Exterior isoperimetry and a mesoscale regularity theorem

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joint with
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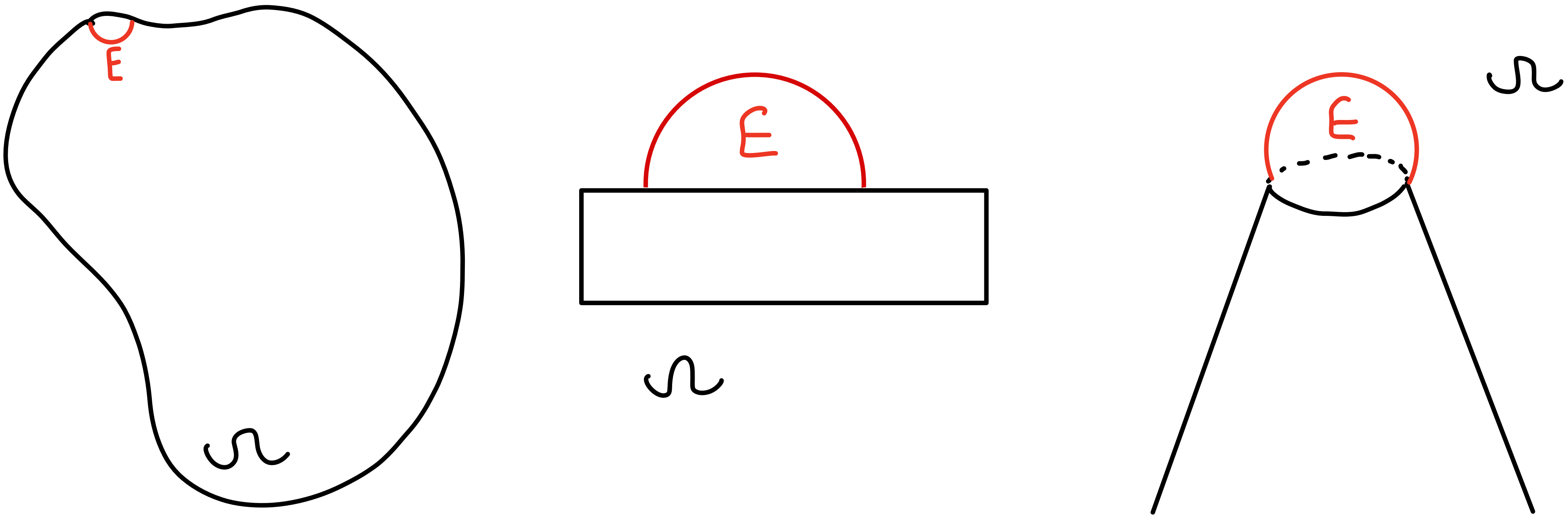
Background: Relative isoperimetry

$\Omega \subset \mathbb{R}^{n+1}$ open (container)

$$\inf\{P(E;\Omega) : |E| = v,\ E \subset \Omega\}$$



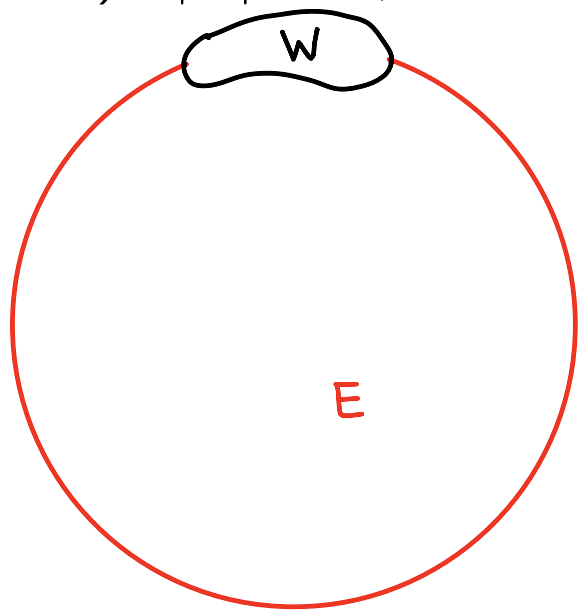
How does the shape/energy of minimizers depend on the container Ω?

Fall '10, Choe-Ghomi-Ritoré '07/Fusco-Morini '23, Fonseca-Fusco-Leoni-Morini '23

Exterior relative isoperimetry: results

$\Omega \subset \mathbb{R}^{n+1} = W^c$, W closed

$$\inf\{P(E; W^c) : |E| = v,\ E \subset W^c\}$$



- Maggi-N.: W compact
  - Resolution of exterior isoperimetric sets at large volumes v
  - Proof based on "mesoscale flatness criterion" that extends classical blow-up/blow-down criteria
- Fusco-Morini-Maggi-N.: W unbounded, convex
  - Characterize dependence of the minimum energy on W (for various v)

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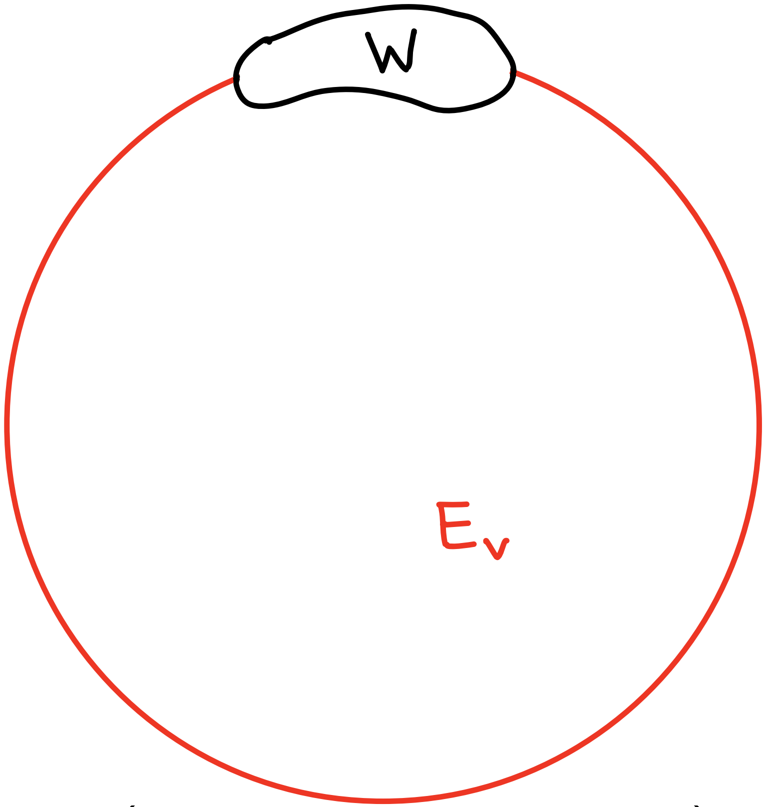
Outline

Exterior Isoperimetry

$W \subset \mathbb{R}^{n+1}$ compact

$$\mathcal{I}_{W^c}(v) = \inf\{P(E; W^c) : |E| = v,\ E \subset W^c\}$$

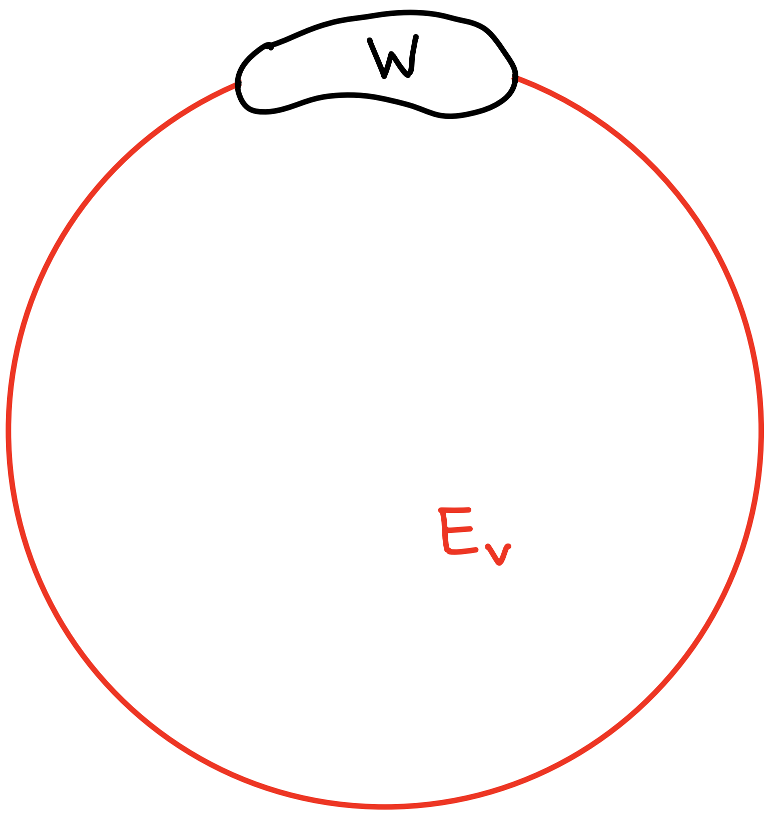


- Basic information
  - Minimizers E_v exist (use compactness of W), have constant mean curvature
  - $\nu_{E_v} \cdot \nu_W = 0$ if $\partial W \in C^{1,1}$ (Young's law)

Large Volume Exterior Isoperimetry

$W \subset \mathbb{R}^{n+1}$ compact, $v^{1/(n+1)} \gg \mathrm{diam}(W)$

$$\mathcal{I}_{W^c}(v) = \inf\{P(E; W^c) : |E| = v,\ E \subset W^c\}$$



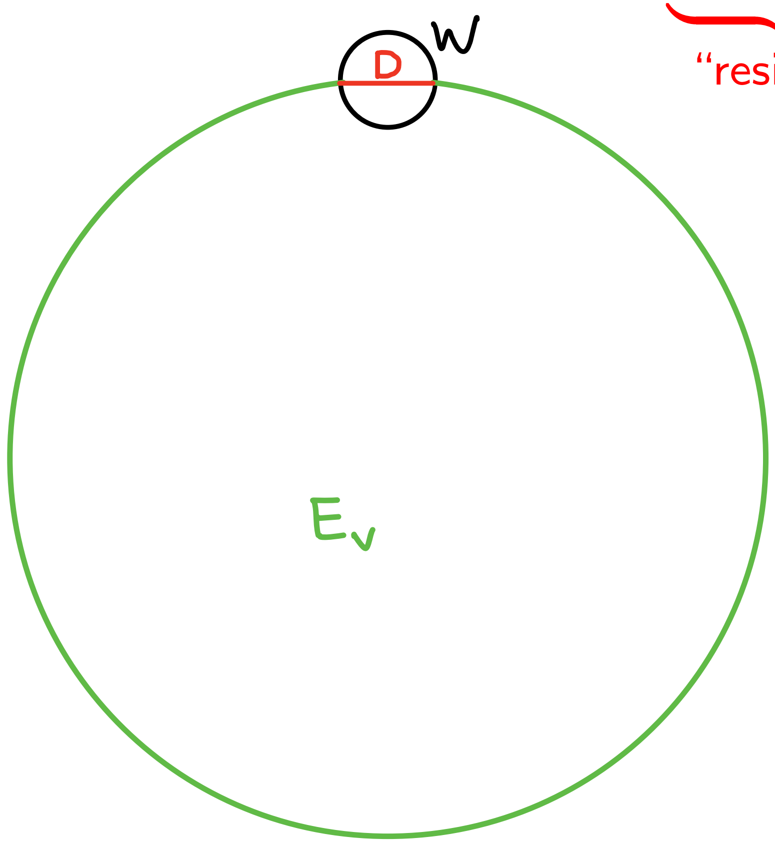
Question: Can we obtain *W-specific* energetic and geometric info for minimizers?

Outline

Does $\mathcal{I}_{W^c}$ admit an asymptotic expansion as $v \to \infty$?

Example: $W = B_1(0) \implies$ minimizers E_v of $\mathcal{I}_{W^c}(v)$ are large balls with $\partial E_v \cap \partial W \approx$ equator

$$\mathcal{I}_{W^c}(v) - c_n v^{n/(n+1)} \overset{v\to\infty}{\to} \underbrace{-\mathcal{H}^n(D)}_{\text{"residue"}} .$$



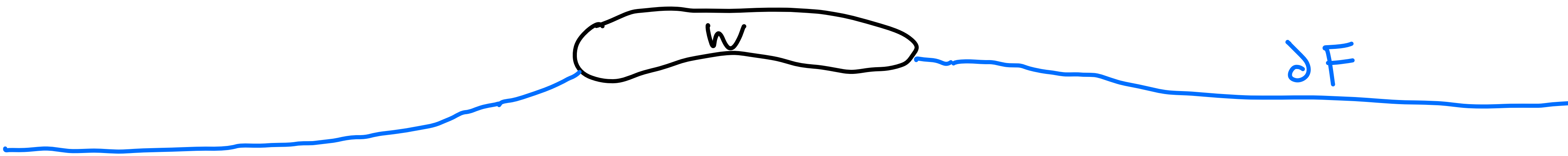
Here residue $= -$ (area of the largest slice of W).

The Asymptotic Expansion and Isoperimetric Residue

Theorem (Maggi-N.)

$$\mathcal{I}_{W^c}(v) - c_n v^{n/(n+1)} \overset{v\to\infty}{\longrightarrow} -\mathcal{R}(W)$$

- "$\mathcal{R}(W) \approx$ Plateau problem outside W with boundary data at ∞"

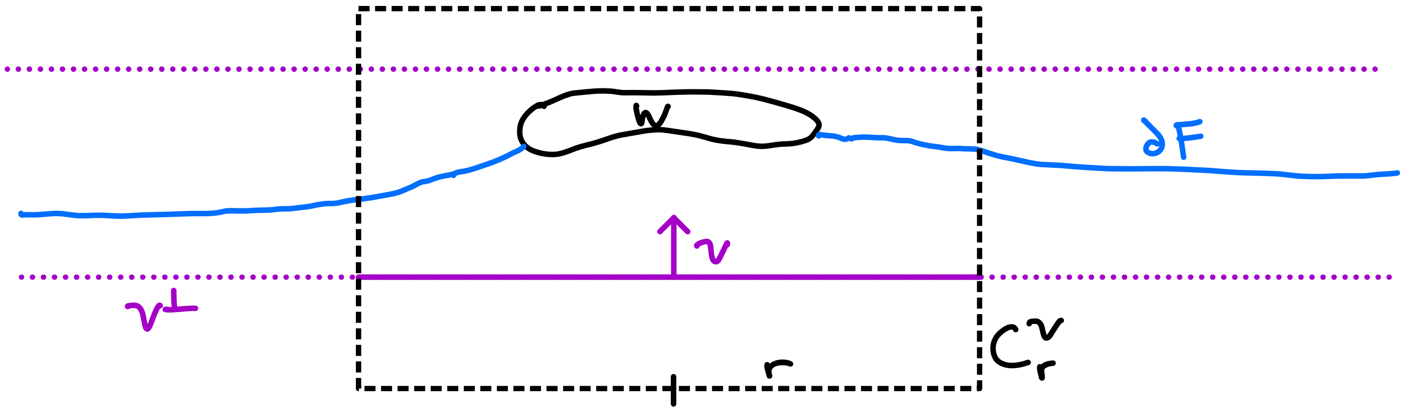


The Asymptotic Expansion and Isoperimetric Residue

Theorem (Maggi-N.)

$$\mathcal{I}_{W^c}(v) - c_n v^{n/(n+1)} \overset{v \to \infty}{\to} -\mathcal{R}(W)$$

- "$\mathcal{R}(W) \approx$ Plateau problem outside W with boundary data at ∞"
- A pair (F, ν), $\nu \in \mathbb{S}^n$ is admissible for $\mathcal{R}(W)$ if:
  - $\mathbf{proj}_{\nu^\perp}(\partial F) = \nu^\perp$ (weak version of graphicality)
  - ∂F is contained in a finite height slab parallel to $\nu^\perp$ (planar at ∞)

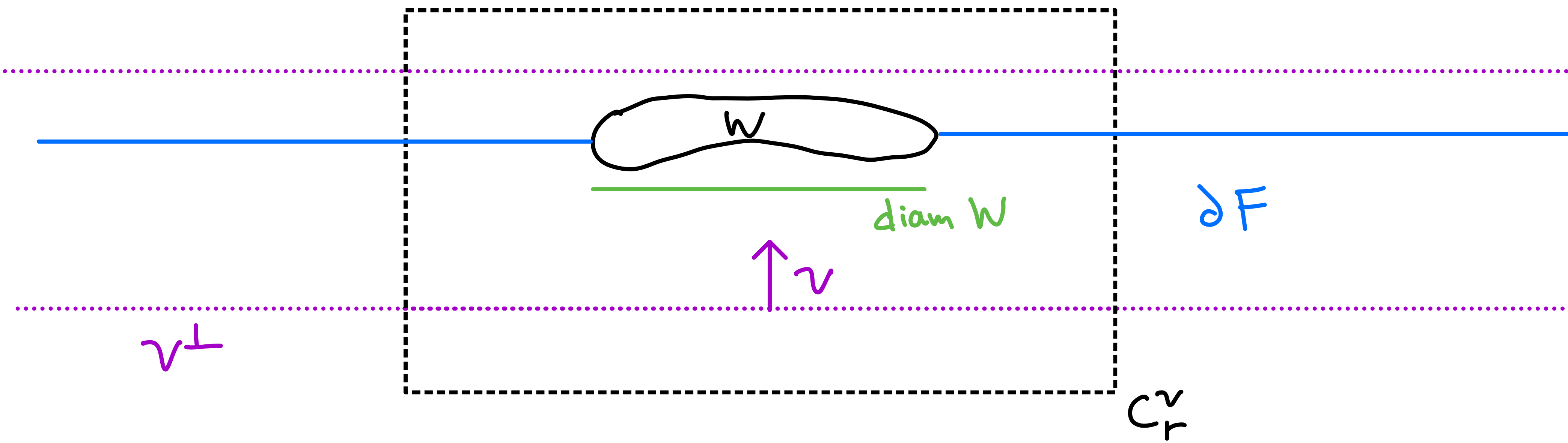


$$0 < \mathcal{R}(W) = \sup_{(F,\nu)} \underbrace{\lim_{r \to \infty} \omega_n r^n - P(F; C_r^\nu \setminus W)}_{\text{large scale cylindrical area "deficit"}}$$

Isoperimetric Residues - An Example

$W \subset \mathbb{R}^2$ compact, connected $\implies \mathcal{R}(W) = \operatorname{diam} W$

$$2r - P(F; C_r^\nu \setminus W) \leq \operatorname{diam} W$$

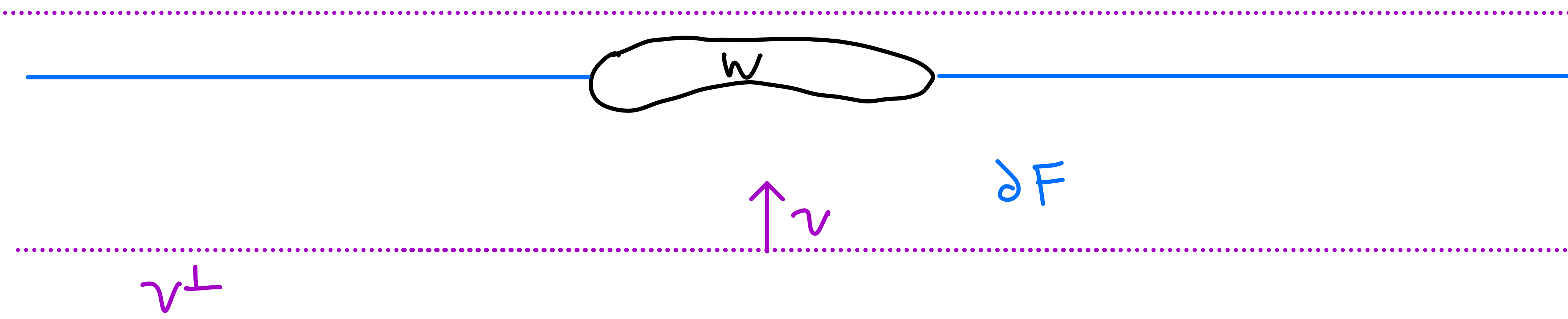


Further Results on Isoperimetric Residues

Theorem (Maggi-N. '22)

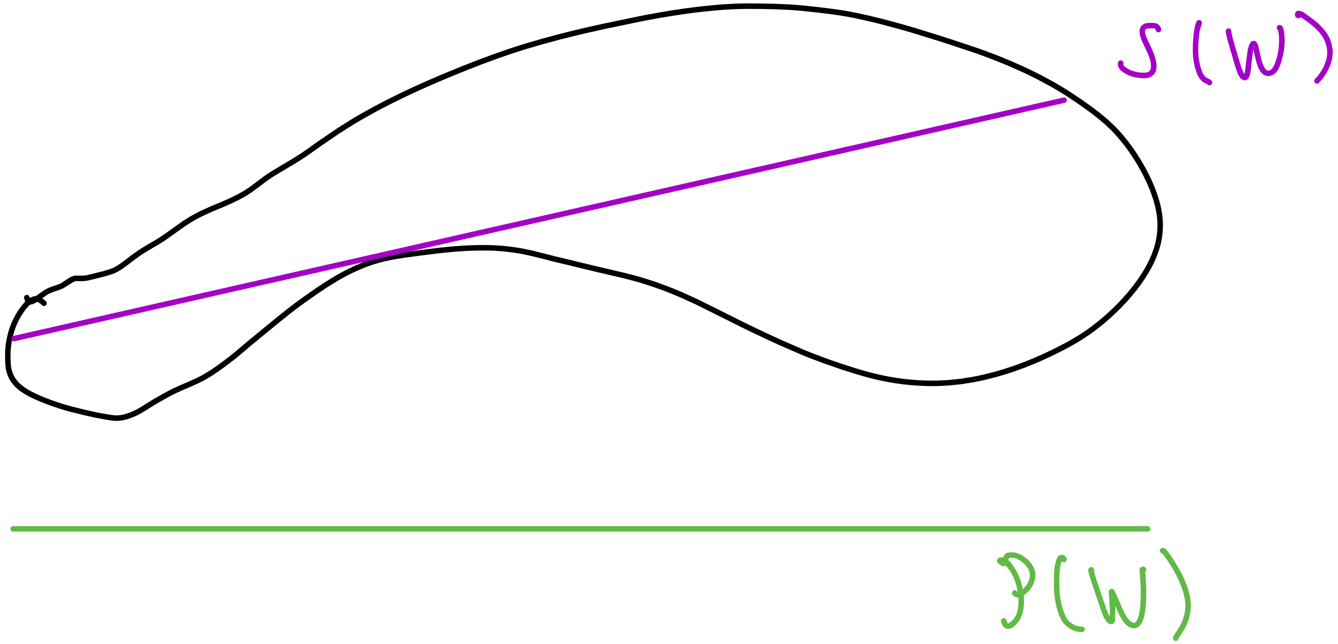
- *Any L^1_{loc}-limit F of minimizers for $\mathcal{I}_{W^c}(v)$ optimizes $\mathcal{R}(W)$*
- *$\partial F \setminus B_R(0)$ is the graph of a sol. $u : \nu^\perp \to \mathbb{R}$ of the minimal surface equation, with*

$$u(x) = a + \frac{b}{|x|^{n-2}} + \frac{c \cdot x}{|x|^n} + \mathrm{O}(|x|^{-n}) \quad \text{as } |x| \to \infty$$



Further properties of $\mathcal{R}(W)$

- $\mathcal{S}(W) \leq \mathcal{R}(W) \leq \mathcal{P}(W)$
  - $\mathcal{S}(W) := \sup\{\mathcal{H}^n(W \cap H) : H \text{ a hyperplane}\}$ (largest slice by a plane)
  - $\mathcal{P}(W) := \sup\{\mathcal{H}^n(\mathbf{proj}_{\nu^\perp}(W)) : \nu \in \mathbb{S}^n\}$ (largest proj. onto a plane)



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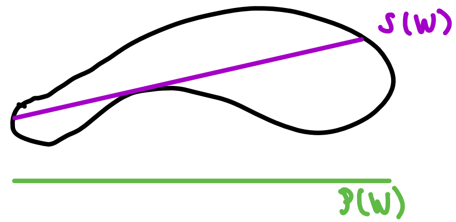
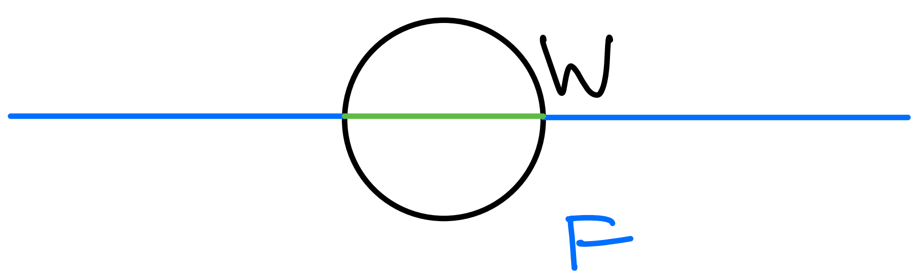
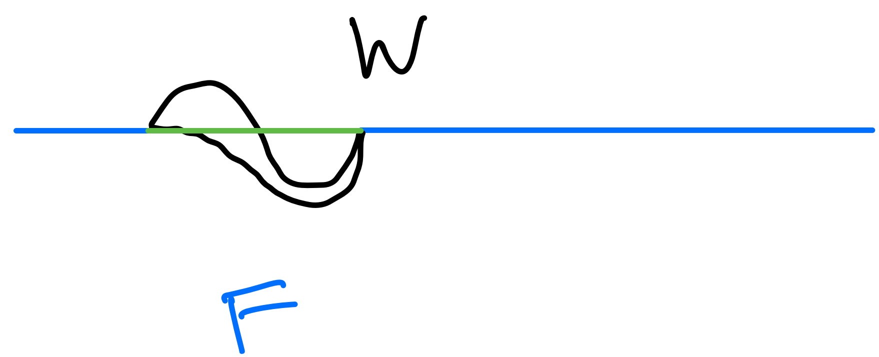


- Among sets of a fixed diameter, balls have the largest residue.
- Characterization of equality cases
  - W contains a $(n-1)$-dim. sphere S of diameter $\mathrm{diam}(W)$
  - $W^c \setminus$ (ext. hypersurface to S) has 2 unbounded connected components





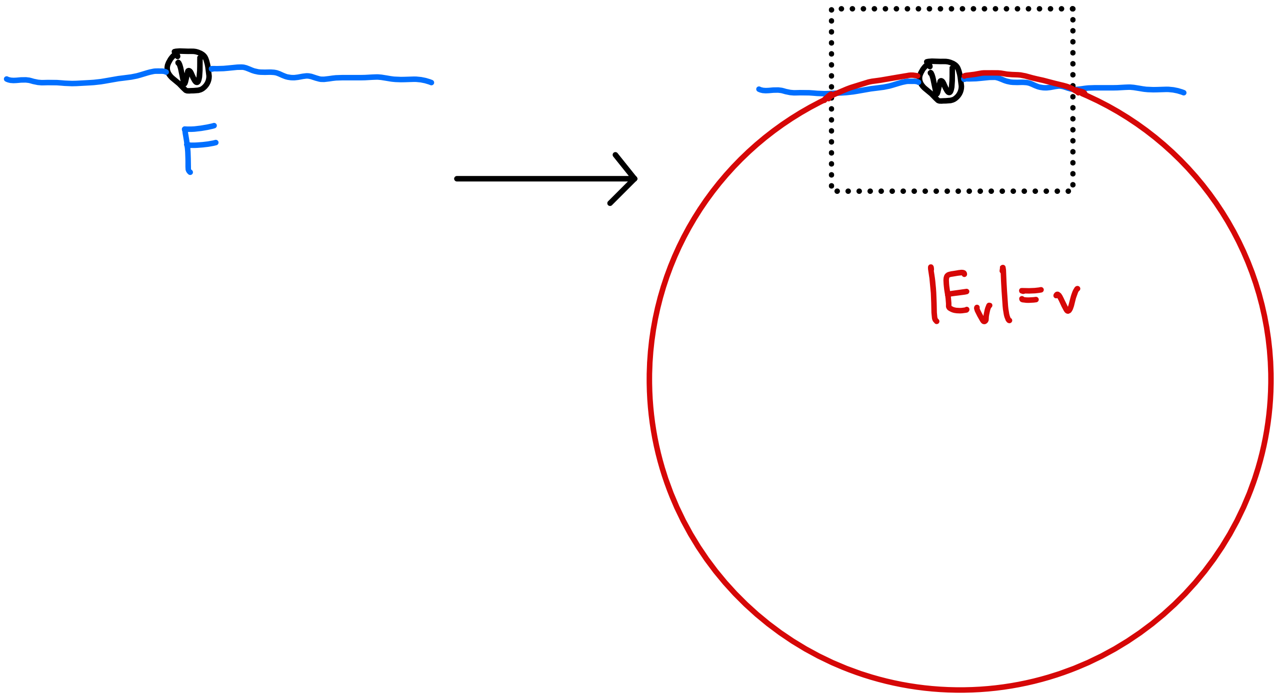
Asymptotic Expansion: Idea of Proof

Need to prove

$$\limsup_{v\to\infty} \mathcal{I}_{W^c}(v) - c_n v^{n/(n+1)} \leq -\mathcal{R}(W) \quad \text{and}$$

$$\liminf_{v\to\infty} \mathcal{I}_{W^c}(v) - c_n v^{n/(n+1)} \geq -\mathcal{R}(W)\,.$$

- Upper bound (ansatz): for asymptotically planar F, we can "glue" large balls to F



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- Upper bound (ansatz): for asymptotically planar F, we can "glue" large balls to F
- Lower bound: If every limit F of minimizers E_v is asymptotically planar (i.e. satisfies the ansatz), then

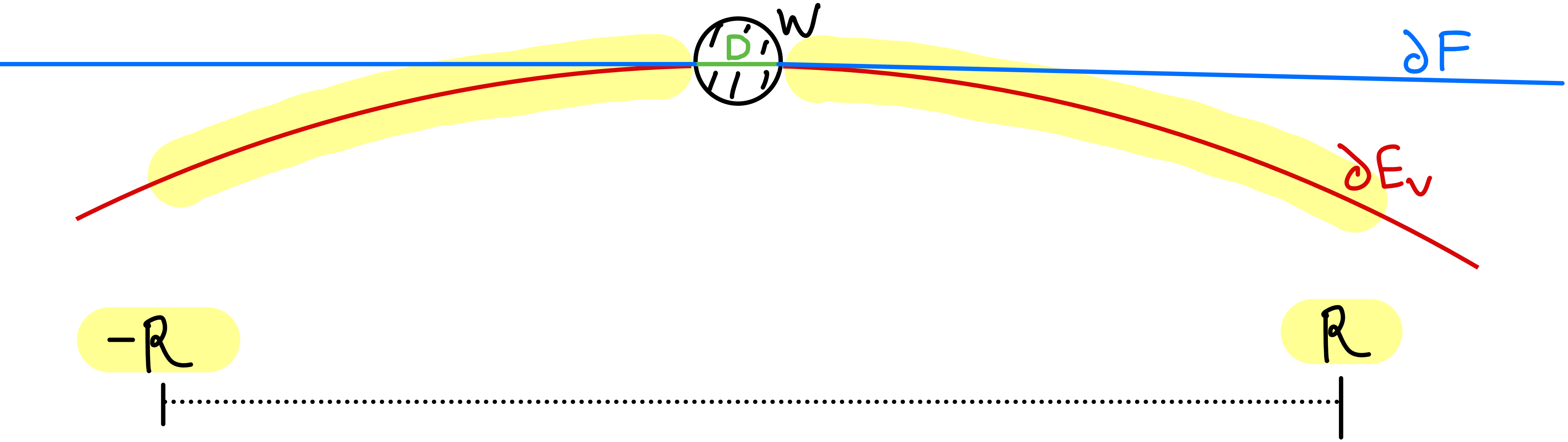
$$\liminf_{v\to\infty} \mathcal{I}_{W^c}(v) - c_n v^{n/(n+1)} \ge -\mathcal{R}(W)$$

  - To close the argument, we must study the behavior at infinity of such F.

Outline

Description of Minimizers Close to W

- Subsequential limits of minimizers E_v of $\mathcal{I}_{W^c}(v)$ converge **locally** to optimizers of $\mathcal{R}(W) \implies \partial E_v$ is **locally** a normal graph over an asymptotically planar minimal surface



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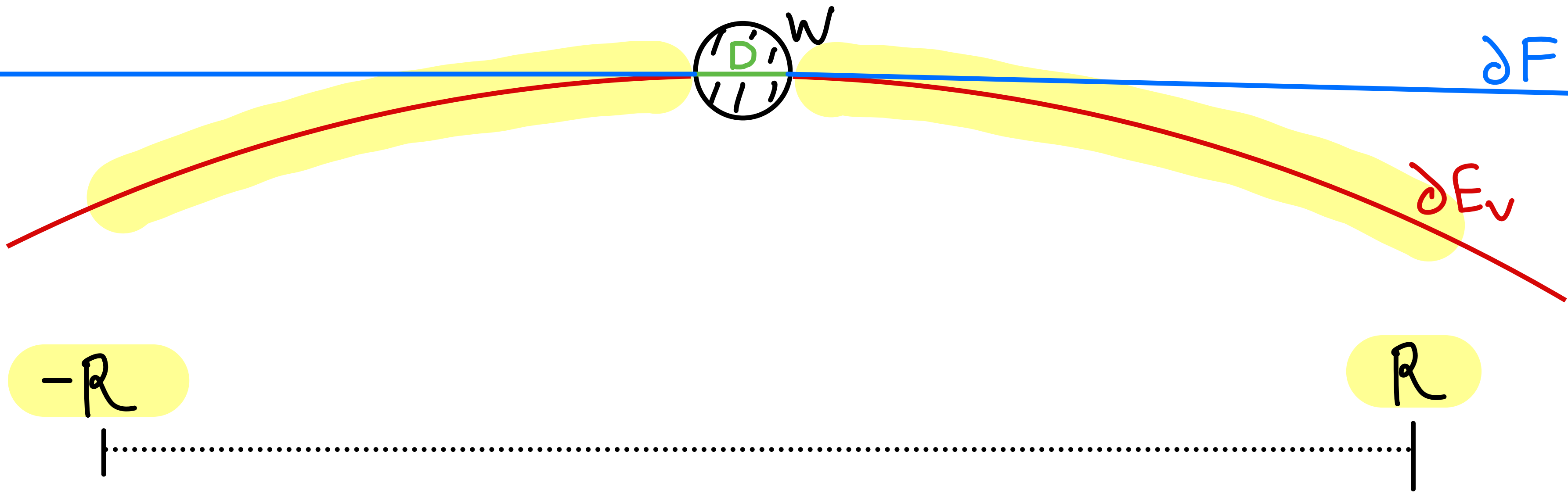


- $\mathbb{R}^3$ (physical case) + W^{int} is open with $C^{1,1}$ boundary
  $\implies \partial F$ is regular up to ∂W
  $\implies \partial E_v \cap B_R(0)$ can be parametrized over ∂F

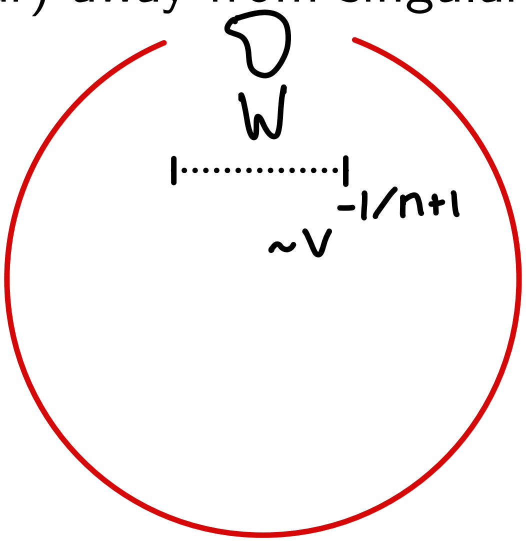
Cannot "push" graphicality over ∂F further outside $B_R(0)$ in a quantitative way.

Description of Minimizers Far from W

- Rescale $E_v \to E_v/v^{1/(n+1)}$

$$\implies P(E_v/v^{1/(n+1)}) \leq P(B_{r_n}(x)) + \text{small error}$$

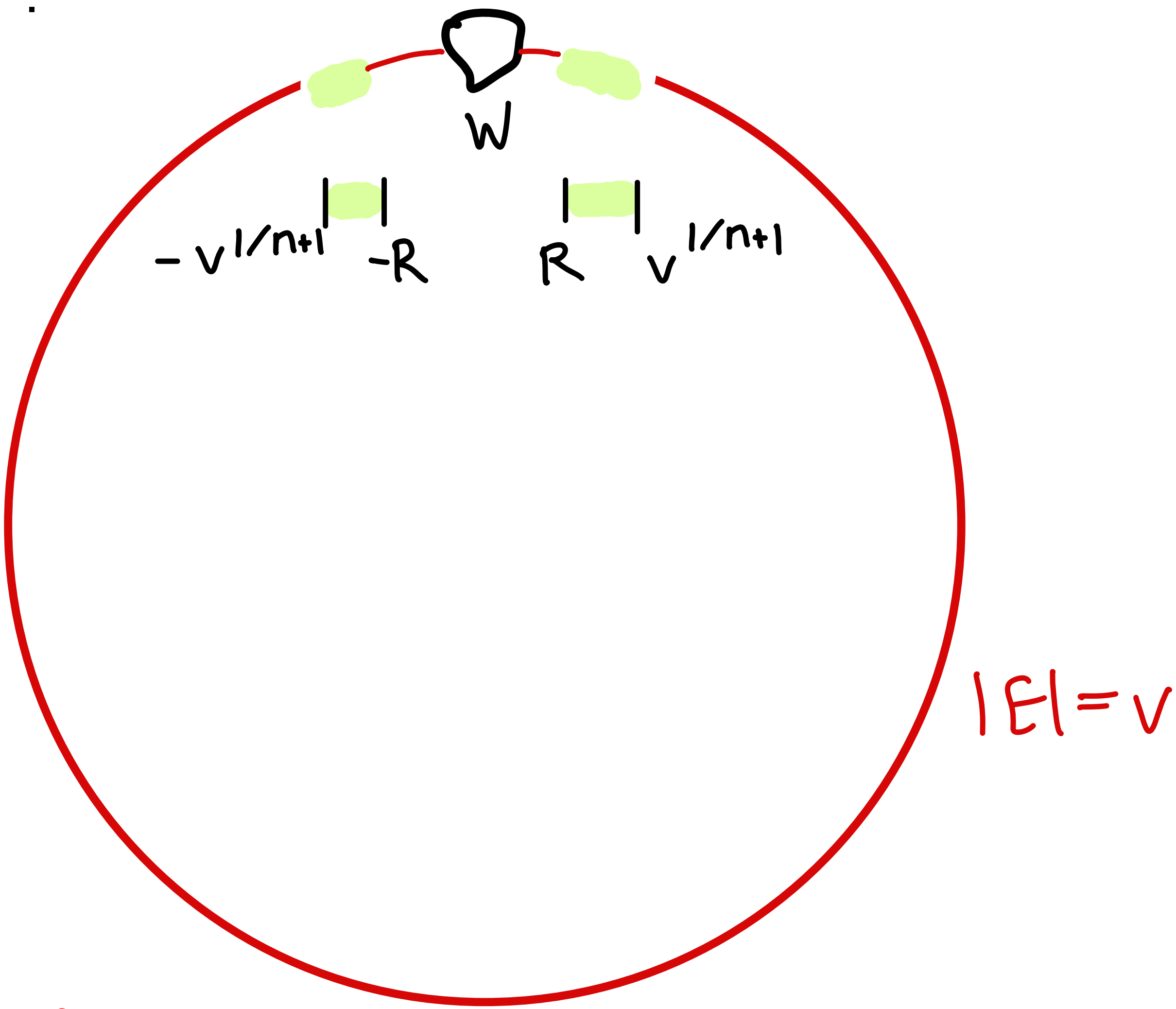
- Perimeter bound $\implies$
  - L^1-proximity to a large ball (quantitative isoperimetry)
  - C^2-parametrization over a large sphere (L^1-prox. + const. mean curvature). . . (far) away from singularities of $H_{\partial E}$, i.e. along W



Cannot "push" graphicality further inside $B_{v^{1/(n+1)}}(0)$ in a quantitative way. . .

What happens in the mesoscale?

Putting these two descriptions together leaves **the mesoscale** undescribed. . .



We need a new flatness criterion!

Outline

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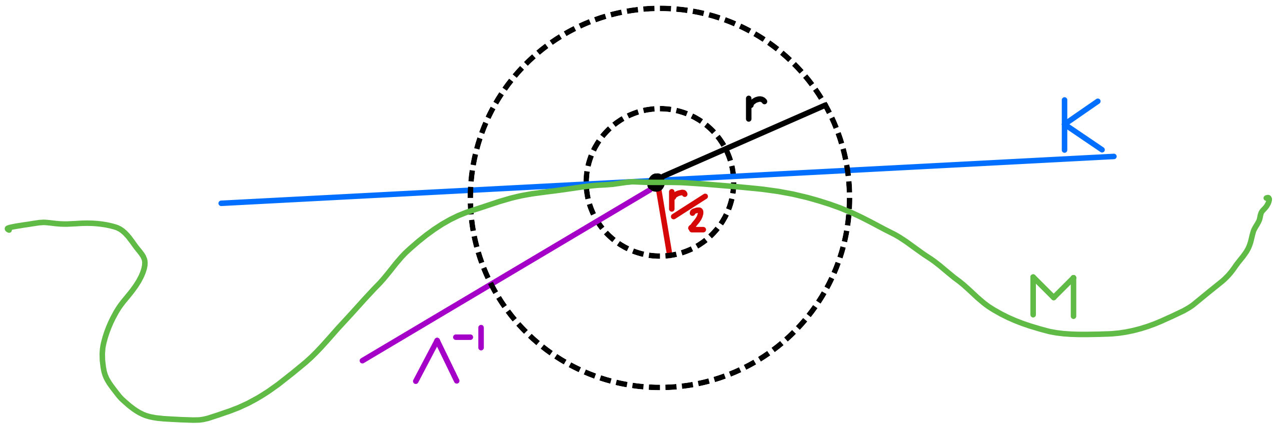
Background: Blow-up Flatness Criteria

Theorem (Allard '72)

$M \subset \mathbb{R}^{n+1}$ is a hypersurface, $|H_M|_\infty \leq \Lambda$, $x \in M$,

$$r < \varepsilon_0/\Lambda \quad \text{and} \quad \frac{\mathcal{H}^n(M \cap B_r(x))}{\omega_n r^n} - 1 < \varepsilon_0$$

then $M \cap B_{r/2}(x)$ is a C^1-graph over a hyperplane K.



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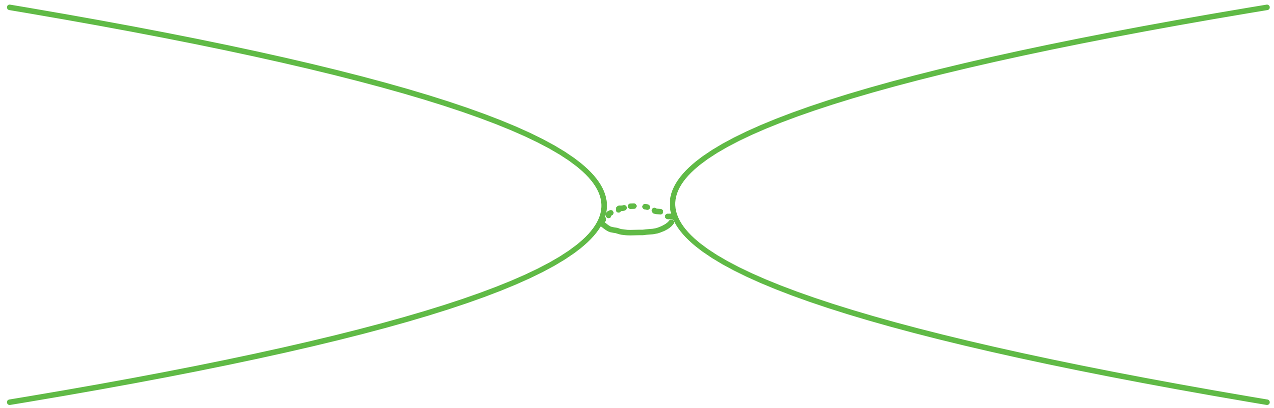
- "Area excess" assumption necessary - e.g. catenoids



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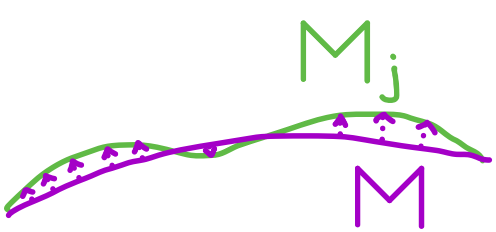
- "Area excess" assumption necessary - e.g. catenoids
- Improved convergence for hypersurfaces $M_j \to M$



$$\begin{cases} |H_{M_j}| \leq \Lambda, \ M \text{ smooth} \\ \mathcal{H}^n(M_j \cap B_\rho(x)) \to \mathcal{H}^n(M \cap B_\rho(x)) \\ \qquad \forall x, \text{ a.e. } \rho \end{cases} \implies M_j \xrightarrow{C^1_{\mathrm{loc}}} M$$

Background: Blow-down Flatness Criteria

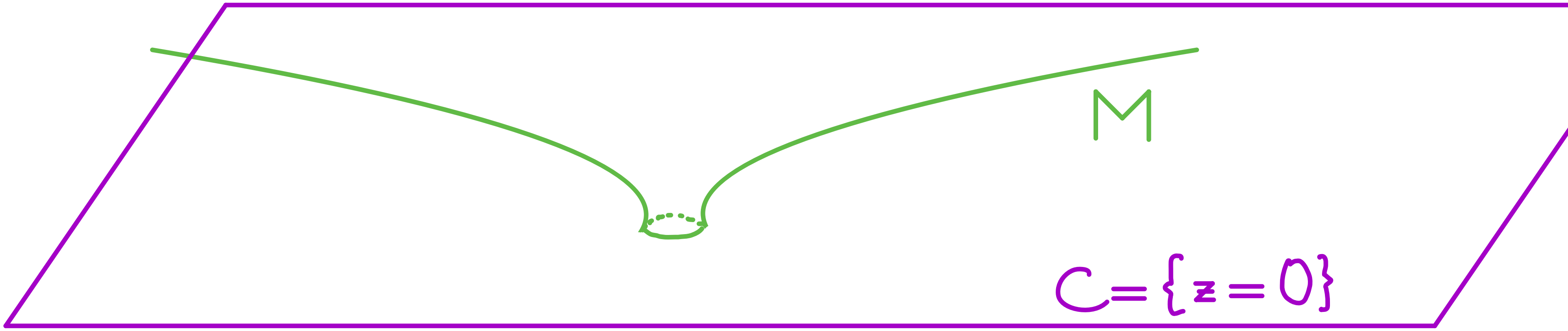
- M an exterior minimal surface in $\mathbb{R}^{n+1} \setminus B_R$
- C is a tangent cone at infinity for M if $\exists\, r_k \to \infty$ such that

$$M/r_k \to C.$$

(monotonicity of $\mathcal{H}^n(M \cap B_r)/r^n \implies$ all blow-downs are cones)

- Example: For $M =$ half-catenoid in $\mathbb{R}^3$,

$$M/r \overset{r\to\infty}{\to} \{z = 0\}.$$



Is the tangent cone at infinity unique?

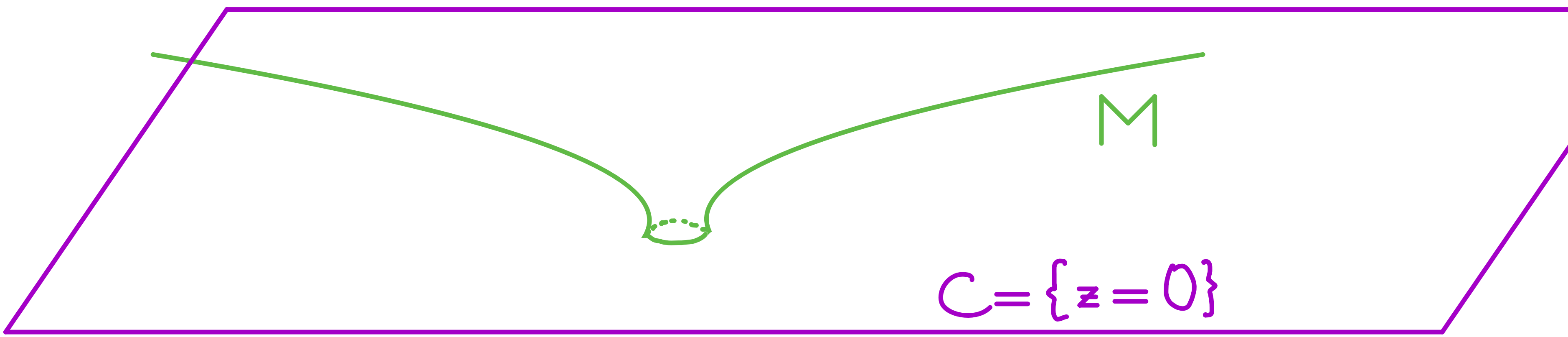
Background: Blow-down Flatness Criteria

Theorem (Allard-Almgren '81, Simon '83)

M minimal in $\mathbb{R}^{n+1} \setminus B_R$, $M/r_k \to K$ (hyperplane) for some $r_k \to \infty$

$$\Downarrow$$

$M/r \overset{r\to\infty}{\to} K$, and $M \setminus B_{r'}$ is a C^1-graph of some $u : K \setminus B_{r'} \to \mathbb{R}$.



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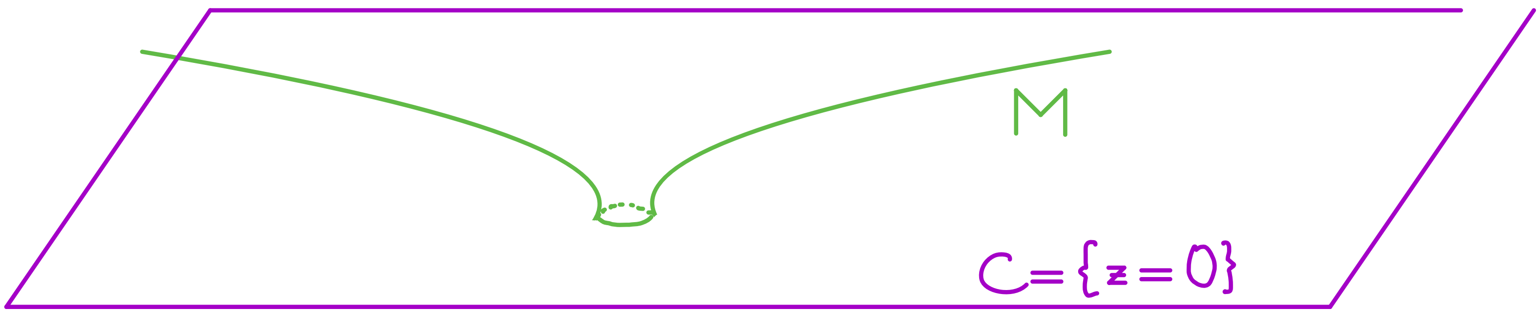
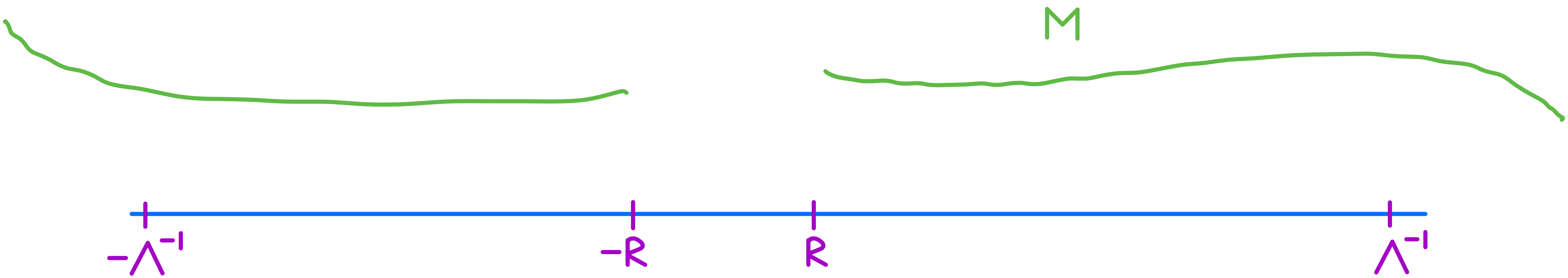


- Higher co-dimension & general (multiplicity one) cones?
  - A-A '81: integrable cones with iso. sing. $\implies$ poly. decay
  - S '83: integrable/non-integrable cones with iso. sing. $\implies$ poly./log decay
  - Engelstein-Spolaor-Velichkov '19: Uniqueness via (log)-epiperimetric inequality for almost area-minimizing currents

Mesoscale Flatness Criterion

- $\mathcal{M}_{R,\Lambda}$ = hypersurfaces $M \subset \mathbb{R}^{n+1} \setminus B_R$ with

$$\text{bdry}\, M \subset \partial B_R, \qquad |H_M|_\infty \le \Lambda \text{ on } B_{\Lambda^{-1}} \setminus \overline{B}_R$$



- Monotone quantity (may change signs)

$$\delta(r) := \frac{\mathcal{H}^n(M \cap (B_r \setminus B_R))}{r^n} - \omega_n + \underbrace{\text{bdry term}}_{\approx \frac{R^n}{r^n}} + \underbrace{\text{curvature term}}_{\approx \Lambda r}$$

Mesoscale Flatness Criterion

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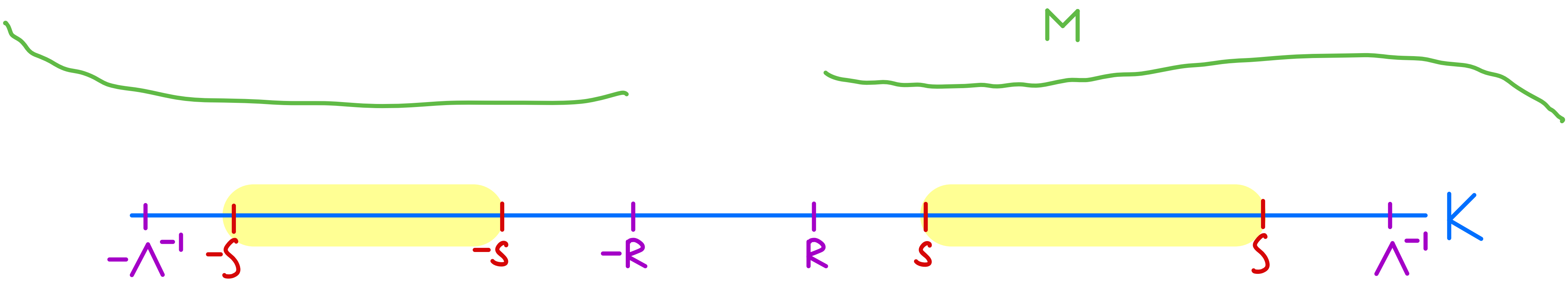

Theorem (Maggi-N. '22)

$\exists\, \varepsilon_0(n)$ *such that if* $M \in \mathcal{M}_{R,\Lambda}$ *and for some* $s, S > 0$,

$$\begin{cases} \dfrac{R}{\varepsilon_0} \leq s < \dfrac{S}{2} \leq \dfrac{\varepsilon_0}{\Lambda}, \\ |\delta(s)|, |\delta(S)| \leq \varepsilon_0, \end{cases}$$

then

$\exists$ *hyperplane* K *s.t.* M *is a* C^1*-graph over* K *on* $B_{S/2} \setminus B_s$.

Mesoscale Flatness Criterion: Further Remarks

Theorem

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- Statement contains other flatness criteria over hyperplanes
  - $R = 0 \longleftrightarrow$ Allard '72 $\delta \geq 0$
  - $\Lambda = 0 \longleftrightarrow$ Allard-Almgren '81, Simon '83, '85, '87 $\delta \leq 0$

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- Also covers surfaces for which δ can change sign and neither blow-ups/blow-downs exist

Idea of Proof

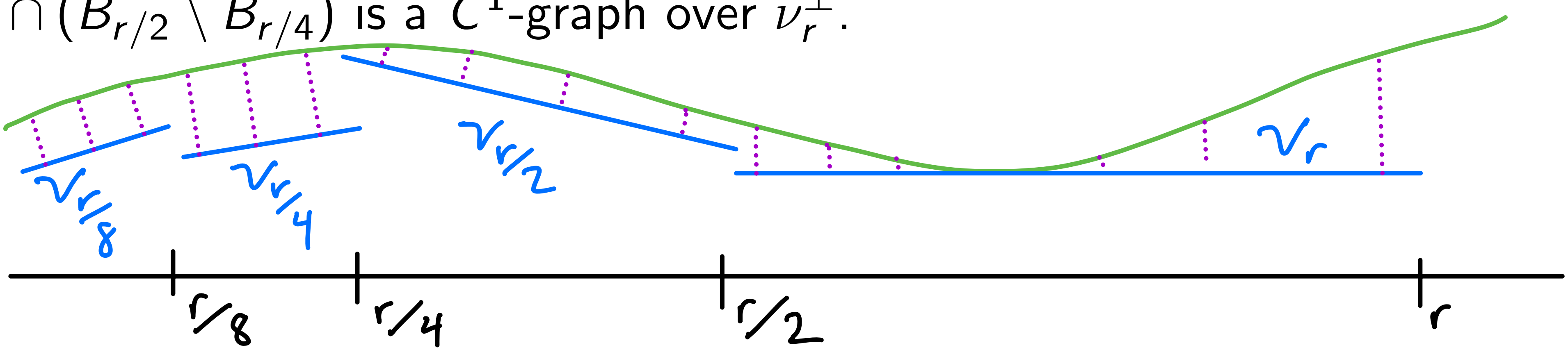
Starting point (improved convergence): If for some $r > 0$, a hypersurface $M \in \mathcal{M}_{R,\Lambda}$ satisfies

$$r \underbrace{|H_M|_\infty}_{=\Lambda} \ll 1 \qquad |\delta(r)|, |\delta(r/8)| \ll 1, \qquad \frac{R}{\varepsilon_0} \le \frac{r}{8} \le r \le \frac{\varepsilon_0}{\Lambda}$$

then $M \cap (B_{r/2} \setminus B_{r/4})$ is a small C^1-graph over a plane.

$$\Downarrow$$

On every dyadic annulus $B_{r/4} \setminus B_{r/2}$ there is some $\nu_r \in \mathbb{S}^n$ such that $M \cap (B_{r/2} \setminus B_{r/4})$ is a C^1-graph over $\nu_r^\perp$.



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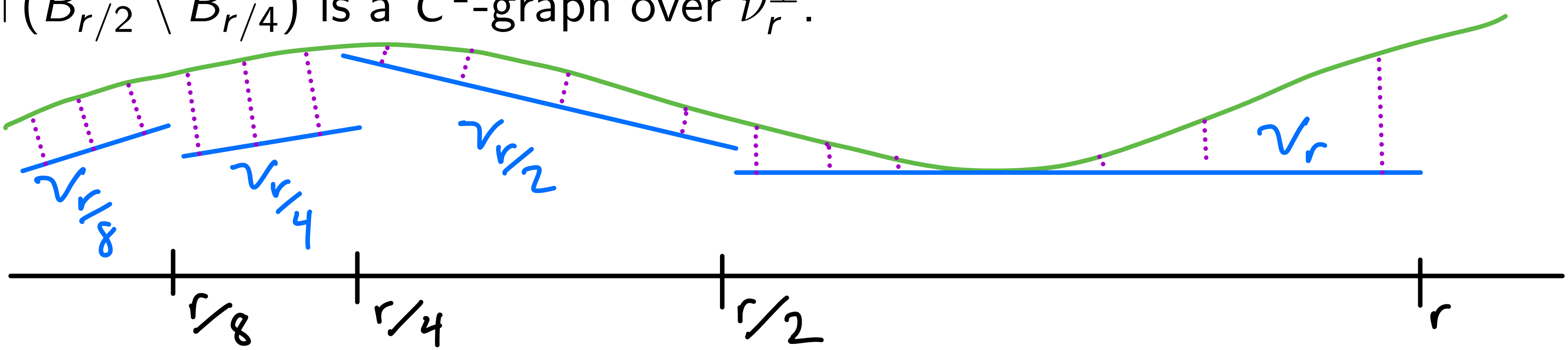


We need to show these normals do not oscillate too much as r varies.

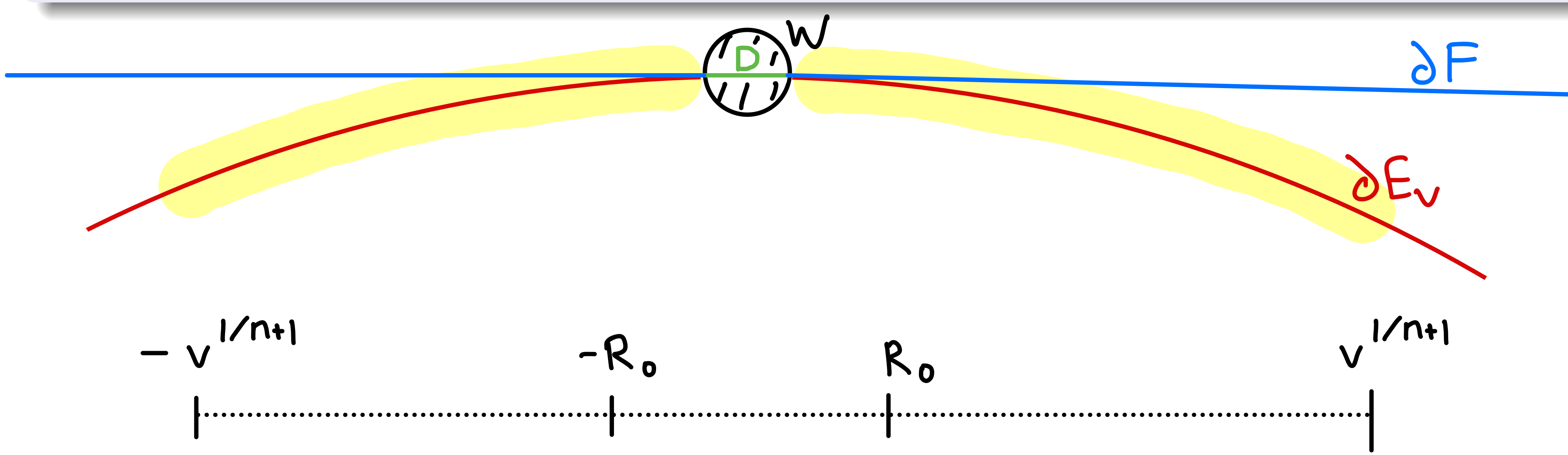
Outline

Geometric Resolution of Minimizers

Theorem (Maggi-N. '22)

There exists R_0, R_1 such that if E_v is a minimizer for large v

- $\partial E_v \cap (B_{R_1 v^{1/(n+1)}} \setminus B_{R_0})$ *is a normal graph over* ∂F, *where* (F, ν) *optimal for* $\mathcal{R}(W)$



Geometric Resolution of Minimizers

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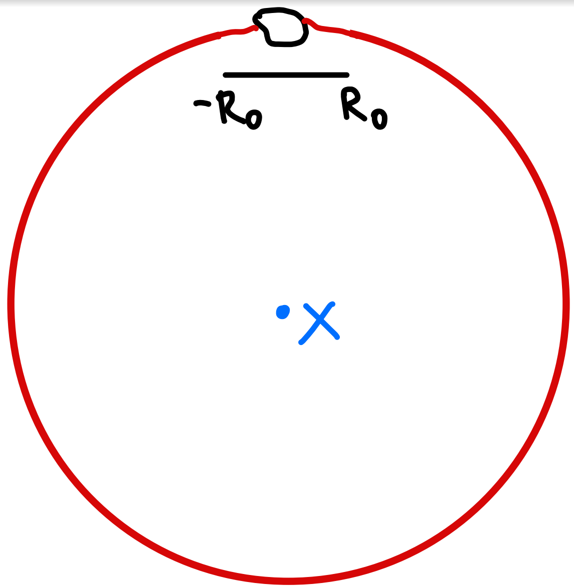
There exists R_0, R_1 such that if E_v is a minimizer for large v

- $\partial E_v \cap (B_{R_1 v^{1/(n+1)}} \setminus B_{R_0})$ *is a normal graph over* ∂F, *where* (F, ν) *optimal for* $\mathcal{R}(W)$
- E_v *determines* $x \in \mathbb{R}^{n+1}$ *such that* $\partial E_v \setminus B_{R_0}$ *is a normal graph over* $\partial B_{r_n v^{1/(n+1)}}(x)$



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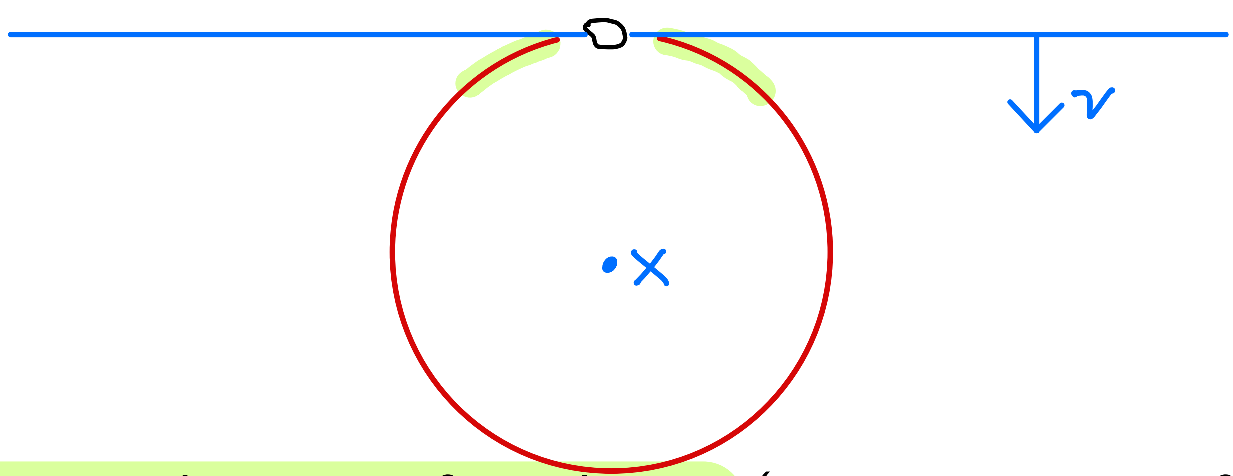

- Overlapping domains of resolution (i.e. agreement of descriptions of ∂E_v close/far from W)
  - $|x/|x| - \nu| \to 0$

Outline

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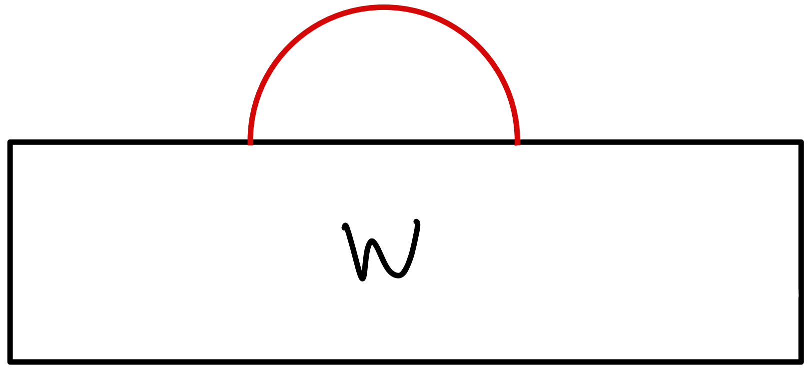
Background

- Literature: Choe-Ghomi-Ritoré '07, Fusco-Morini '21
  - W closed, convex with non-empty interior:

$$P(E;W^c) \geq (n+1)\left(\frac{\omega_{n+1}}{2}\right)^{n/(n+1)} |E|^{n/(n+1)}$$

with "$=$" iff E is a half-ball supported on a facet of W.



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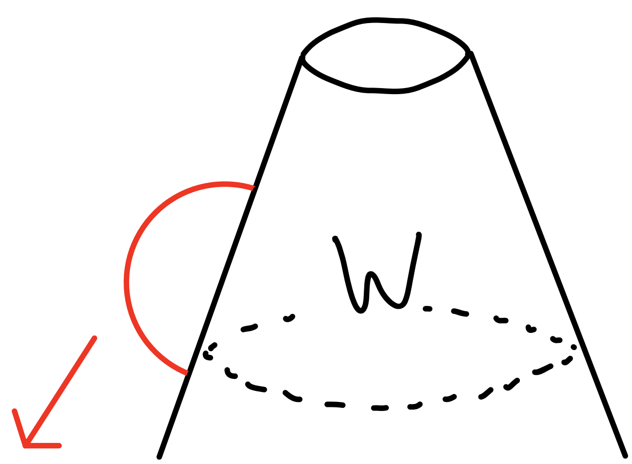
with "$=$" iff E is a half-ball supported on a facet of W.
- Minimizers may not exist when W is unbounded (Fonseca-Fusco-Leoni-Morini '23)
  - Minimizing sequences may run off to infinity in search of facets
  - Upshot: The isoperimetric problem outside an unbounded convex set W is strongly influenced by the large scale behavior of W.



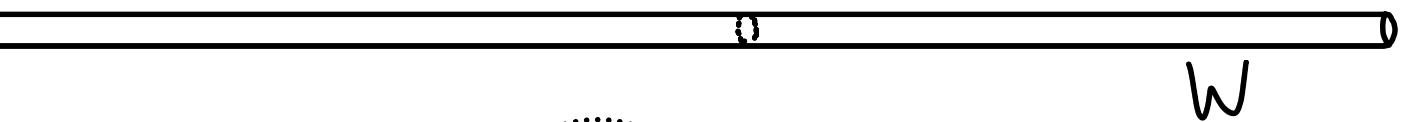
Outline

Asymptotic Dimension of a Convex Set W

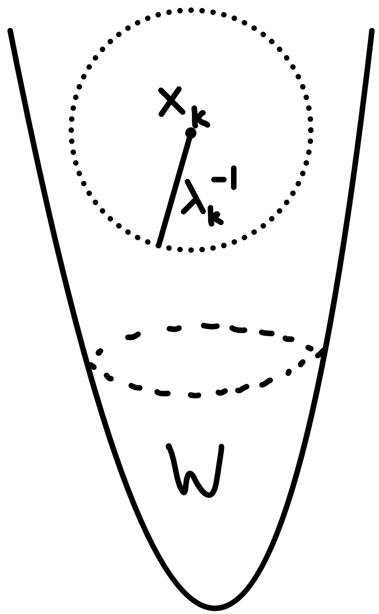
$d^*(W) \approx$ largest affine dimension of any sequence of blow-downs+translations of W

More precisely,

$d^*(W) =$ max. affine dimension of any limit $\lambda_k(W - x_k)$, $\lambda_k \to 0$, $x_k \in W$



$d^*(W) = 1$



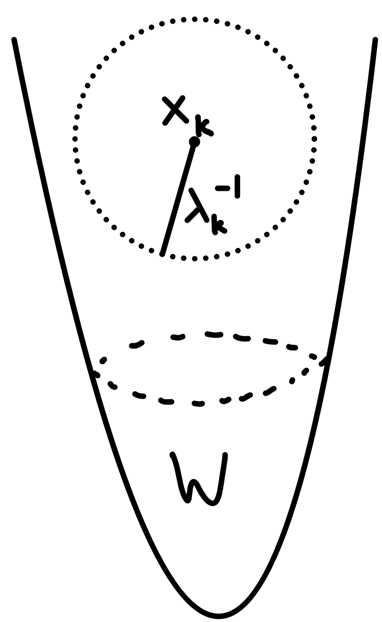
$d^*(W) = 3$

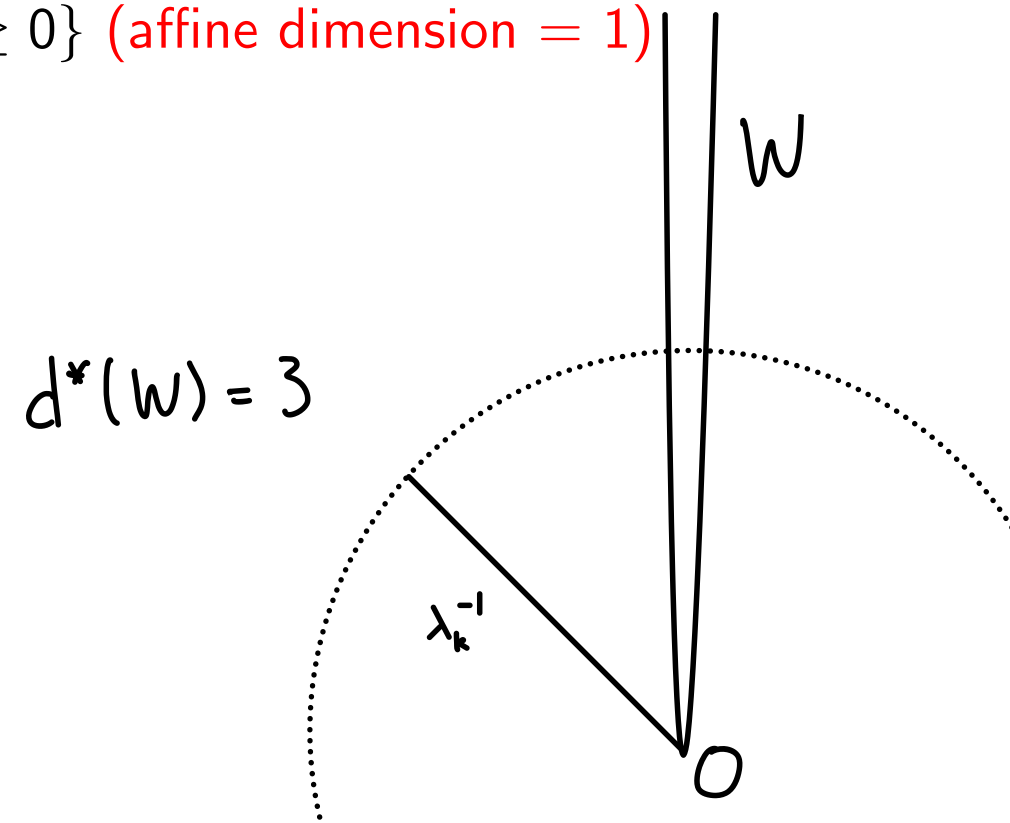
$d^*(W) = k \implies W \subset \mathbb{R}^k \times W'$ for compact $W' \subset \mathbb{R}^{n+1-k}$

Comparison with other notions

$d^*(W) =$ max. affine dimension of any limit $\lambda_k(W - x_k)$, $\lambda_k \to 0$, $x_k \in W$

- If $x_k = x \in W$, we recover the affine dimension of the classical **recession cone**.
- Example: $W = \{z \geq x^2 + y^2\}$
  - Recession cone $= \{(0,0,z) : z \geq 0\}$ (affine dimension $= 1$)
  - $d^*(W) = 3$





$d^*(W)$ and the isoperimetric profile

Theorem (Fusco-Maggi-Morini-N.)

Let $W \subset \mathbb{R}^{n+1}$ be unbounded and convex.

- *If $d^*(W) \geq n$, then the isoperimetric profile $\mathcal{I}_{W^c}(v)$ of W coincides with that of a halfspace. (half-balls optimal at all volumes)*
- *If $d^*(W) < n$, then*

$$\lim_{v \to \infty} \mathcal{I}_{W^c}(v)/v^{n/(n+1)} = P(B)\,,$$

(i.e. $\approx$ balls are optimal for large volumes).

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- Corollary: If W is strictly convex and $d^*(W) \geq n$, then there is non-existence for every volume.
- When $d^*(W) < n$, we have preliminary estimates on the isoperimetric residues, i.e. the second order term in the energy expansion in v.

Future directions

- Other mesoscale flatness criteria
  - In arbitrary co-dimension for multiplicity one cones with isolated singularity at 0
  - Higher multiplicity cones?
  - Cones with non-isolated singularities?

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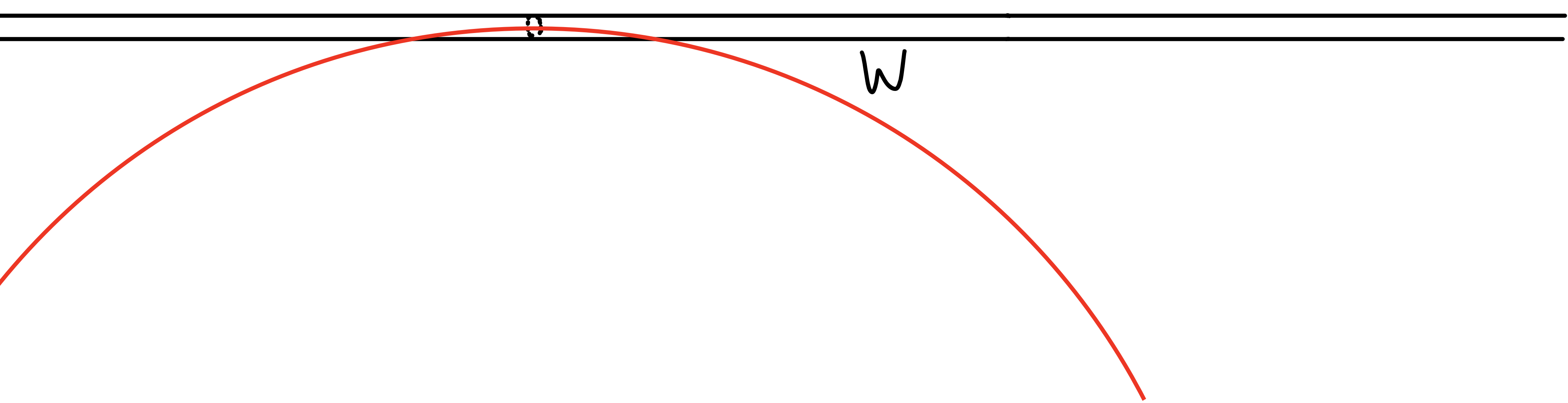
  - Example: $W = \mathbb{R} \times \{y^2 + z^2 \leq 1\} \subset \mathbb{R}^3$



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Thank you!