Moving Interfaces that Separate Loose and Compact Phases of Elastic Aggregates:

A Mechanism for Drastic Increase or Reduction in Macroscopic Deformation

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This research was initiated during a CNA-supported visit of L. Deseri in Fall, 2010.

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- to identify compact and loose phases for a broad class of elastic aggregates
- to describe moving interfaces that separate loose and compact phases

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- Aggregate in compact phase is stiff (presence of friction between pieces)
- Interfaces between the phases are easily produced
- Loose phase often appears in narrow bands in constrained aggregates

Tools for modelling aggregates/granular materials such as sand:

- classical continuum mechanics: aggregate is a "(visco)-plastic body" (stability, flow)

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- continua with microstructure: aggregate is a continuum with additional fields reflecting submacroscopic structure

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Continuum mechanics based on multiscale geometry
Structured motions: $(g, G) : \mathcal{B} \times \mathbb{R} \longrightarrow \mathcal{E} \times Lin\mathcal{V}$

- $\mathcal{B} \subset \mathcal{E}$ ...reference configuration of the body

Approximation Theorem: (Del Piero & Owen-1993) For each t there exists $n \longmapsto f_n(\cdot, t)$ injective and piecewise smooth such that

$$\lim_{n \longrightarrow \infty} f_n(\cdot, t) = g(\cdot, t), \quad \lim_{n \longrightarrow \infty} \nabla f_n(\cdot, t) = G(\cdot, t).$$

Corollary: $M(\cdot, t)$ is a limit of averages of $[f_n](\cdot, t) \otimes \nu$.
$[f_n]$...jump in f_n
Choksi & Fonseca (1997)... SBV versions (without Accomodation Inequality and injectivity)

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- $M(\cdot, t) := \nabla g(\cdot, t) - G(\cdot, t)$ ... deformation due to disarrangements at time t (submacroscopic slips and void formation)

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(Classical) Elasticity: $\psi(X,t) = \Psi_c(\nabla g(X,t))$...Helmholtz free energy

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$\psi(X,t) = \Psi(G(X,t), M(X,t))$

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Elasticity with purely dissipative disarrangements

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$\leftrightarrow$"Disarrangements do not contribute to the energy stored in the body."

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NOTE: "$S = 0$" is a solution of (cr), (di), (fi)

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Elasticity with purely dissipative disarrangements

As above: $\psi(X,t) = \Psi(G(X,t))$

Assumptions on the Helmholtz free energy response Ψ:

- Ψ is of class C^2 on Lin^+ (smoothness)

Consequence of a Theorem of Mizel -1999: There exists $\zeta_{\min} > 0$ such that

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- $\lim_{|G| \longrightarrow \infty} \Psi(G) = \lim_{\det G \longrightarrow 0} \Psi(G) = +\infty$ ...growth

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g ... given macroscopic deformation with $\zeta_{\min}^3 \leq \det \nabla g$
Q ... any rotation-valued field
Loose phase for g **:** $(g, \zeta_{\min} Q)$

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Submacroscopic view via the Approximation Theorem: Each injective, piecewise smooth f_n (with $f_n \approx g$, $\nabla f_n \approx \zeta_{\min} Q$) divides $\mathcal{B}$ into non-overlapping pieces that undergo approximately

We may think of the body in the loose phase for g **as an aggregate that deforms according to** g **with pieces that deform according to** f_n**.**

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- pure dilatations of amount $\zeta_{\min}$,
- rotations corresponding to the field Q, and
- translations that do not cause interpenetration of the pieces.

We may think of the body in the loose phase for g as an aggregate that deforms according to g with pieces that deform according to f_n.

g .... given macroscopic deformation

Compact phase for g: $(g, \nabla g)$

- Whether or not the macroscopic deformation g satisfies $\zeta_{\min}^3 \leq \det \nabla g$, the classical deformation $(g, \nabla g)$ satisfies the Accomodation Inequality

Submacroscopic view of $(g, \nabla g)$ **via the Approximation Theorem**: Take $f_n = g$ for every n. Submacroscopic and macroscopic views agree at all stages of approximation.

We may think of the body in the compact phase for g **as an aggregate in which both the pieces and the aggregate deform according to** g**.**

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- $S = D_G \Psi(\nabla g) \neq 0$, in general.
- $M = \nabla g - \nabla g = 0$ in the compact phase: no submacroscopic slips or formation of voids arise via $(g, \nabla g)$.

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Compact phase for g: $(g, \nabla g)$...no disarrangements; available for every g

Field relations all are satisfied identically (with equality) except for:

- $\rho_0 \ddot{g} = \mathrm{div}(D_G \Psi(\nabla g)) + b$ ...balance of linear momentum

NOTE: $\dot{\psi} = S \cdot \nabla \dot{g}$ ...no internal dissipation.

Loose phase for g: $(g, \zeta_{\min} Q)$ ... energy minimizer; disarrangements; phase is available if volume change for g is large enough

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GOAL: To study moving interfaces that separate the loose and compact phases of elastic aggregates.

A first step:

- for special deformations in the compact phase, chosen so that shock waves are absent, identify planar interfaces $t = \hat{t}(X)$ in $\mathcal{B} \times \mathbb{R}$ that can separate the loose and compact phases

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- $(g, G) : \mathcal{B} \times \mathbb{R} \longrightarrow \mathcal{E} \times Lin\mathcal{V}$ ... structured motion

standard form

$$\rho_0 \ddot{g} = \operatorname{div} S + b$$
$$\dot{\varepsilon} = S \cdot \nabla \dot{g} - \operatorname{div} q + r$$
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divergence form $(\dot{b} = 0)$

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Constitutive assumptions:

- ψ, ε are functions of G, θ, and so therefore is $\eta = (\varepsilon - \psi)/\theta$

NOTE:

In what follows, we'll assume the temperature field is constant, so that

$$\nabla\theta = 0 = q, \quad \dot{\theta} = 0,$$

and, consequently, the Second Law becomes $D_G\Psi \cdot \dot{M} \geq 0$, which is satisfied with equality in both the loose and compact phases.

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NOTE:

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2. $\varepsilon = \Psi - \theta D_\theta\Psi$, so that Ψ and the heat-flux response function determine all the others.

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Jump conditions on a parametric space-time hypersurface $\mathcal{I}$: $X \mapsto (X, \hat{t}(X))$ with orientation given by the space-time normal field $X \mapsto (-\nabla\hat{t}(X), 1)$.

Assume that the motion

- is determined by the compact phase $(g_c, \nabla g_c)$ for a macroscopic motion g_c on one side of $\mathcal{I}$

$$[S](X, \hat{t}(X))(-\nabla\hat{t}(X)) - \rho_0[\dot{g}](X, \hat{t}(X)) = 0$$

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- is determined by the loose phase $(g_\ell, \zeta_{\min} I)$ for a macroscopic motion g_ℓ on the opposite side, so that

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Special assumptions on g_c and g_ℓ: both are homogeneous motions composed with a (time-dependent) translation:

- $g_\ell(X,t) = X_0 + F(I + \xi_\ell a \otimes n)(X - X_0) + tv_\ell + \frac{t^2}{2\rho_0} b$

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$$\zeta^3_{\min} \leq \det \nabla g_\ell = \det F(1 + \xi_\ell \, a \cdot n).$$

- The only possible source of dissipation is the motion of the phase boundary $\mathcal{I}_t = \{X \in \mathcal{B} \mid \hat{t}(X) = t\}$ in the reference configuration.

Implications of the jump conditions

Among the implications of the jump conditions when $\hat{t}$ is affine are:

- The phase boundary $\mathcal{I}_t$ in the reference configuration has normal n, and, if $b \neq 0$, the traction $D_G \Psi(F(I + \xi_c a \otimes n), \theta)n$ on the phase boundary is a linear combination of Fa and $Fa \times b$.

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- Let $N_{a,n}$ be the bounded, possibly singleton interval of numbers ξ_c such that $D_G\Psi(F(I+\xi_c a\otimes n),\theta)n\cdot Fa = 0$. For $\xi_c \notin N_{a,n}$, ξ_c determines ξ_ℓ, $\Xi = |\nabla\hat{t}|^{-1}$, and $v_c - v_\ell$, e.g.,

$$\xi_\ell = \xi_c - \frac{2(\Psi_c - \theta(D_\theta\Psi)_c) - 2(\Psi_\ell - \theta(D_\theta\Psi)_\ell)}{(D_G\Psi)_c\, n\cdot Fa},$$

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- If a loose-to-compact transition occurs, i.e, $(-\nabla \hat{t}(X), 1)$ points into the compact phase, then $(D_\theta\Psi)_c \leq (D_\theta\Psi)_\ell$ (reverse for compact-to-loose transition; equality for a "reversible" transition).

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- If a loose-to-compact transition occurs, i.e, $(-\nabla\hat{t}(X),1)$ points into the compact phase, then $(D_\theta\Psi)_c \leq (D_\theta\Psi)_\ell$ (reverse for compact-to-loose transition; equality for a "reversible" transition).
- The Accomodation Inequality in the loose phase takes the form:

$$\frac{\zeta_{\min}^3}{\det F} - 1 \leq a\cdot n\left\{\xi_c - \frac{2(\Psi_c - \theta(D_\theta\Psi)_c) - 2(\Psi_\ell - \theta(D_\theta\Psi)_\ell)}{(D_G\Psi)_c\, n\cdot Fa}\right\}.$$

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Sufficient conditions for a loose-to-compact transition

Assume

- The closed interval $N_{a,n} := \{ \xi_c \mid D_G \Psi(F(I + \xi_c a \otimes n), \theta) n \cdot Fa = 0 \}$ is a singleton $\{\xi_0\}$.

Then there exists an open interval I of the form $(\xi_0 - \delta, \xi_0)$ or $(\xi_0, \xi_0 + \delta)$ such that for every $\xi_c \in I$ the body admits a moving planar interface that transforms material in the loose phase $(g_\ell, \zeta_{\min} I)$, with

$$\xi_\ell = \xi_c - \frac{2(\Psi_c - \theta(D_\theta \Psi)_c) - 2(\Psi_\ell - \theta(D_\theta \Psi)_\ell)}{(D_G \Psi_c) n \cdot Fa},$$

into the compact phase $(g_c, \nabla g_c)$. Moreover, $\lim\limits_{\xi_c \longrightarrow \xi_0} |\xi_\ell| = \infty$, so that there is a drastic reduction in the level of deformation as a material point is transformed from the loose phase to the compact phase.

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Sufficient conditions for a loose-to-compact transition

Assume

- The closed interval $N_{a,n} := \{\xi_c \mid D_G\Psi(F(I+\xi_c a\otimes n),\theta)n\cdot Fa = 0\}$ is a singleton $\{\xi_0\}$.
- $D_\theta\Psi(F(I+\xi_0 a\otimes n),\theta) < D_\theta\Psi(\zeta_{\min} I,\theta)$

Then there exists an open interval I of the form $(\xi_0-\delta,\xi_0)$ or $(\xi_0,\xi_0+\delta)$ such that for every $\xi_c \in I$ the body admits a moving planar interface that transforms material in the loose phase $(g_\ell, \zeta_{\min} I)$, with

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- $D_\theta\Psi(F(I+\xi_0 a\otimes n),\theta) < D_\theta\Psi(\zeta_{\min} I,\theta)$
- $b = 0$

Then there exists an open interval I of the form $(\xi_0-\delta,\xi_0)$ or $(\xi_0,\xi_0+\delta)$ such that for every $\xi_c\in I$ the body admits a moving planar interface that transforms material in the loose phase $(g_\ell,\zeta_{\min} I)$, with

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into the compact phase $(g_c,\nabla g_c)$. Moreover, $\lim_{\xi_c\longrightarrow\xi_0}|\xi_\ell| = \infty$, so that there is a drastic reduction in the level of deformation as a material point is transformed from the loose phase to the compact phase.

Sufficient conditions for a compact-to-loose transition

Assume

- The closed interval $N_{a,n} := \{\xi_c \mid D_G\Psi(F(I+\xi_c a\otimes n),\theta)n\cdot Fa = 0\}$ is a singleton $\{\xi_0\}$.

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Moreover, $\lim_{\xi_c\longrightarrow\xi_0}|\xi_\ell| = \infty$, so that deformation drastically increases from the compact phase to the loose phase.

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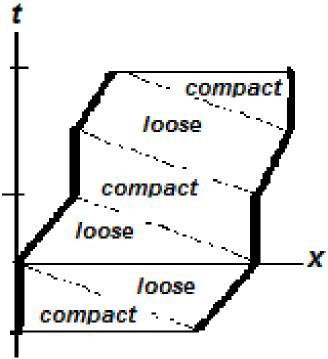
Sufficient conditions for a reversible transition

Assume $b = 0$ and there exists $\hat{\xi}_c \notin N_{a,n}$ such that $D_\theta \Psi(F(I + \hat{\xi}_c a \otimes n), \theta) = D_\theta \Psi(\zeta_{\min} I, \theta)$ and $\frac{\zeta_{\min}^3}{\det F} - 1 \leq a \cdot n \{ \hat{\xi}_c - \frac{2(\Psi_c - \Psi_\ell)}{(D_G \Psi)_c n \cdot Fa} |_{\xi_c = \hat{\xi}_c} \}$. Then both the compact-to-loose and the loose-to-compact transitions corresponding to the structured deformations $(g_c, \nabla g_c)$ and $(g_\ell, \zeta_{\min} Q)$ are available to the body for $\xi_c = \hat{\xi}_c$ and for $\xi_\ell = \hat{\xi}_c - \frac{2(\Psi_c - \Psi_\ell)}{(D_G \Psi)_c n \cdot Fa} |_{\xi_c = \hat{\xi}_c}$.



Other issues addressed in present research:

- Illustrative example: $\Psi(G, \theta) = \frac{\alpha(\theta)}{2}(\det G)^{-2} + \frac{\beta(\theta)}{2} G \cdot G$
  Simple shears in each phase: $F = I$, $a \cdot n = 0$:

$$g_c(X, t) = X_0 + (I + \xi_c a \otimes n)(X - X_0) + t v_c + \frac{t^2}{2\rho_0} b$$

$$g_\ell(X, t) = X_0 + (I + \xi_\ell a \otimes n)(X - X_0) + t v_c + \frac{t^2}{2\rho_0} b$$

  e.g.: Sufficient conditions for loose-to-compact transition stated earlier become:

$$b \cdot n = 0, \ \ \alpha(\theta) < \beta(\theta), \ \alpha'(\theta) + 3\beta'(\theta) > 0$$

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- Plane progressive waves with small associated deformations in the compact phase

$$g_c(X,t) = X_0 + F_c(X - X_0) + \varphi((X - X_0) \cdot n + st)\, e + \tfrac{t^2}{2\rho_0} b$$

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- Same as above, while admitting other phases:
  $D_G \Psi(G(X)) \ (\nabla g(X) - G(X))^T = 0$...consistency.

Supplementary relations:

$$|v_\ell - v_c| = \{2(\Psi_c - \theta(D_\theta\Psi)_c)/\rho_0 - 2(\Psi_\ell - \theta(D_\theta\Psi)_\ell)/\rho_0\}^{1/2}$$

...relative speed of phases

$$\Xi = \frac{|(D_G\Psi)_c\, n\cdot Fa| \;/\; |Fa|}{\{2\rho_0(\Psi_c - \theta(D_\theta\Psi)_c) - 2\rho_0(\Psi_\ell - \theta(D_\theta\Psi)_\ell)\}^{1/2}}$$

... speed of interface

$$D = \Xi\,|(D_\theta\Psi)_c - (D_\theta\Psi)_\ell|$$

.... rate of dissipation by interface