Relaxation results for an optimal design problem with perimeter penalization

Graça Carita, Elvira Zappale

University of Evora, University of Salerno

CNA seminar, CMU

Outline

- C.- Zappale: 3D-2D dimension reduction for a nonlinear optimal design problem with perimeter penalization.
  CRAS 2012

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- C.- Zappale: 3D-2D dimension reduction for a nonlinear optimal design problem with perimeter penalization.
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- C.- Zappale: Relaxation in the case of linear growth.
  To appear in Proc. Roy. Soc. of Edinburgh.

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- More general models
- Next steps: different growths

3D-2D dimension reduction: setting the problem

For $\varepsilon > 0$ let

$$\Omega(\varepsilon) := \omega \times (-\varepsilon, \varepsilon), \qquad \omega \subset \mathbb{R}^2$$

and $F_\varepsilon : BV(\Omega(\varepsilon); \{0,1\}) \times W^{1,p}(\Omega(\varepsilon); \mathbb{R}^3) \to [0, +\infty]$, $p > 1$

$$F_\varepsilon(\chi, v) \ : \ = \frac{1}{\varepsilon}\left(\int_{\Omega(\varepsilon)} \left(\chi_{E(\varepsilon)} W_1 + (1 - \chi_{E(\varepsilon)}) W_2 \right) (\nabla v) dx \right.$$
$$\left. - \int_{\Omega(\varepsilon)} f_\varepsilon \cdot v \ dx + \mathrm{Per}(E(\varepsilon); \Omega(\varepsilon)) \right)$$

$$\lambda := \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)}(x)\, dx \rightsquigarrow \quad \text{volume fraction}$$

$E(\varepsilon) \subset \Omega(\varepsilon)$ has finite perimeter, $f_\varepsilon \in L^{p'}(\Omega(\varepsilon))$.
$W_i : \mathbb{R}^{3x3} \to \mathbb{R}$ continuous satisfying

$$\alpha |\xi|^p \leq W_i(\xi) \leq \beta(1 + |\xi|^p), \qquad \forall\, \xi \in \mathbb{R}^{3\times 3},\ i = 1, 2, \tag{1}$$

for some $\alpha,\ \beta > 0$.

Optimal Design Problem

$$\mathcal{C}_\varepsilon(\chi) = -\inf_{v \in W^{1,p}} \left\{ \frac{1}{\varepsilon} \left(\int_{\Omega(\varepsilon)} \left(\chi_{E(\varepsilon)} W_1 + (1-\chi_{E(\varepsilon)}) W_2 \right) (\nabla v) dx - \int_{\Omega(\varepsilon)} f \cdot v \; dx + \mathrm{Per}\,(E(\varepsilon); \Omega(\varepsilon)) \right) \right\}$$

The best optimal design would be

$$-\sup_{\chi} \left\{ -\mathcal{C}_\varepsilon(\chi) : \chi \in BV(\Omega(\varepsilon); \{0,1\}), \; \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)} \; dx = \lambda \right\}$$

Problem

$$\inf_{(\chi,u)} \left\{ F_\varepsilon(\chi, v) : v = 0 \text{ on } \partial\omega \times (-\varepsilon,\varepsilon), \; \frac{1}{\mathcal{L}^3(\Omega(\varepsilon))} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)} \; dx = \lambda, \; v \in W^{1,p}(\Omega(\varepsilon); R^d), \; \chi \in BV(\Omega(\varepsilon); \{0,1\}) \right\}$$

3D-2D dimension reduction

- **Question:** What happens as $\varepsilon \to 0^+$?

3D-2D dimension reduction

- **Question:** What happens as $\varepsilon \to 0^+$?
- Rescale the energies F_ε

$$\Omega := \omega \times (-1,1)$$

$$E_\varepsilon := \{(x_1,x_2,x_3) \in \Omega : (x_1,x_2,\varepsilon x_3) \in E(\varepsilon)\}$$

$$u(x_1,x_2,x_3) := v(x_1,x_2,\varepsilon x_3)$$

$$f(x_1,x_2,x_3) := f_\varepsilon(x_1,x_2,\varepsilon x_3)$$

$$\chi(x_1,x_2,x_3) := \chi_{E(\varepsilon)}(x_1,x_2,\varepsilon x_3)$$

$$\frac{1}{\varepsilon}\mathrm{Per}\,(E(\varepsilon);\Omega(\varepsilon)) = \frac{1}{\varepsilon}\left|D\chi_{E(\varepsilon)}\right|(\Omega(\varepsilon)) = \left|\left(D_\alpha\chi\Big|\frac{1}{\varepsilon}D_3\chi\right)\right|(\Omega)$$

Rescaled problem

Problem

$$\inf_{\substack{u\in W^{1,p}\left(\Omega;\mathbb{R}^3\right)\\ \chi\in BV(\Omega;\{0,1\})}}\left\{G_\varepsilon(\chi,u) : u=0 \text{ on } \partial\omega\times(-1,1),\ \frac{1}{\mathcal{L}^3(\Omega)}\int_\Omega \chi dx=\lambda\right\}$$

where

$$G_\varepsilon(\chi,u) := \int_\Omega V(\chi,(\nabla_\alpha u|\frac{1}{\varepsilon}\nabla_3 u))\ dx - \int_\Omega f\cdot u\ dx + |(D_\alpha\chi|\frac{1}{\varepsilon}D_3\chi)|(\Omega)$$

with

$$V(\chi,\nabla u) := (\chi W_1 + (1-\chi) W_2)(\nabla u).$$

Compactness

Define $J_\varepsilon : L^1\left(\Omega; \{0,1\}\right) \times L^p\left(\Omega; \mathbb{R}^3\right) \to [0,+\infty]$ by

$$J_\varepsilon(\chi, u) := \begin{cases} G_\varepsilon(\chi, u) & \text{in } BV\left(\Omega; \{0,1\}\right) \times W^{1,p}\left(\Omega; \mathbb{R}^3\right), \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

- Bounded admissible sequences for problem (2) are compact in $L^1\left(\Omega; \{0,1\}\right) \times L^p\left(\Omega; \mathbb{R}^3\right)$.

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- Bounded admissible sequences for problem (2) are compact in $L^1\left(\Omega; \{0,1\}\right) \times L^p\left(\Omega; \mathbb{R}^3\right)$.
- The limit is independent of the x_3 variable

Characterization of the Gamma limit

- Let $\overline{W}_i : \mathbb{R}^{3\times 2} \to [0,+\infty)$ be given by

$$\overline{W}_i\left(\overline{F}\right) := \inf_{z\in\mathbb{R}^3} W_i\left(\overline{F}|z\right), \qquad \overline{F} \in \mathbb{R}^{3\times 2}, \quad i = 1, 2.$$

and $\overline{V} : \{0,1\} \times \mathbb{R}^{3\times 2} \to [0,+\infty)$ such that

$$\overline{V}\left(\chi, \overline{F}\right) := \chi \overline{W}_1\left(\overline{F}\right) + (1-\chi)\,\overline{W}_2\left(\overline{F}\right),$$

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- $Q\overline{V}$ $\cdots$ quasiconvexification of $\overline{V}$ in the second variable.

$$Q\overline{V}\left(\chi, \overline{F}\right) := \inf\left\{\int_{Q'} \overline{V}\left(\chi, \overline{F} + \nabla_\alpha\varphi\right) dx_\alpha : \varphi \in W_0^{1,p}\left(Q'; \mathbb{R}^3\right)\right\},$$

Q' unit cube in $\mathbb{R}^2$.

Main result

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz set and let $W_i : R^{3\times 3} \to [0, +\infty)$, $i = 1, 2$, be continuous functions satisfying (3). Then

$$\Gamma - \lim_{\varepsilon \to 0^+} (L^1(\Omega; \{0,1\}) \times L^p(\Omega; \mathbb{R}^3)) J_\varepsilon(\chi, u) = J_0(\chi, u)$$

for every $(\chi, u) \in L^1(\Omega; \{0,1\}) \times L^p(\Omega; \mathbb{R}^3)$.

$J_0 : L^1(\Omega; \{0,1\}) \times L^p(\Omega; \mathbb{R}^3) \to [0, +\infty]$

$$J_0(\chi, u) := \begin{cases} G_0(\chi, u) & \text{if } (\chi, u) \in BV(\omega; \{0,1\}) \times W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$G_0(\chi, u) := 2\int_\omega Q\overline{V}(\chi, \nabla_\alpha u)\, dx_\alpha - \int_{-1}^{1}\int_\omega f \cdot u\, dx_\alpha dx_3 + 2|D\chi|(\omega).$$

Lower bound

Let

$$J\left(\chi,u\right) := \Gamma - \lim_{\varepsilon\to 0^+} \left(L^1\left(\Omega;\{0,1\}\right)\times L^p\left(\Omega;R^3\right)\right) J_\varepsilon\left(\chi,u\right)$$

For every $(\chi,u)\in BV(\omega;\{0,1\})\times W^{1,p}(\omega;\mathbb{R}^3)$, we claim that

$$\begin{aligned} J(\chi,u) \quad \geq \quad & 2\int_\omega Q\overline{V}\left(\chi\left(x_\alpha\right),\nabla_\alpha u\left(x_\alpha\right)\right)dx_\alpha \\ & -\int_{-1}^{1}\int_\omega f\left(x_\alpha,x_3\right)u\left(x_\alpha\right)dx_\alpha dx_3 + 2\left|D_\alpha\chi\right|(\omega). \end{aligned}$$

Let $\{(\chi_\varepsilon,u_\varepsilon)\}\subset L^1\left(\Omega;\{0,1\}\right)\times L^p\left(\Omega;\mathbb{R}^3\right)$ such that $(\chi_\varepsilon,u_\varepsilon)\to(\chi,u)$ w.r.t. the strong topology of $L^1\left(\Omega;\{0,1\}\right)\times L^p\left(\Omega;\mathbb{R}^3\right)$.

- for the forces it is ok (because $u_\varepsilon\to u$ in $L^p\left(\Omega;\mathbb{R}^3\right)$)

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Let $\{(\chi_\varepsilon, u_\varepsilon)\} \subset L^1(\Omega; \{0,1\}) \times L^p\left(\Omega; \mathbb{R}^3\right)$ such that $(\chi_\varepsilon, u_\varepsilon) \to (\chi, u)$ w.r.t. the strong topology of $L^1(\Omega; \{0,1\}) \times L^p\left(\Omega; \mathbb{R}^3\right)$.

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- lower semicontinuity of the total variation

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- for the forces it is ok (because $u_\varepsilon \to u$ in $L^p\left(\Omega; \mathbb{R}^3\right)$)
- lower semicontinuity of the total variation
- Bulk energy $\rightsquigarrow$ use Decomposition Lemma for scaled gradients

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- for the forces it is ok (because $u_\varepsilon \to u$ in $L^p(\Omega; \mathbb{R}^3)$)
- lower semicontinuity of the total variation
- Bulk energy $\rightsquigarrow$ use Decomposition Lemma for scaled gradients
- Superadditivity of lim inf .

Gamma-Convergence result (Babadjian - Francfort)

Let $G_\varepsilon : L^p(\Omega;\mathbb{R}^3) \to [0,+\infty]$ be given by

$$G_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W(x_\alpha, (\nabla_\alpha u \,|\, \tfrac{1}{\varepsilon}\nabla_3 u\,))dx - \int_\Omega f \cdot u dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^3), \\ +\infty & \text{otherwise} \end{cases}$$

where $W : \omega \times \mathbb{R}^{3\times 3} \to [0,+\infty)$ is a Carathéodory function satisfying the following growth condition

$$\frac{1}{C}|\xi|^p - C \leq W(x_\alpha, \xi) \leq C(1 + |\xi|^p)$$

for a.e. $x_\alpha \in \omega$ and for every $\xi \in \mathbb{R}^{3\times 3}$ and some constant $C > 0$.

Gamma-Convergence result (Babadjian - Francfort)

Theorem

Under the above assumptions $\{G_\varepsilon\}$ Γ*-converges, with respect to* L^p*-strong convergence, to the functional*

$$G(u) := \begin{cases} 2\displaystyle\int_\omega \underline{W}(x_\alpha, \nabla_\alpha u)dx_\alpha - \int_{-1}^{1}\int_\omega f \cdot u dx_\alpha dx_3 & \text{if } u \in W^{1,p}(\omega;\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

where the function $\underline{W}$ *is given by*

$$\underline{W}(x_0, \overline{F}) \ := \inf_{\substack{\lambda>0 \\ \varphi \in W^{1,p}(Q;\mathbb{R}^3)}} \left\{ \frac{1}{2}\int_Q W(x_0, \overline{F} + \nabla_\alpha \varphi, \lambda \nabla_3 \varphi)dx : \varphi \equiv 0 \text{ on } \partial Q' \times (-1,1) \right\},$$

where $Q := Q' \times (-1, 1)$.

Upper bound

For every $\chi \in L^1(\Omega; \{0,1\})$

$$J(\chi, u) \leq \liminf_{\varepsilon \to 0^+} J_\varepsilon(\chi, u_\varepsilon) \text{ if } u_\varepsilon \to u \text{ in } L^p(\Omega; \mathbb{R}^3).$$

Study the asymptotic behaviour

$$\int_\Omega (\chi W_1(\nabla_\alpha u_\varepsilon, \frac{1}{\varepsilon}\nabla_3 u_\varepsilon) + (1-\chi) W_2(\nabla_\alpha u_\varepsilon, \frac{1}{\varepsilon}\nabla_3 u_\varepsilon)) dx - \int_\Omega f \cdot u_\varepsilon dx.$$

Define

$$W(x_\alpha, F) := \chi(x_\alpha) W_1(F) + (1 - \chi(x_\alpha))\, W_2(F).$$

$$Q\overline{W}\left(x_\alpha, \overline{F}\right) = \underline{W}\left(x_\alpha, \overline{F}\right),$$

where $\overline{W}$ is defined by $\overline{W}(x_\alpha, \overline{F}) := \inf_{c \in \mathbb{R}^3} W(x_\alpha, (\overline{F}|c))$.

- Applying Gamma convergence result of Babadjian and Francfort there exists a sequence $\{u_\varepsilon\}$ such that

$$\lim_{\varepsilon\to 0^+}\left(\int_\Omega W(x_\alpha,(\nabla_\alpha u_\varepsilon \frac{1}{\varepsilon}\nabla_3 u_\varepsilon))dx - \int_\Omega f\cdot u_\varepsilon dx\right)$$
$$\leq \quad 2\int_\omega Q\overline{W}(x_\alpha,\nabla_\alpha u)dx_\alpha - \int_{-1}^{1}\int_\omega f\cdot u dx_\alpha dx_3.$$

Upper bound

- Applying Gamma convergence result of Babadjian and Francfort there exists a sequence $\{u_\varepsilon\}$ such that

$$\lim_{\varepsilon\to 0^+}\left(\int_\Omega W(x_\alpha,(\nabla_\alpha u_\varepsilon \frac{1}{\varepsilon}\nabla_3 u_\varepsilon))dx - \int_\Omega f\cdot u_\varepsilon dx\right)$$
$$\leq \; 2\int_\omega Q\overline{W}(x_\alpha,\nabla_\alpha u)dx_\alpha - \int_{-1}^{1}\int_\omega f\cdot u dx_\alpha dx_3.$$

- The proof is concluded by observing that

$$\overline{W}(x_0,\overline{F}) = \chi(x_0)\overline{W_1}(\overline{F}) + (1-\chi(x_0))\overline{W_2}(\overline{F}) = \overline{V}(\chi(x_0),\overline{F}),$$
$$Q\overline{W}(x_0,\overline{F}) = Q\overline{V}(\chi(x_0),\overline{F}).$$

for every $(x_0,\overline{F})\in\omega\times\mathbb{R}^{3\times 2}$.

Relaxation p=1: settings

Let $F : BV\left(\Omega;\{0,1\}\right) \times W^{1,1}\left(\Omega;\mathbb{R}^d\right) \to [0,+\infty]$,

$$F(\chi,u) := \int_\Omega \left(\chi_E W_1 + (1-\chi_E) W_2\right)(\nabla u)dx - \int_\Omega f \cdot u\ dx + \mathrm{Per}\,(E;\Omega)$$

where $\mathrm{Per}\,(E;\Omega) = |D\chi_E|\,(\Omega)$, $\quad f \in L^\infty\left(\Omega;\mathbb{R}^d\right)$.
$W_i : \mathbb{R}^{dxN} \to \mathbb{R}$ continuous satisfying

$$\alpha\,|\xi| \leq W_i(\xi) \leq \beta\,(1+|\xi|)\,, \qquad \forall\ \xi \in \mathbb{R}^{d\times N},\ i=1,2, \tag{3}$$

for some $\alpha,\ \beta > 0$.

$$\inf_{\substack{u\in W^{1,1}(\Omega;\mathbb{R}^d)\\ \chi_E\in BV(\Omega;\{0,1\})}} \left\{\int_\Omega \left(\chi_E W_1(\nabla u) + (1-\chi_E) W_2\right)(\nabla u)dx + \mathrm{Per}(E;\Omega)\right.$$

$$\left. u = u_0 \text{ on } \partial\Omega \ \right\}$$

Relaxation p=1

Relaxation in $BV(\Omega;\{0,1\}) \times W^{1,1}(\Omega;\mathbb{R}^d)$

$$\mathcal{F}_{\mathcal{OD}}(\chi,u;A) = \inf\left\{ \varliminf_{n\to\infty} \left(\int_A (\chi_n W_1 + (1-\chi_n) W_2)(\nabla u_n)\,dx \right.\right.$$

$$\left. + |D\chi_n|(A) \right) : \{(\chi_n,u_n)\} \subseteq BV(A;\{0,1\}) \times W^{1,1}(A;\mathbb{R}^d)$$

$$\left. \begin{array}{l} \chi_n \overset{*}{\rightharpoonup} \chi \text{ in } BV(A;\{0,1\}) \\ u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d). \end{array} \right\}$$

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$$V(q,z) := qW_1(z) + (1-q)W_2(z).$$

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$$V(q,z) := qW_1(z) + (1-q)W_2(z).$$

- $QV \cdots$ quasiconvex envelope of V,
  $QV^\infty \cdots$ recession function of QV, namely,

$$QV^\infty(q,z) := \lim_{t\to\infty} \frac{QV(q,tz)}{t}.$$

Integral representation

Theorem

Let $\Omega \subset \mathbb{R}^N$ *be a bounded open set and let* $W_i : \mathbb{R}^{d\times N} \to [0, +\infty)$, $i = 1, 2$, *be continuous functions satisfying* (3). *Then for every* $(\chi, u) \in L^1(\Omega; \{0, 1\}) \times L^1(\Omega; \mathbb{R}^d)$

$$\mathcal{F}_{\mathcal{OD}}(\chi, u; A) = \begin{cases} \overline{F_{\mathcal{OD}}}(\chi, u; A) & \text{if } (\chi, u) \in BV(\Omega; \{0, 1\}) \times BV(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\overline{F_{\mathcal{OD}}}(\chi, u; A) \; : \; = \int_A QV(\chi, \nabla u)\, dx + \int_A QV^\infty\left(\chi, \frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_{J_{(\chi,u)}\cap A} K_2(\chi^+, \chi^-, u^+, u^-)\, d\mathcal{H}^{N-1}$$

Integral representation

The interaction is described through the following density

$$K_2\left(a,b,c,d,\nu\right) := \inf\left\{\int_{Q_\nu} QV^\infty\left(\chi\left(x\right),\nabla u\left(x\right)\right)dx + \left|D\chi\right|\left(Q_\nu\right) : \right.$$
$$\left.\left(\chi,u\right)\in \mathcal{A}_2(a,b,c,d,\nu)\right\},$$

$$\mathcal{A}_2\left(a,b,c,d,\nu\right) := \left\{\left(\chi,u\right)\in BV\left(Q_\nu;\{0,1\}\right)\times W^{1,1}\left(Q_\nu;\mathbb{R}^d\right) : \right.$$
$$\left(\chi(y),u\left(y\right)\right) = \left(a,c\right) \text{ if } y\cdot\nu = \frac{1}{2},$$
$$\left(\chi(y),u\left(y\right)\right) = \left(b,d\right) \text{ if } y\cdot\nu = -\frac{1}{2},$$
$$\left.\left(\chi,u\right) \text{ are } 1-\text{periodic in } \nu_1,\ldots,\nu_{N-1} \text{ directions}\right\},$$

for $(a,b,c,d,\nu)\in\{0,1\}\times\{0,1\}\times\mathbb{R}^d\times\mathbb{R}^d\times S^{N-1}$, with Q_ν the unit cube, centered at the origin, with axes parallel to $\{\nu_1,\nu_2,\ldots,\nu_{N-1},\nu\}$.

Integral representation

- The surface energy can be specialized as follows

$$
\begin{aligned}
&\int_{J_{(\chi,u)}} K_2(\chi^+,\chi^-,u^+,u^-,\nu_{(\chi,u)})\,d\mathcal{H}^{N-1}\\
&= \int_{J_u \setminus J_\chi} QV^\infty(\chi,(u^+-u^-)\otimes \nu_u)\,d\mathcal{H}^{N-1}\\
&+|D\chi|(\Omega \cap (J_\chi \setminus J_u)) + \int_{J_\chi \cap J_u} K_2(\chi^+,\chi^-,u^+,u^-,\nu_{(\chi,u)})\,d\mathcal{H}^{N-1}
\end{aligned}
$$

- **Key idea:** consider the couple (χ, u) as a unique BV-field U,

$$\nabla U = (0, \nabla u) \quad \text{and } D^c U = (0, D^c u)\,.$$

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1. Barroso-Bouchittée-Buttazzo-Fonseca, ARMA (1996):
Relaxation in BV for

$$u \in SBV \to \int_\Omega f(x, \nabla u)\, dx + \int_{J_u} g(x, [u](x), \nu_u)\, d\mathcal{H}^{N-1}.$$

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  2. Bouchitté-Fonseca-Mascarenhas ARMA (1998): deals with more general models.

Lower bound

- Observe that

$$\overline{F_{\mathcal{OD}}}(\chi, u) \geq \varliminf_{n\to\infty} \int_\Omega (\chi_n W_1(\nabla u_n) + (1-\chi_n) W_2(\nabla u_n))\, dx. \qquad (4)$$

For Bulk and Cantor parts we exploit Fonseca-Müller SIAM JoMA (1992) and ARMA (1993).

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- Observe that

$$\overline{F_{\mathcal{OD}}}(\chi, u) \geq \underline{\lim}_{n\to\infty} \int_\Omega (\chi_n W_1(\nabla u_n) + (1-\chi_n) W_2(\nabla u_n))\, dx. \qquad (4)$$

For Bulk and Cantor parts we exploit Fonseca-Müller SIAM JoMA (1992) and ARMA (1993).

- (4) could go much lower than $\overline{F_{\mathcal{OD}}}(\chi, u)$.

Upper bound

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- We compute "Bulk" and "Cantor" parts exploiting the "Global Method for Relaxation" by Bouchitté-Fonseca-Mascarenhas 1998.
- For the jump we compute it directly again mixing Ambrosio-Braides JMPA (1993) and Fonseca-Rybka Proc. Roy. Soc. Edinburgh (1992).

Comparison between the densities

- Clearly $K_2 \geq K$, where K is the following density

$$K(a,b,c,d,\nu) := \inf \left\{ \int_{Q_\nu} \left(QV^\infty(v(x), \nabla u(x)) + |\nabla v(x)| \right) dx : \right.$$
$$\left. (v,u) \in \mathcal{A}(a,b,c,d,\nu) \right\},$$

where

$$\mathcal{A}(a,b,c,d,\nu) := \left\{ (v,u) \in W^{1,1}\left(Q_\nu; \mathbb{R}^{1+d}\right) : \right.$$
$$(v(y), u(y)) = (a,c) \text{ if } y \cdot \nu = \tfrac{1}{2},\ (v(y), u(y)) = (b,d) \text{ if } y \cdot \nu = -\tfrac{1}{2}$$
$$\left. (v,u) \text{ are } 1 - \text{periodic in } \nu_1, \ldots, \nu_{N-1} \text{ directions} \right\}.$$

Comparison between the densities

- Recall Fonseca-Muller ARMA (1993)

$$\inf\left\{\varliminf_{n\to\infty}\left(\int_\Omega (v_n W_1 + (1-v_n) W_2)(\nabla u_n)\, dx + \int_\Omega |\nabla v_n| dx\right.\right.$$
$$: \{(v_n, u_n)\} \subset W^{1,1}(\Omega;[0,1]) \times W^{1,1}(\Omega;\mathbb{R}^d)$$
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- The result has been applied for $U\equiv(v,u)$, now with $U\in W^{1,1}(\Omega;\mathbb{R}^{1+d})$.

What about $K_2\leq K$?

More general models

- Relaxation in $SBV_0(\Omega;\mathbb{R}^m) \times W^{1,1}(\Omega;\mathbb{R}^d)$

$$
\begin{aligned}
\mathcal{F}(v,u) \quad = \quad & \inf\Bigg\{ \underline{\lim}_{n\to\infty} \left(\int_\Omega f(v_n, \nabla u_n)\, dx + \int_{J_{v_n}} g\left(v_n^+, v_n^-, \nu_{v_n}\right) d\mathcal{H}^{N-1} \right. \\
& \{u_n\} \subset W^{1,1}\left(\Omega;\mathbb{R}^d\right),\ \{v_n\} \subset SBV_0(\Omega;\mathbb{R}^m), \\
& \left. \begin{array}{l} u_n \to u \text{ in } L^1\left(\Omega;\mathbb{R}^d\right) \\ v_n \to v \text{ in } L^1(\Omega;\mathbb{R}^m) \end{array} \right\}
\end{aligned}
$$

More general models

Recall that

$$SBV_0(\Omega;\mathbb{R}^m) := \left\{ v \in BV(\Omega;\mathbb{R}^m) : D^c v = 0,\ \nabla v = 0,\ \mathcal{H}^{N-1}(J_v) < \infty \right\}$$

i.e. $v \in SBV_0(\Omega;\mathbb{R}^m)$ iff there exists a Borel partition $\{E_i\}$ of Ω and a sequence $\{v_i\} \subset \mathbb{R}^m$ such that

$$v = \sum_{i=1}^{\infty} v_i \chi_{E_i} \text{ a.e. } x \in \Omega,$$

$$\mathcal{H}^{N-1}(J_v \cap \Omega) = \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{H}^{N-1}(\partial^* E_i \cap \Omega),$$

$$(v^+, v^-, \nu_v) \equiv (v_i, v_j, \nu_i) \text{ a.e. } x \in \partial^* E_i \cap \partial^* E_j \cap \Omega,$$

ν_i being the unit normal to $\partial^* E_i \cap \partial^* E_j$.

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4. there exist $\alpha \in (0,1)$, and $C, L > 0$ such that

$$t|\xi| > L \ \Rightarrow \ \left| f^\infty(v,\xi) - \frac{f(v,t\xi)}{t} \right| \leq C\frac{|\xi|^{1-\alpha}}{t^\alpha}, \ \forall (v,\xi) \ \ \forall t > 0,$$

with f^∞ the recession function of f w.r.t. ξ.

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3. $g(\lambda,\theta,\nu) = g(\theta,\lambda,-\nu)$, for every $(\lambda,\theta,\nu) \in \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1}$.

General result

Theorem

Let $\Omega \in \mathcal{A}(\Omega)$ *and* $f : \mathbb{R}^m \times \mathbb{R}^{d \times N} \to [0, +\infty[$ *be a function satisfying* $(F_1) - (F_4)$ *and* $g : \mathbb{R}^m \times \mathbb{R}^m \times S^{N-1} \to [0, +\infty[$ *satisfying* $(G_1) - (G_3)$. *Then for every* $(v, u) \in L^1(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^d)$

$$\mathcal{F}(v, u; \Omega) = \begin{cases} \overline{F_0}(v, u; \Omega) & \text{if } (v, u) \in SBV_0(\Omega; \mathbb{R}^m) \times BV(\Omega; \mathbb{R}^d), \\ +\infty & \text{otherwise} \end{cases}$$

where $\overline{F_0} : SBV_0(\Omega; \mathbb{R}^m) \times BV(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, +\infty]$ *is given by*

$$\overline{F_0}(v, u; A) := \int_A Qf(v, \nabla u)\, dx + \int_A Qf^\infty\left(v, \frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_{J_{(v,u)} \cap A} K_3(v^+, v^-, u^+, u^-, \nu)\, d\mathcal{H}^{N-1}.$$

Energy density

Let $K_3 : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1} \to [0, +\infty[$ be defined as

$$K_3(a,b,c,d,\nu) := \inf \left\{ \int_{Q_\nu} Qf^\infty(v(x), \nabla u(x))dx \right.$$
$$\left. + \int_{J_v \cap Q_\nu} g(v^+(x), v^-(x), \nu(x)) d\mathcal{H}^{N-1} : (v,u) \in \mathcal{A}_3(a,b,c,d,\nu) \right\}$$

where

$$\mathcal{A}_3(a,b,c,d,\nu) := \Big\{ (v,u) \in SBV_0(Q_\nu; \mathbb{R}^m) \cap L^\infty(Q_\nu; \mathbb{R}^m) \times W^{1,1}(Q_\nu; \mathbb{R}$$
$$(v(y), u(y)) = (a,c) \text{ if } y \cdot \nu = \tfrac{1}{2},$$
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Auxiliary Lemma

Lemma

Let $Q := [0,1]^N$ *and*

$$v_0(y) := \begin{cases} a & \text{if } x_N > 0, \\ b & \text{if } x_N < 0, \end{cases} \qquad u_0(y) := \begin{cases} c & \text{if } x_N > 0, \\ d & \text{if } x_N < 0. \end{cases}$$

Let $\{v_n\} \subset SBV_0(\Omega : \mathbb{R}^m)$ *and* $\{u_n\} \subset W^{1,1}(\Omega; \mathbb{R}^d)$, *such that* $v_n \to v_0$ *in* $L^1(Q; \mathbb{R}^m)$ *and* $u_n \to u_0$ *in* $L^1(Q; \mathbb{R}^d)$. *There exists* $\{(\zeta_n, \xi_n)\} \in \mathcal{A}_3(a, b, c, d, e_N)$ *such that* $\zeta_n = v_0$ *on* ∂Q, $\zeta_n \to v_0$ *in* $L^1(Q; \mathbb{R}^m)$, $\xi_n = \rho_{i(n)} * u_0$ *on* ∂Q, $\xi_n \to u_0$ *in* $L^1(Q; \mathbb{R}^d)$

$$\begin{aligned} & \overline{\lim_{n\to\infty}} \left(\int_Q Qf(\zeta_n, \nabla \xi_n)\, dx + \int_{J_{\zeta_n} \cap Q} g(\zeta_n^+, \zeta_n^-, \nu_{\zeta_n})\, d\mathcal{H}^{N-1} \right) \\ \leq\ & \underline{\lim_{n\to\infty}} \left(\int_Q Qf(v_n, \nabla u_n)\, dx + \int_{J_{v_n} \cap Q} g(v_n^+, v_n^-, \nu_{v_n})\, d\mathcal{H}^{N-1} \right) \end{aligned}$$

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  - Jump part: comes from an explicit calculation mixing techniques of Ambrosio-Braides and Fonseca-Muller.

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$$+ \int_{J_v \cap J_u} K_3(v^+, v^-, u^+, u^-, \nu_{(v,u)})d\mathcal{H}^{N-1}.$$
- $\mathcal{R}g$ is the BV-elliptic envelope of g!

In progress: GAP

Let $W_i : \mathbb{R}^{d\times N} \to \mathbb{R}$ be continuous such that $\exists\ \alpha, \beta > 0$

$$\alpha|\xi|^p \le W_i(\xi) \le \beta(1+|\xi|^q) \quad \forall \xi \in \mathbb{R}^{d\times N}, \quad 1 < p < q < \frac{Np}{N-1}, \; i = 1,2$$

$$F(\chi, u) := \int_\Omega \chi_E W_1(\nabla u) + (1-\chi_E) W_2(\nabla u)\, dx + |D\chi_E|(\Omega).$$

$$\begin{aligned}\mathcal{F}(\chi, u; A) := \inf \Big\{ \liminf_{n\to\infty} F(\chi_n, u_n; A) : \{u_n\} \subset W^{1,q}\left(A; \mathbb{R}^d\right), \\ \{\chi_n\} \subset BV(A; \{0,1\})\ \ u_n \rightharpoonup u \text{ in } W^{1,p}\left(A; \mathbb{R}^d\right), \\ \chi_n \overset{*}{\rightharpoonup} \chi \text{ in } BV(A; \{0,1\}) \Big\}\end{aligned}$$