \subseteq \mathbb{R}^n$ is open, connected

$\Omega := X \times [0,\infty)$

$u \in C^{2,1}(\Omega; \mathbb{R}^k) \cap C(\bar{\Omega}; \mathbb{R}^k)$ satisfies

(Simple) $\quad u_t = \underbrace{\Delta u}_{\text{Diffusion}} + \underbrace{\phi(u)}_{\text{Reaction}}$

(Full) $\quad u_t = D(x,t,u) \sum_{i,j} a_{ij}(x,t)\, u_{x_i x_j} + \sum_i M_i(x,t,u)\, u_{x_i} + \phi(x,t,u)$

"mixing matricies"

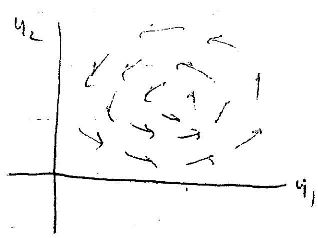
We will focus on the simple ~~equation~~ PDE

<u>One Interpretation/Example</u>: X = island (assume Neumann boundary conditions)

$u_1(x,t)$ = Deer population density at time t

$u_2(x,t)$ = Wolf population density at time t

Suppose $u \mapsto \phi(u)$ looks like



(Predator-Prey system)

Then for the PDE $u_t = \Delta u + \phi(u)$ we have the "Blob Picture"

L is image of X under mapping $x \mapsto u(x,t)$)

<u>Idea</u>: Blob <u>contracts</u> and moves <u>along vector</u> field

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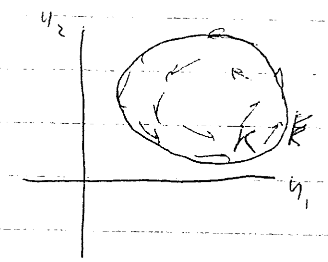
<u>Weak Maximum Principle</u>:

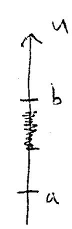
- Let $K \subset \mathbb{R}^k$ be closed, <u>convex</u>
- Let ϕ be <u>Lipschitz</u> and "<u>point inwards</u>" on ∂K



Then if $u \in K$ on $\partial\Omega \Rightarrow u \in K$ on all of Ω

(OR "a blob that starts in K will not leave K")

(<u>Note</u>: This generalizes usual 1-D weak maximum principle for $u_t = u_{xx}$)

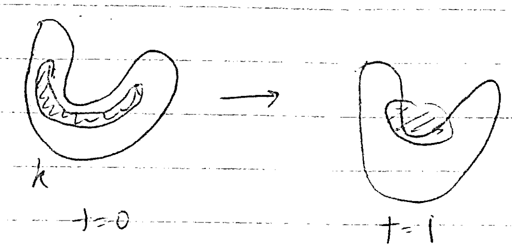


- Weak Max Principle for (Full) PDE is due to (1979) Chueh, Conley, and Smoller
  (need some regularity assumptions on Ω and u)

  (also some extra assumptions on K are needed which depend on the mixing matrices D and M if doing the (Full) PDE)

These conditions are needed:

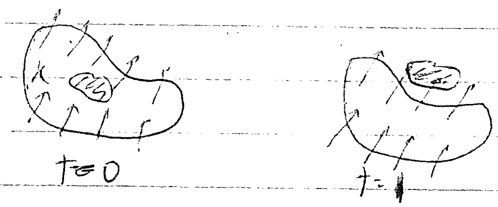
· If K is not convex:



Blob might contract outside the set

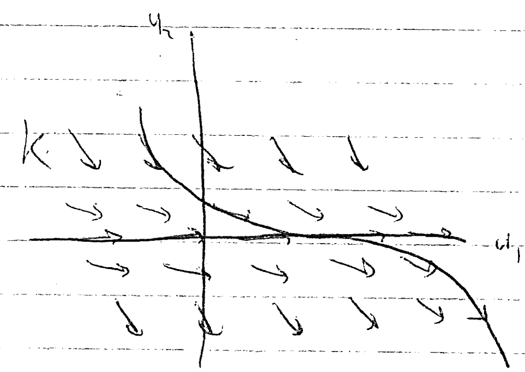
If Time

· If ϕ doesn't point inward:



· If ϕ is not Lipschitz: Consider $\phi(u) = \begin{pmatrix} 1 \\ -\sqrt{|u_2|} \end{pmatrix}$

and let $K = \{u : u_2 \geq 0\}$



← an integral curve of $\phi(u)$

So a "point-blob" traveling along this integral curve will exit K

Strong Maximum Principle:

$\boxed{\text{Assume } u(x,t) \in K \text{ for all } (x,t) \in \bar{\Omega}}$ and also K convex, ϕ is Lipschitz and points inward.

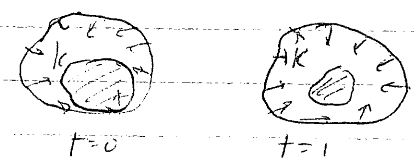
(if weak maximum principle holds we just need $u(x,t) \in K$ for $(x,t) \in \partial\Omega$)

Then $u(x_0,t_0) \in \partial K$ for $(x_0,t_0) \in \Omega$ (interior)

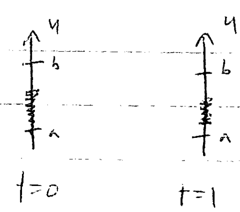
$\Longrightarrow u(x,t) \in \partial K$ for all $(x,t) \in \bar{\Omega}$

Interpretations:

1. The only way the blob is touching the bndry of K at some later time is if it was always on the bndry
2. "Blob will contract off the boundary"



Note: This is the usual Strong Max Principle in 1D



(1975) Weinberger proved this for (Simple) assuming ∂K satisfies "slab condition"

(1990) X. Wang proved this for (Full) assuming ∂K is piecewise C^2

(2010) I proved this for arbitrary convex K

Wang's Proof: Let $d(z)$ = dist from z to ∂K

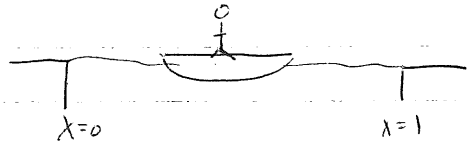
$\Rightarrow$ Apply 1D Strong Max Principle to $d(u(x,t))$

Needs ∂K to be C^2 to get regularity on $d(z)$

Idea: Extend this argument using viscosity solutions

(Very Rough) Review of Viscosity Solutions:

Example Problem:
(Man in Boat)



$u(x) =$ fuel needed to get to shore
(can travel 1 unit of distance per 1 unit of fuel)

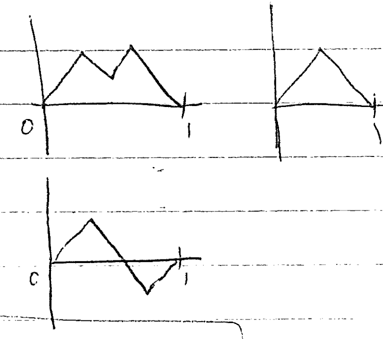
Dynamic Programming:

$$u(x) = \min\left(u(x+h),\, u(x-h)\right) + h$$

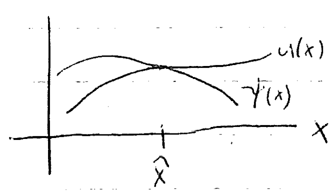
Rearrange, $h \to 0$

$$\begin{cases} |u'(x)| = 1 \\ u(0) = 0,\ u(1) = 0 \end{cases}$$

No classical solution. If we instead only ask that $|u'(x)| = 1$ a.e. get many solns

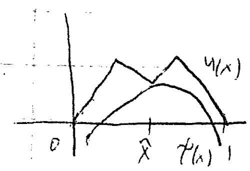


Defn: A continuous function $u(x)$ is a viscosity super solution to $F(x, u, \nabla u, \nabla^2 u) = 0$ if for each $\psi(x) \in C^\infty$, if ψ touches u from below at $\hat{x}$ then $F(\hat{x}, \psi(\hat{x}), \nabla\psi(\hat{x}), \nabla^2\psi(\hat{x})) \geq 0$



- Visc sub-soln defined analogously
- Visc soln if both sub & super soln

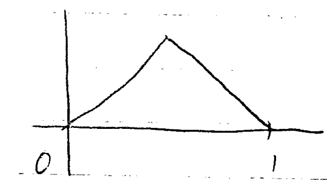
In our example, a viscosity solution cannot have "valleys"



$\Rightarrow |\psi'(\hat{x})| = 0 < 1$

$\Rightarrow$ BAD

$\Longrightarrow$ So



is the unique viscosity solution!

(In general for control problems the viscosity solution is the correct one)

Proof of SMP: $u_t = \Delta u + \phi(u)$

Know: $u(x,t) \in K$ for all $(x,t) \in \bar{\Omega}$

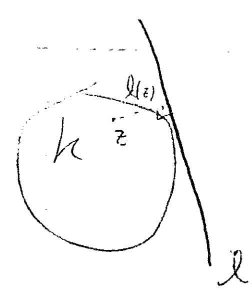
Let $\bar{d}(x,t) := d(u(x,t))$

$\boxed{\text{Claim: } \bar{d}_t - \Delta \bar{d} + C\bar{d} \geq 0 \text{ in viscosity sense}}$

Then we are done by the strong max principle for viscosity solutions (e.g. (2004) Da-Lio)

Pf of Claim: Consider a supporting hyperplane ℓ and let $\ell(z)$ = dist from z to ℓ

Let $\bar{\ell}(x,t) = \ell(u(x,t))$



Then $\bar{\ell}_t(x,t) = \nabla \ell \cdot u_t = \nabla \ell \cdot (\Delta u(x,t) + \phi(u(x,t)))$

$= \Delta \bar{\ell}(x,t) + \nabla \ell \cdot \phi(u(x,t))$

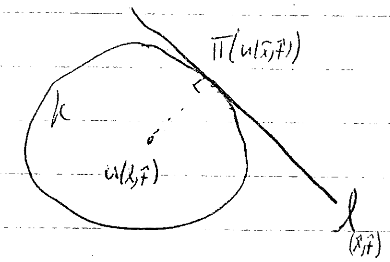
Holds for any supporting hyperplane ℓ!

possibly non-unique

Now, for a given point $(\hat{x},\hat{t})$,

there exists a hyperplane $l_{(\hat{x},\hat{t})}$ ~~such that~~ and a point $\Pi(u(\hat{x},\hat{t})) \in \partial K$

such that $d(u(\hat{x},\hat{t})) = l_{(\hat{x},\hat{t})}(u(\hat{x},\hat{t})) = |u(\hat{x},\hat{t}) - \Pi(u(\hat{x},\hat{t}))|$



So for this hyperplane,

...t a differential equality, ...ly true at exact point $(\hat{x},\hat{t})$

$$\partial_t \bar{l}_{(\hat{x},\hat{t})}(\hat{x},\hat{t}) = \Delta \bar{l}_{(\hat{x},\hat{t})}(\hat{x},\hat{t}) + \nabla l_{(\hat{x},\hat{t})} \cdot \phi(u(\hat{x},\hat{t}))$$

$$= \Delta \bar{l}_{(\hat{x},\hat{t})}(\hat{x},\hat{t}) + \;\;'' \;\; - \nabla l_{(\hat{x},\hat{t})} \cdot \phi(\Pi(u(\hat{x},\hat{t}))) + \underbrace{\nabla l_{(\hat{x},\hat{t})} \cdot \phi(\Pi(u(\hat{x},\hat{t})))}_{\geq 0}$$

$$\geq \Delta \bar{l}_{(\hat{x},\hat{t})}(\hat{x},\hat{t}) - C\,|\underbrace{u(\hat{x},\hat{t}) - \Pi(u(\hat{x},\hat{t}))}_{\bar{l}_{(\hat{x},\hat{t})}(\hat{x},\hat{t})}|$$

(b/c ϕ is inward pointing)

Now suppose Ψ touches $\bar{d}$ from below at $(\hat{x},\hat{t})$

then Ψ touches $\bar{l}_{(\hat{x},\hat{t})}$ from below at $(\hat{x},\hat{t})$ as well

$$\Rightarrow \partial_t \Psi(\hat{x},\hat{t}) = \partial_t \bar{l}_{(\hat{x},\hat{t})}(\hat{x},\hat{t}) \quad \text{AND} \quad \Delta\Psi(\hat{x},\hat{t}) \leq \Delta \bar{l}_{(\hat{x},\hat{t})}(\hat{x},\hat{t})$$

So $\Psi_t(\hat{x},\hat{t}) - \Delta\Psi(\hat{x},\hat{t}) + C\,\Psi(\hat{x},\hat{t}) \geq 0$

Since Ψ was arbitrary, $\bar{d}$ is a visc supersolution $\boxed{}$