Periodic Homogenization of Non-variational Interface Motions

Peter Morfe

University of Chicago

Center for Nonlinear Analysis Seminar
Carnegie Mellon University
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Overview

(1) Motivation/Statement of the Problem

(2) Main Result

(3) Averaging in Codimension One

(4) Making Sense of the Homogenized Equation

(5) Conclusion/Future Directions

Necessary preliminaries from geometry

Given an open set E with smooth boundary, we let

$$\begin{aligned} n_{\partial E} = &\text{ normal vector to } \partial E \\ A_{\partial E} = &\text{ second fundamental form associated with } \partial E \\ &\quad (\text{symmetrix matrix-valued function on } \partial E) \end{aligned}$$

If $n_{\partial E}$ is "first derivative of ∂E," then $A_{\partial E}$ is "second derivative/Hessian matrix" of ∂E
(It measures how how curved ∂E is)

Can compute $A_{\partial E} = -Dn_{\partial E}$

Interface motion in heterogeneous media

Question of interest in materials science/math: how do microscopic heterogeneities affect macroscopic motion of interfaces (e.g. phase/grain boundaries, domain walls)?

A toy model of solidification:

(i) $E^\epsilon(t)$ "melt" phase at time t, (ii) $\mathbb{R}^d \setminus E^\epsilon(t)$ "solid" phase at time t
(iii) phase boundary $\partial E^\epsilon(t)$ moves with normal velocity

$$V_{\partial E^\epsilon(t)} = \mathrm{tr}\left(a(\epsilon^{-1} x, n_{\partial E^\epsilon(t)}) A_{\partial E^\epsilon(t)} \right)$$

$a : \mathbb{R}^d \times S^{d-1} \to \mathbb{R}^{d \times d}_{\mathrm{sym}}$ models the heterogeneities in the material, $\lambda \mathrm{Id} \leq a \leq \Lambda \mathrm{Id}$, $\mathbb{Z}^d$-periodic, i.e. $a(x+k, n) = a(x, n)$ for $k \in \mathbb{Z}^d$

Parabolic scaling: $E(t) = \epsilon^{-1} E^\epsilon(\epsilon^2 t)$ satisfies same equation with $\epsilon = 1$

Macroscopic behavior? $\iff$ Limit of $E^\epsilon(t)$ as $\epsilon \to 0$?

Non-variational model: energy dissipation is missing

Main Result: Homogenization & Discontinuities

Recall the geometric flow of interest:

$$V_{\partial E^{\epsilon}(t)} = \mathrm{tr}\left(a(\epsilon^{-1}x, n_{\partial E^{\epsilon}(t)}) A_{\partial E^{\epsilon}(t)}\right) \tag{1}$$

Notation: $S^{d-1} = \{e \in \mathbb{R}^d \mid \|e\| = 1\}$, $\mathbb{R}\mathbb{Z}^d = \{\alpha k \mid \alpha \in \mathbb{R},\ k \in \mathbb{Z}^d\}$.

Theorem (M. '20)

There is a homogenized tensor $\bar{a} : S^{d-1} \setminus \mathbb{R}\mathbb{Z}^d \to \mathbb{R}^{d \times d}_{sym}$ *such that*

(i) *Given any* $E \subseteq \mathbb{R}^d$ *open, there is a unique level set solution* $\{E(t)\}_{t \geq 0}$ *of the homogenized flow*

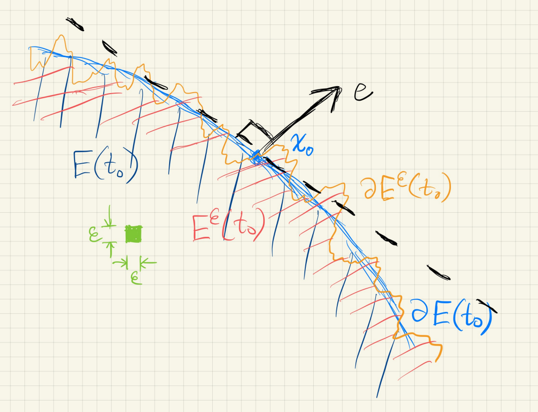
$$V_{\partial E(t)} = tr\left(\bar{a}(n_{\partial E(t)}) A_{\partial E(t)}\right), \quad E(0) = E$$

(ii) *If* $\{E^{\epsilon}(t)\}_{t \geq 0}$ *is the level set solution of* (1) *with initial set* $E^{\epsilon}(0) = E$, *then* $E^{\epsilon}(t) \to E(t)$ *as* $\epsilon \to 0$.

(iii) *If* $d \geq 3$, *then* $\bar{a}$ *generically has no well-defined limit at any direction* $e \in S^{d-1} \cap \mathbb{R}\mathbb{Z}^d$. *(If* $d = 2$, *then* $\bar{a}$ *extends continuously to* S^1 *in general.)*

Heuristics

Assume $E^{\epsilon}(t) \to E(t)$ as $\epsilon \to 0$. **Goal:** derive equation for $\{E(t)\}_{t \geq 0}$



Let $x_0 \in \partial E(t_0)$

Suppose $\partial E(t_0)$ is smooth there

Recall: close to x_0, the boundary looks like the graph of a paraboloid

Set $e = n_{\partial E(t_0)}(x_0)$

To first order, $\partial E^{\epsilon}(t)$ sees a along the black hyperplane pictured

Question: Average of periodic function on a hyperplane?

Averaging over Planes

Let $f : \mathbb{R}^d \to \mathbb{R}$ be continuous and $\mathbb{Z}^d$-periodic, i.e. $f(x+k) = f(x)$ if $k \in \mathbb{Z}^d$

Given an $e \in S^{d-1}$ and $s \in \mathbb{R}$, define $H(s,e) = \{x \in \mathbb{R}^d \mid x \cdot e = s\}$.

Question: What happens if we average f over $H(s,e)$?

Lemma

For any $e \in S^{d-1}$, $s \in \mathbb{R}$, and $x_0 \in H(s,e)$, the average

$$Avg(f,e,s) := \lim_{R\to\infty} (\omega_{d-1} R^{d-1})^{-1} \int_{B(x_0,R)\cap H(s,e)} f(y)\, dA(y)$$

exists and is independent of x_0. Furthermore, we have:

(i) *If $e \in \mathbb{R}\mathbb{Z}^d$, then $s' \mapsto Avg(f,e,s')$ is periodic in $\mathbb{R}$ with period r_e.*

(ii) *If $e \notin \mathbb{R}\mathbb{Z}^d$, then $Avg(f,e,s) = \int_{[0,1)^d} f(x)\, dx$ independently of s.*

Proof of (i) and (ii): Fourier series expansion.

(ii) is a manifestation of unique ergodicity

Averaging over Planes, cont.

If we think of $e \mapsto \mathrm{Avg}(f, e, \cdot)$ as a function of e, it is discontinuous — **but not too discontinuous**

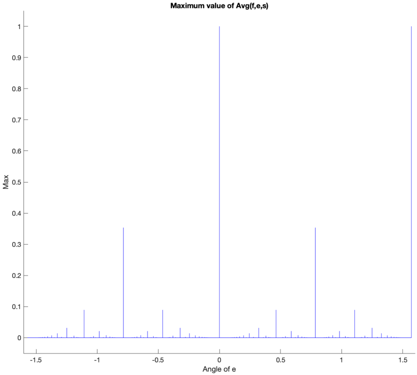
Demonstration: let $f(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \|k\|^{-3} \cos(2\pi k \cdot x)$

Notice $0 = \int_{[0,1)^2} f(x)\, dx$

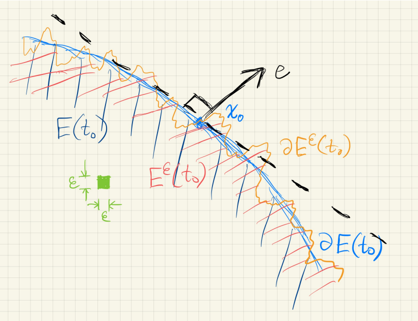
So $\mathrm{Avg}(f, e, \cdot) \equiv 0$ for **irrational** e

$\mathrm{Avg}(f, e, \cdot)$ is non-zero for all **rational** e

But $\max |\mathrm{Avg}(f, e, \cdot)| > \delta$ for at most finitely many e (BV-like property)



Back to the heuristic derivation



Averaging occurs if $e \notin \mathbb{R}\mathbb{Z}^d$

There is a probability measure $\bar{\mu}_e$ on the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ such that

$$\bar{a}(e) = \int_{\mathbb{T}^d} a(y, e)\, d\bar{\mu}_e(y)$$

and, in the picture,

$$V_{\partial E(t_0)}(x_0) = \text{tr}\left(\bar{a}(e) A_{\partial E(t_0)}(x_0)\right)$$

($\bar{\mu}_e$ is supported in $[0, 1)^d$ and AC w.r.t. Lebesgue)

$\bar{\mu}_e$ is the **invariant measure** of the diffusion equation (formally) satisfied by the fluctuation of the height h^ϵ around the black line in the figure

Rational Directions

The approach above fails if $e \in \mathbb{R}\mathbb{Z}^d$

In dimensions $d \geq 3$, $\bar{a}$ is generically discontinuous at each $e \in \mathbb{R}\mathbb{Z}^d$

Similar issues arise in other periodic homogenization problems involving surfaces
(see Feldman and Kim '17, Feldman and Smart '19)

Oscillation of $\bar{a}$ has the same **"BV-like" behavior** we saw in $\mathrm{Avg}(f, e, \cdot)$

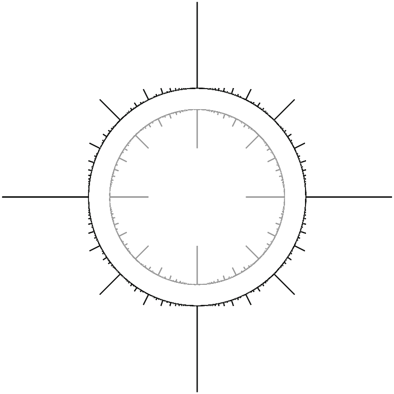


Figure from Feldman, Smart 2019

In what sense is the limiting equation satisfied?

We argued that **if** $E^\epsilon(t) \to E(t)$, **then**

$$V_{\partial E(t)} = \text{tr}\left(\bar{a}(n_{\partial E(t)}) A_{\partial E(t)}\right) \quad \text{whenever } n_{\partial E(t)} \notin \mathbb{R}\mathbb{Z}^d$$

Question: What about when $n_{\partial E(t)} \in \mathbb{R}\mathbb{Z}^d$?

How to deal with discontinuities?

Viscosity solutions theory: $\{E(t)\}_{t \geq 0}$ solves the equation if

$$V_{\partial E(t)}(x) \leq \lim_{\delta \to 0} \sup \left\{ \text{tr}\left(\bar{a}(e) A_{\partial E(t)}(x)\right) \mid \|e - n_{\partial E(t)}(x)\| < \delta \right\},$$
$$V_{\partial E(t)}(x) \geq \lim_{\delta \to 0} \inf \left\{ \text{tr}\left(\bar{a}(e) A_{\partial E(t)}(x)\right) \mid \|e - n_{\partial E(t)}(x)\| < \delta \right\}$$

How to prove this holds?

Introduce a new notion of solution adapted to what we **do** know

Solutions of the flow "in irrational directions"

Very roughly, the idea is that a flow $\{E(t)\}_{t \geq 0}$ is a **solution "in irrational directions"** if there is a $K > 0$ such that

$$\begin{aligned} V_{\partial E(t)} &= \operatorname{tr}\left(\bar{a}(n_{\partial E(t)}) A_{\partial E(t)}\right) \quad \text{whenever } n_{\partial E(t)} \notin \mathbb{R}\mathbb{Z}^d \\ |V_{\partial E(t)}| &\leq K \|A_{\partial E(t)}\| \end{aligned}$$

We just proved the "hard part," the equation for $n_{\partial E(t)} \notin \mathbb{R}\mathbb{Z}^d$

The second part follows from sending $\epsilon \to 0$ in the trivial inequality

$$|V_{\partial E^\epsilon(t)}| = \left|\operatorname{tr}\left(a(\epsilon^{-1}x, n_{\partial E^\epsilon(t)}) A_{\partial E^\epsilon(t)}\right)\right| \leq \|a\|_{L^\infty} \|A_{\partial E^\epsilon(t)}\|$$

Rough meaning of inequality: "Normal velocity is controlled by curvature"

Key point: velocity vanishes wherever boundary is flat, that is,

$$\text{if} \quad A_{\partial E(t_0)}(x_0) = 0, \quad \text{then} \quad V_{\partial E(t_0)}(x_0) = 0$$

Solution "in irrational directions" $\implies$ Viscosity solution

Suppose $\{E(t)\}_{t\geq 0}$ is a "solution in irrational directions" and fix (x_0, t_0)

Recall we want to show that

$$V_{\partial E(t_0)}(x_0) \leq \lim_{\delta\to 0} \sup \left\{ \operatorname{tr}\left(\bar{a}(e) A_{\partial E(t_0)}(x_0)\right) \mid \|e - n_{\partial E(t)}(x_0)\| < \delta \right\}$$
$$V_{\partial E(t_0)}(x_0) \geq \lim_{\delta\to 0} \inf \left\{ \operatorname{tr}\left(\bar{a}(e) A_{\partial E(t_0)}(x_0)\right) \mid \|e - n_{\partial E(t)}(x_0)\| < \delta \right\}$$

We already know this when $n_{\partial E(t_0)}(x_0) \notin \mathbb{R}\mathbb{Z}^d$ so assume it is rational

Two cases:

$$\text{(i) } A_{\partial E(t_0)}(x_0) = 0 \quad \text{and} \quad \text{(ii) } A_{\partial E(t_0)}(x_0) \neq 0$$

Case (i): If $A_{\partial E(t_0)})(x_0) = 0$, then our previous observation says $V_{\partial E(t_0)}(x_0) = 0$

Thus, desired conclusion is immediate — all sides are zero

Proof of viscosity solution property, cont.

Case (ii): If $A_{\partial E(t_0)}(x_0) \neq 0$, then $n_{\partial E(t_0)}(x) \notin \mathbb{R}\mathbb{Z}^d$ for some x closeby

By assumption, we have

$$V_{\partial E(t_0)}(x) = \mathrm{tr}\left(\bar{a}(n_{\partial E(t_0)}(x))A_{\partial E(t_0)}(x)\right), \quad \lim_{x \to x_0} n_{\partial E(t_0)}(x) = n_{\partial E(t_0)}(x_0)$$

(assuming $\partial E(t_0)$ is smooth)

Therefore,

$$\begin{aligned} V_{\partial E(t_0)}(x_0) &= \lim_{x \to x_0} V_{\partial E(t_0)}(x) \\ &= \lim_{x \to x_0} \mathrm{tr}\left(\bar{a}(n_{\partial E(t_0)}(x))A_{\partial E(t_0)}(x)\right) \\ &\leq \lim_{\delta \to 0} \sup \left\{\mathrm{tr}\left(\bar{a}(e)A_{\partial E(t)}(x_0)\right) \mid \|e - n_{\partial E(t)}\| < \delta\right\} \end{aligned}$$

Opposite inequality follows the same way

Recap

We saw that **if** $E^\epsilon(t) \to E(t)$ as $\epsilon \to 0$, **then**

$$V_{\partial E(t)} = \mathrm{tr}\left(\bar{a}(n_{\partial E(t)})A_{\partial E(t)}\right)$$

interpreted using viscosity solutions theory

But why does $E^\epsilon(t)$ converge as $\epsilon \to 0$?
Answer: More viscosity theory... (this part is **standard**)

Analogous to Leray-Schauder fixed point thm, where

"a priori estimate $\implies$ existence"

In viscosity solutions theory,

"a priori differential equation $\implies$ convergence"

(Moral: compactness comes almost "for free")

Future directions

Qualitative behavior: We derived a geometric flow with discontinuous coefficients

$$V_{\partial E(t)} = \operatorname{tr}\left(\bar{a}(n_{\partial E(t)})A_{\partial E(t)}\right)$$

(1) What can be said about regularity of solution $E(t)$?
(2) How do discontinuities manifest themselves where $n_{\partial E(t)} \in \mathbb{R}\mathbb{Z}^d$?

Quantify the convergence: How close is $E^\epsilon(t)$ to $E(t)$?

Forced problem: Add a periodic force α to the flow and look at the hyperbolically scaled problem

$$V_{\partial E^\epsilon(t)} = \epsilon \operatorname{tr}\left(a(\epsilon^{-1}x, n_{\partial E^\epsilon(t)})A_{\partial E^\epsilon(t)}\right) + \alpha(\epsilon^{-1}x)$$

What can be said about the double limit $\epsilon \to 0$, $\alpha \to 0$? What if $\alpha = O(\epsilon)$?

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