

Relaxation results for an optimal design problem with perimeter penalization

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CNA seminar, CMU

- C.- Zappale: 3D-2D dimension reduction for a nonlinear optimal design problem with perimeter penalization.
CRAS 2012

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- C.- Zappale: Relaxation in the case of linear growth.
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- More general models

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- More general models
- Next steps: different growths

3D-2D dimension reduction: setting the problem

For $\varepsilon > 0$ let

$$\Omega(\varepsilon) := \omega \times (-\varepsilon, \varepsilon), \quad \omega \subset \mathbb{R}^2$$

and $F_\varepsilon : BV(\Omega(\varepsilon); \{0, 1\}) \times W^{1,p}(\Omega(\varepsilon); \mathbb{R}^3) \rightarrow [0, +\infty]$, $p > 1$

$$F_\varepsilon(\chi, v) := \frac{1}{\varepsilon} \left(\int_{\Omega(\varepsilon)} \left(\chi_{E(\varepsilon)} W_1 + (1 - \chi_{E(\varepsilon)}) W_2 \right) (\nabla v) dx - \int_{\Omega(\varepsilon)} f_\varepsilon \cdot v dx + \text{Per}(E(\varepsilon); \Omega(\varepsilon)) \right)$$

$$\lambda := \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)}(x) dx \rightsquigarrow \text{volume fraction}$$

$E(\varepsilon) \subset \Omega(\varepsilon)$ has finite perimeter, $f_\varepsilon \in L^{p'}(\Omega(\varepsilon))$.

$W_i : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ continuous satisfying

$$\alpha |\xi|^p \leq W_i(\xi) \leq \beta (1 + |\xi|^p), \quad \forall \xi \in \mathbb{R}^{3 \times 3}, \quad i = 1, 2, \quad (1)$$

for some $\alpha, \beta > 0$.

Optimal Design Problem

$$\mathcal{C}_\varepsilon(\chi) = - \inf_{v \in W^{1,p}} \left\{ \frac{1}{\varepsilon} \left(\int_{\Omega(\varepsilon)} \left(\chi_{E(\varepsilon)} W_1 + (1 - \chi_{E(\varepsilon)}) W_2 \right) (\nabla v) dx - \int_{\Omega(\varepsilon)} f \cdot v dx + \text{Per}(E(\varepsilon); \Omega(\varepsilon)) \right) \right\}$$

The best optimal design would be

$$- \sup_{\chi} \left\{ -\mathcal{C}_\varepsilon(\chi) : \chi \in BV(\Omega(\varepsilon); \{0, 1\}), \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)} dx = \lambda \right\}$$

Problem

$$\inf_{(\chi, u)} \left\{ F_\varepsilon(\chi, v) : v = 0 \text{ on } \partial\omega \times (-\varepsilon, \varepsilon), \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)} dx = \lambda, \right. \\ \left. v \in W^{1,p}(\Omega(\varepsilon); \mathbb{R}^d), \chi \in BV(\Omega(\varepsilon); \{0, 1\}) \right\}$$

- Question: What happens as $\varepsilon \rightarrow 0^+$?

3D-2D dimension reduction

- Question: What happens as $\varepsilon \rightarrow 0^+$?
- Rescale the energies F_ε

$$\Omega := \omega \times (-1, 1)$$

$$E_\varepsilon := \{(x_1, x_2, x_3) \in \Omega : (x_1, x_2, \varepsilon x_3) \in E(\varepsilon)\}$$

$$u(x_1, x_2, x_3) := v(x_1, x_2, \varepsilon x_3)$$

$$f(x_1, x_2, x_3) := f_\varepsilon(x_1, x_2, \varepsilon x_3)$$

$$\chi(x_1, x_2, x_3) := \chi_{E(\varepsilon)}(x_1, x_2, \varepsilon x_3)$$

$$\frac{1}{\varepsilon} \text{Per}(E(\varepsilon); \Omega(\varepsilon)) = \frac{1}{\varepsilon} \left| D\chi_{E(\varepsilon)} \right|(\Omega(\varepsilon)) = \left| \left(D_\alpha \chi \Big| \frac{1}{\varepsilon} D_3 \chi \right) \right|(\Omega)$$

Problem

$$\inf_{\substack{u \in W^{1,p}(\Omega; \mathbb{R}^3) \\ \chi \in BV(\Omega; \{0,1\})}} \left\{ G_\varepsilon(\chi, u) : u = 0 \text{ on } \partial\omega \times (-1, 1), \frac{1}{\mathcal{L}^3(\Omega)} \int_\Omega \chi dx = \lambda \right\}$$

where

$$G_\varepsilon(\chi, u) := \int_\Omega V(\chi, (\nabla_\alpha u | \frac{1}{\varepsilon} \nabla_3 u)) dx - \int_\Omega f \cdot u dx + |(D_\alpha \chi | \frac{1}{\varepsilon} D_3 \chi)|(\Omega)$$

with

$$V(\chi, \nabla u) := (\chi W_1 + (1 - \chi) W_2)(\nabla u).$$

Define $J_\varepsilon : L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ by

$$J_\varepsilon(\chi, u) := \begin{cases} G_\varepsilon(\chi, u) & \text{in } BV(\Omega; \{0, 1\}) \times W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (2)$$

- Bounded admissible sequences for problem (2) are compact in $L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$.

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- Bounded admissible sequences for problem (2) are compact in $L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$.
- The limit is independent of the x_3 variable

Characterization of the Gamma limit

- Let $\overline{W}_i : \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$ be given by

$$\overline{W}_i(\overline{F}) := \inf_{z \in \mathbb{R}^3} W_i(\overline{F}|z), \quad \overline{F} \in \mathbb{R}^{3 \times 2}, \quad i = 1, 2.$$

and $\overline{V} : \{0, 1\} \times \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$ such that

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- $Q\overline{V} \dots$ quasiconvexification of \overline{V} in the second variable.

$$Q\overline{V}(\chi, \overline{F}) := \inf \left\{ \int_{Q'} \overline{V}(\chi, \overline{F} + \nabla_\alpha \varphi) dx_\alpha : \varphi \in W_0^{1,p}(Q'; \mathbb{R}^3) \right\},$$

Q' unit cube in \mathbb{R}^2 .

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz set and let $W_i : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$, $i = 1, 2$, be continuous functions satisfying (3). Then

$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} (L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)) J_\varepsilon(\chi, u) = J_0(\chi, u)$$

for every $(\chi, u) \in L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$.

$$J_0 : L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$$

$$J_0(\chi, u) := \begin{cases} G_0(\chi, u) & \text{if } (\chi, u) \in BV(\omega; \{0, 1\}) \times W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$G_0(\chi, u) := 2 \int_{\omega} Q\bar{V}(\chi, \nabla_{\alpha} u) dx_{\alpha} - \int_{-1}^1 \int_{\omega} f \cdot u dx_{\alpha} dx_3 + 2|D\chi|(\omega).$$

Lower bound

Let

$$J(\chi, u) := \Gamma - \lim_{\varepsilon \rightarrow 0^+} (L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)) J_\varepsilon(\chi, u)$$

For every $(\chi, u) \in BV(\omega; \{0, 1\}) \times W^{1,p}(\omega; \mathbb{R}^3)$, we claim that

$$\begin{aligned} J(\chi, u) &\geq 2 \int_{\omega} Q\bar{V}(\chi(x_\alpha), \nabla_\alpha u(x_\alpha)) dx_\alpha \\ &\quad - \int_{-1}^1 \int_{\omega} f(x_\alpha, x_3) u(x_\alpha) dx_\alpha dx_3 + 2 |D_\alpha \chi|(\omega). \end{aligned}$$

Let $\{(\chi_\varepsilon, u_\varepsilon)\} \subset L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$ such that $(\chi_\varepsilon, u_\varepsilon) \rightarrow (\chi, u)$ w.r.t. the strong topology of $L^1(\Omega; \{0, 1\}) \times L^p(\Omega; \mathbb{R}^3)$.

- for the forces it is ok (because $u_\varepsilon \rightarrow u$ in $L^p(\Omega; \mathbb{R}^3)$)

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- lower semicontinuity of the total variation

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- for the forces it is ok (because $u_\varepsilon \rightarrow u$ in $L^p(\Omega; \mathbb{R}^3)$)
- lower semicontinuity of the total variation
- Bulk energy \rightsquigarrow use Decomposition Lemma for scaled gradients

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For every $(\chi, u) \in BV(\omega; \{0, 1\}) \times W^{1,p}(\omega; \mathbb{R}^3)$, we claim that

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- for the forces it is ok (because $u_\varepsilon \rightarrow u$ in $L^p(\Omega; \mathbb{R}^3)$)
- lower semicontinuity of the total variation
- Bulk energy \rightsquigarrow use Decomposition Lemma for scaled gradients
- Superadditivity of \liminf .

Gamma-Convergence result (Babadjian - Francfort)

Let $G_\varepsilon : L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ be given by

$$G_\varepsilon(u) := \begin{cases} \int_{\Omega} W(x_\alpha, (\nabla_\alpha u | \frac{1}{\varepsilon} \nabla_3 u)) dx - \int_{\Omega} f \cdot u dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise} \end{cases}$$

where $W : \omega \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ is a Carathéodory function satisfying the following growth condition

$$\frac{1}{C} |\xi|^p - C \leq W(x_\alpha, \xi) \leq C(1 + |\xi|^p)$$

for a.e. $x_\alpha \in \omega$ and for every $\xi \in \mathbb{R}^{3 \times 3}$ and some constant $C > 0$.

Gamma-Convergence result (Babadjian - Francfort)

Theorem

Under the above assumptions $\{G_\varepsilon\}$ Γ -converges, with respect to L^p -strong convergence, to the functional

$$G(u) := \begin{cases} 2 \int_{\omega} \underline{W}(x_\alpha, \nabla_\alpha u) dx_\alpha - \int_{-1}^1 \int_{\omega} f \cdot u dx_\alpha dx_3 & \text{if } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

where the function \underline{W} is given by

$$\underline{W}(x_0, \bar{F}) : = \inf_{\substack{\lambda > 0 \\ \varphi \in W^{1,p}(Q; \mathbb{R}^3)}} \left\{ \frac{1}{2} \int_Q W(x_0, \bar{F} + \nabla_\alpha \varphi, \lambda \nabla_3 \varphi) dx : \varphi \equiv 0 \right. \\ \left. \text{on } \partial Q' \times (-1, 1) \right\},$$

where $Q := Q' \times (-1, 1)$.

Upper bound

For every $\chi \in L^1(\Omega; \{0, 1\})$

$$J(\chi, u) \leq \liminf_{\varepsilon \rightarrow 0^+} J_\varepsilon(\chi, u_\varepsilon) \text{ if } u_\varepsilon \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^3).$$

Study the asymptotic behaviour

$$\int_{\Omega} (\chi W_1(\nabla_\alpha u_\varepsilon, \frac{1}{\varepsilon} \nabla_3 u_\varepsilon) + (1 - \chi) W_2(\nabla_\alpha u_\varepsilon, \frac{1}{\varepsilon} \nabla_3 u_\varepsilon)) dx - \int_{\Omega} f \cdot u_\varepsilon dx.$$

Define

$$W(x_\alpha, F) := \chi(x_\alpha) W_1(F) + (1 - \chi(x_\alpha)) W_2(F).$$

$$Q\overline{W}(x_\alpha, \overline{F}) = \underline{W}(x_\alpha, \overline{F}),$$

where \overline{W} is defined by $\overline{W}(x_\alpha, \overline{F}) := \inf_{c \in \mathbb{R}^3} W(x_\alpha, (\overline{F}|_c))$.

- Applying Gamma convergence result of Babadjian and Francfort there exists a sequence $\{u_\varepsilon\}$ such that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\Omega} W(x_\alpha, (\nabla_\alpha u_\varepsilon \frac{1}{\varepsilon} \nabla_3 u_\varepsilon)) dx - \int_{\Omega} f \cdot u_\varepsilon dx \right) \\ & \leq 2 \int_{\omega} Q\overline{W}(x_\alpha, \nabla_\alpha u) dx_\alpha - \int_{-1}^1 \int_{\omega} f \cdot u dx_\alpha dx_3. \end{aligned}$$

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- The proof is concluded by observing that

$$\begin{aligned} \overline{W}(x_0, \overline{F}) &= \chi(x_0) \overline{W}_1(\overline{F}) + (1 - \chi(x_0)) \overline{W}_2(\overline{F}) = \overline{V}(\chi(x_0), \overline{F}), \\ Q\overline{W}(x_0, \overline{F}) &= Q\overline{V}(\chi(x_0), \overline{F}). \end{aligned}$$

for every $(x_0, \overline{F}) \in \omega \times \mathbb{R}^{3 \times 2}$.

Relaxation $p=1$: settings

Let $F : BV(\Omega; \{0, 1\}) \times W^{1,1}(\Omega; \mathbb{R}^d) \rightarrow [0, +\infty]$,

$$F(\chi, u) := \int_{\Omega} (\chi_E W_1 + (1 - \chi_E) W_2) (\nabla u) dx - \int_{\Omega} f \cdot u dx + \text{Per}(E; \Omega)$$

where $\text{Per}(E; \Omega) = |D\chi_E|(\Omega)$, $f \in L^\infty(\Omega; \mathbb{R}^d)$.

$W_i : \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ continuous satisfying

$$\alpha |\xi| \leq W_i(\xi) \leq \beta (1 + |\xi|), \quad \forall \xi \in \mathbb{R}^{d \times N}, \quad i = 1, 2, \quad (3)$$

for some $\alpha, \beta > 0$.

$$\inf_{\substack{u \in W^{1,1}(\Omega; \mathbb{R}^d) \\ \chi_E \in BV(\Omega; \{0,1\})}} \left\{ \int_{\Omega} (\chi_E W_1(\nabla u) + (1 - \chi_E) W_2) (\nabla u) dx + \text{Per}(E; \Omega) \right. \\ \left. u = u_0 \text{ on } \partial\Omega \right\}$$

Relaxation in $BV(\Omega; \{0, 1\}) \times W^{1,1}(\Omega; \mathbb{R}^d)$

$$\mathcal{F}_{OD}(\chi, u; A) = \inf \left\{ \liminf_{n \rightarrow \infty} \left(\int_A (\chi_n W_1 + (1 - \chi_n) W_2) (\nabla u_n) dx \right. \right. \\ \left. \left. + |D\chi_n|(A) \right) : \{(\chi_n, u_n)\} \subseteq BV(A; \{0, 1\}) \times W^{1,1}(A; \mathbb{R}^d) \right.$$

$$\left. \begin{array}{l} \chi_n \xrightarrow{*} \chi \text{ in } BV(A; \{0, 1\}) \\ u_n \rightarrow u \text{ in } L^1(\Omega; \mathbb{R}^d) \end{array} \right\}$$

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- $QV \dots$ quasiconvex envelope of V ,
- $QV^\infty \dots$ recession function of QV , namely,

$$QV^\infty(q, z) := \lim_{t \rightarrow \infty} \frac{QV(q, tz)}{t}.$$

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $W_i : \mathbb{R}^{d \times N} \rightarrow [0, +\infty)$, $i = 1, 2$, be continuous functions satisfying (3). Then for every $(\chi, u) \in L^1(\Omega; \{0, 1\}) \times L^1(\Omega; \mathbb{R}^d)$

$$\mathcal{F}_{OD}(\chi, u; A) = \begin{cases} \overline{\mathcal{F}_{OD}}(\chi, u; A) & \text{if } (\chi, u) \in BV(\Omega; \{0, 1\}) \times BV(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \overline{\mathcal{F}_{OD}}(\chi, u; A) : &= \int_A QV(\chi, \nabla u) \, dx + \int_A QV^\infty \left(\chi, \frac{dD^c u}{d|D^c u|} \right) d|D^c u| \\ &+ \int_{J_{(\chi, u)} \cap A} K_2(\chi^+, \chi^-, u^+, u^-) \, d\mathcal{H}^{N-1} \end{aligned}$$

Integral representation

The interaction is described through the following density

$$K_2(a, b, c, d, \nu) := \inf \left\{ \int_{Q_\nu} QV^\infty(\chi(x), \nabla u(x)) dx + |D\chi|(Q_\nu) : \right. \\ \left. (\chi, u) \in \mathcal{A}_2(a, b, c, d, \nu) \right\},$$

$$\mathcal{A}_2(a, b, c, d, \nu) := \left\{ (\chi, u) \in BV(Q_\nu; \{0, 1\}) \times W^{1,1}(Q_\nu; \mathbb{R}^d) : \right. \\ (\chi(y), u(y)) = (a, c) \text{ if } y \cdot \nu = \frac{1}{2}, \\ (\chi(y), u(y)) = (b, d) \text{ if } y \cdot \nu = -\frac{1}{2}, \\ \left. (\chi, u) \text{ are } 1\text{-periodic in } \nu_1, \dots, \nu_{N-1} \text{ directions} \right\},$$

for $(a, b, c, d, \nu) \in \{0, 1\} \times \{0, 1\} \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$, with Q_ν the unit cube, centered at the origin, with axes parallel to $\{\nu_1, \nu_2, \dots, \nu_{N-1}, \nu\}$.

- The surface energy can be specialized as follows

$$\begin{aligned} & \int_{J_{(\chi,u)}} K_2(\chi^+, \chi^-, u^+, u^-, v_{(\chi,u)}) d\mathcal{H}^{N-1} \\ &= \int_{J_u \setminus J_\chi} QV^\infty(\chi, (u^+ - u^-) \otimes v_u) d\mathcal{H}^{N-1} \\ &+ |D\chi|(\Omega \cap (J_\chi \setminus J_u)) + \int_{J_\chi \cap J_u} K_2(\chi^+, \chi^-, u^+, u^-, v_{(\chi,u)}) d\mathcal{H}^{N-1} \end{aligned}$$

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$$\nabla U = (0, \nabla u) \quad \text{and} \quad D^c U = (0, D^c u).$$

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- 1 Barroso-Bouchitté-Buttazzo-Fonseca, ARMA (1996):
Relaxation in BV for

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- 2 Bouchitté-Fonseca-Mascarenhas ARMA (1998): deals with more general models.

Lower bound

- Observe that

$$\overline{F_{OD}}(\chi, u) \geq \liminf_{n \rightarrow \infty} \int_{\Omega} (\chi_n W_1(\nabla u_n) + (1 - \chi_n) W_2(\nabla u_n)) dx. \quad (4)$$

For Bulk and Cantor parts we exploit Fonseca-Müller SIAM JoMA (1992) and ARMA (1993).

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- (4) could go much lower than $\overline{F_{OD}}(\chi, u)$.

Upper bound

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Upper bound

- \mathcal{F}_{OD} is a measure absolutely continuous w.r.t. $\mathcal{L}^N + |D\chi| + |Du|$.
- We compute “Bulk” and “Cantor” parts exploiting the “Global Method for Relaxation” by Bouchitté-Fonseca-Mascarenhas 1998.
- For the jump we compute it directly again mixing Ambrosio-Braides JMPA (1993) and Fonseca-Rybka Proc. Roy. Soc. Edinburgh (1992).

Comparison between the densities

- Clearly $K_2 \geq K$, where K is the following density

$$K(a, b, c, d, \nu) := \inf \left\{ \int_{Q_\nu} (QV^\infty(\nu(x), \nabla u(x)) + |\nabla \nu(x)|) dx : \right. \\ \left. (\nu, u) \in \mathcal{A}(a, b, c, d, \nu) \right\},$$

where

$$\mathcal{A}(a, b, c, d, \nu) := \left\{ (\nu, u) \in W^{1,1}(Q_\nu; \mathbb{R}^{1+d}) : \right. \\ \left. (\nu(y), u(y)) = (a, c) \text{ if } y \cdot \nu = \frac{1}{2}, (\nu(y), u(y)) = (b, d) \text{ if } y \cdot \nu = -\frac{1}{2} \right. \\ \left. (\nu, u) \text{ are } 1\text{-periodic in } \nu_1, \dots, \nu_{N-1} \text{ directions} \right\}.$$

Comparison between the densities

- Recall Fonseca-Muller ARMA (1993)

$$\inf \left\{ \liminf_{n \rightarrow \infty} \left(\int_{\Omega} (v_n W_1 + (1 - v_n) W_2) (\nabla u_n) dx + \int_{\Omega} |\nabla v_n| dx \right. \right. \\ \left. \left. : \{(v_n, u_n)\} \subset W^{1,1}(\Omega; [0, 1]) \times W^{1,1}(\Omega; \mathbb{R}^d) \right. \right. \\ \left. \left. v_n \rightarrow v \text{ in } L^1(\Omega; [0, 1]), u_n \rightarrow u \text{ in } L^1(\Omega; \mathbb{R}^d) \right\}, \dots$$

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- The result has been applied for $U \equiv (v, u)$, now with $U \in W^{1,1}(\Omega; \mathbb{R}^{1+d})$.

What about $K_2 \leq K$?

- Relaxation in $SBV_0(\Omega; \mathbb{R}^m) \times W^{1,1}(\Omega; \mathbb{R}^d)$

$$\mathcal{F}(v, u) = \inf \left\{ \liminf_{n \rightarrow \infty} \left(\int_{\Omega} f(v_n, \nabla u_n) dx + \int_{J_{v_n}} g(v_n^+, v_n^-, \nu_{v_n}) d\mathcal{H}^{N-1} \right) \right. \\ \left. \begin{array}{l} \{u_n\} \subset W^{1,1}(\Omega; \mathbb{R}^d), \quad \{v_n\} \subset SBV_0(\Omega; \mathbb{R}^m), \\ u_n \rightarrow u \text{ in } L^1(\Omega; \mathbb{R}^d) \\ v_n \rightarrow v \text{ in } L^1(\Omega; \mathbb{R}^m) \end{array} \right\}$$

More general models

Recall that

$$SBV_0(\Omega; \mathbb{R}^m) := \left\{ v \in BV(\Omega; \mathbb{R}^m) : D^c v = 0, \nabla v = 0, \mathcal{H}^{N-1}(J_v) < \infty \right\}$$

i.e. $v \in SBV_0(\Omega; \mathbb{R}^m)$ iff there exists a Borel partition $\{E_i\}$ of Ω and a sequence $\{v_i\} \subset \mathbb{R}^m$ such that

$$v = \sum_{i=1}^{\infty} v_i \chi_{E_i} \quad \text{a.e. } x \in \Omega,$$

$$\mathcal{H}^{N-1}(J_v \cap \Omega) = \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{H}^{N-1}(\partial^* E_i \cap \Omega),$$

$$(v^+, v^-, \nu_v) \equiv (v_i, v_j, \nu_i) \quad \text{a.e. } x \in \partial^* E_i \cap \partial^* E_j \cap \Omega,$$

ν_i being the unit normal to $\partial^* E_i \cap \partial^* E_j$.

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$$|f(v_1, \xi) - f(v_2, \xi)| \leq L|v_1 - v_2|(1 + |\xi|) \quad \forall v_1, v_2 \in \mathbb{R}^m, \xi \in \mathbb{R}^{d \times N};$$

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- 4 there exist $\alpha \in (0, 1)$, and $C, L > 0$ such that

$$t|\zeta| > L \Rightarrow \left| f^\infty(v, \zeta) - \frac{f(v, t\zeta)}{t} \right| \leq C \frac{|\zeta|^{1-\alpha}}{t^\alpha}, \quad \forall (v, \zeta) \quad \forall t > 0,$$

with f^∞ the recession function of f w.r.t. ζ .

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- 3 $g(\lambda, \theta, \nu) = g(\theta, \lambda, -\nu)$, for every $(\lambda, \theta, \nu) \in \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1}$.

Theorem

Let $\Omega \in \mathcal{A}(\Omega)$ and $f : \mathbb{R}^m \times \mathbb{R}^{d \times N} \rightarrow [0, +\infty[$ be a function satisfying $(F_1) - (F_4)$ and $g : \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{S}^{N-1} \rightarrow [0, +\infty[$ satisfying $(G_1) - (G_3)$. Then for every $(v, u) \in L^1(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^d)$

$$\mathcal{F}(v, u; \Omega) = \begin{cases} \overline{F}_0(v, u; \Omega) & \text{if } (v, u) \in SBV_0(\Omega; \mathbb{R}^m) \times BV(\Omega; \mathbb{R}^d), \\ +\infty & \text{otherwise} \end{cases}$$

where $\overline{F}_0 : SBV_0(\Omega; \mathbb{R}^m) \times BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ is given by

$$\begin{aligned} \overline{F}_0(v, u; A) & : = \int_A Qf(v, \nabla u) dx + \int_A Qf^\infty\left(v, \frac{dD^c u}{d|D^c u|}\right) d|D^c u| \\ & + \int_{J_{(v,u)} \cap A} K_3(v^+, v^-, u^+, u^-, \nu) d\mathcal{H}^{N-1}. \end{aligned}$$

Energy density

Let $K_3 : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \rightarrow [0, +\infty[$ be defined as

$$K_3(a, b, c, d, \nu) := \inf \left\{ \int_{Q_\nu} Qf^\infty(\nu(x), \nabla u(x)) dx + \int_{J_\nu \cap Q_\nu} g(\nu^+(x), \nu^-(x), \nu(x)) d\mathcal{H}^{N-1} : (\nu, u) \in \mathcal{A}_3(a, b, c, d, \nu) \right\}$$

where

$$\begin{aligned} \mathcal{A}_3(a, b, c, d, \nu) := & \{(\nu, u) \in SBV_0(Q_\nu; \mathbb{R}^m) \cap L^\infty(Q_\nu; \mathbb{R}^m) \times W^{1,1}(Q_\nu; \mathbb{R}) \\ & (\nu(y), u(y)) = (a, c) \text{ if } y \cdot \nu = \frac{1}{2}, \\ & (\nu(y), u(y)) = (b, d) \text{ if } y \cdot \nu = -\frac{1}{2}, \\ & (\nu, u) \text{ are } 1\text{-periodic in } \nu_1, \dots, \nu_{N-1}\}. \end{aligned}$$

Auxiliary Lemma

Lemma

Let $Q := [0, 1]^N$ and

$$v_0(y) := \begin{cases} a & \text{if } x_N > 0, \\ b & \text{if } x_N < 0, \end{cases} \quad u_0(y) := \begin{cases} c & \text{if } x_N > 0, \\ d & \text{if } x_N < 0. \end{cases}$$

Let $\{v_n\} \subset SBV_0(\Omega; \mathbb{R}^m)$ and $\{u_n\} \subset W^{1,1}(\Omega; \mathbb{R}^d)$, such that $v_n \rightarrow v_0$ in $L^1(Q; \mathbb{R}^m)$ and $u_n \rightarrow u_0$ in $L^1(Q; \mathbb{R}^d)$. There exists $\{(\zeta_n, \xi_n)\} \in \mathcal{A}_3(a, b, c, d, e_N)$ such that $\zeta_n = v_0$ on ∂Q , $\zeta_n \rightarrow v_0$ in $L^1(Q; \mathbb{R}^m)$, $\xi_n = \rho_{i(n)} * u_0$ on ∂Q , $\xi_n \rightarrow u_0$ in $L^1(Q; \mathbb{R}^d)$

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \left(\int_Q Qf(\zeta_n, \nabla \xi_n) dx + \int_{J_{\zeta_n} \cap Q} g(\zeta_n^+, \zeta_n^-, v_{\zeta_n}) d\mathcal{H}^{N-1} \right) \\ & \leq \underline{\lim}_{n \rightarrow \infty} \left(\int_Q Qf(v_n, \nabla u_n) dx + \int_{J_{v_n} \cap Q} g(v_n^+, v_n^-, v_{v_n}) d\mathcal{H}^{N-1} \right) \end{aligned}$$

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 - Jump part: comes from an explicit calculation mixing techniques of Ambrosio-Braides and Fonseca-Muller.

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- 3 Case 3- $U \equiv (v, u) \in (SBV_0(\Omega; \mathbb{R}^m) \times BV(\Omega; \mathbb{R}^d)) \cap L^\infty(\Omega; \mathbb{R}^{m+d})$.
- 4 Case 4- $U \equiv (v, u) \in SBV_0(\Omega; \mathbb{R}^m) \times BV(\Omega; \mathbb{R}^d)$.

Comparison between densities

- Comparison between K_2 and K_3 :

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$$\begin{aligned} & \int_{J_{(v,u)}} K_3(v^+, v^-, u^+, u^-, v_{(v,u)}) d\mathcal{H}^{N-1} \\ &= \int_{J_u \setminus J_v} Qf^\infty(v, (u^+ - u^-) \otimes v_u) d\mathcal{H}^{N-1} \\ &+ \int_{J_v \setminus J_u} \mathcal{R}g(v^+, v^-, v_v) d\mathcal{H}^{N-1} \\ &+ \int_{J_v \cap J_u} K_3(v^+, v^-, u^+, u^-, v_{(v,u)}) d\mathcal{H}^{N-1}. \end{aligned}$$

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- $\mathcal{R}g$ is the BV-elliptic envelope of g !

In progress: GAP

Let $W_i : \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ be continuous such that $\exists \alpha, \beta > 0$

$$\alpha |\xi|^p \leq W_i(\xi) \leq \beta(1 + |\xi|^q) \quad \forall \xi \in \mathbb{R}^{d \times N}, \quad 1 < p < q < \frac{Np}{N-1}, \quad i = 1, 2$$

$$F(\chi, u) := \int_{\Omega} \chi_E W_1(\nabla u) + (1 - \chi_E) W_2(\nabla u) dx + |D\chi_E|(\Omega).$$

$$\mathcal{F}(\chi, u; A) := \inf \left\{ \liminf_{n \rightarrow \infty} F(\chi_n, u_n; A) : \begin{aligned} &\{u_n\} \subset W^{1,q}(A; \mathbb{R}^d), \\ &\{\chi_n\} \subset BV(A; \{0, 1\}) \quad u_n \rightharpoonup u \text{ in } W^{1,p}(A; \mathbb{R}^d), \\ &\chi_n \xrightarrow{*} \chi \text{ in } BV(A; \{0, 1\}) \end{aligned} \right\}$$