Passage from the Boltzmann equation with Diffuse Boundary to the Incompressible Euler equation with Heat Convection

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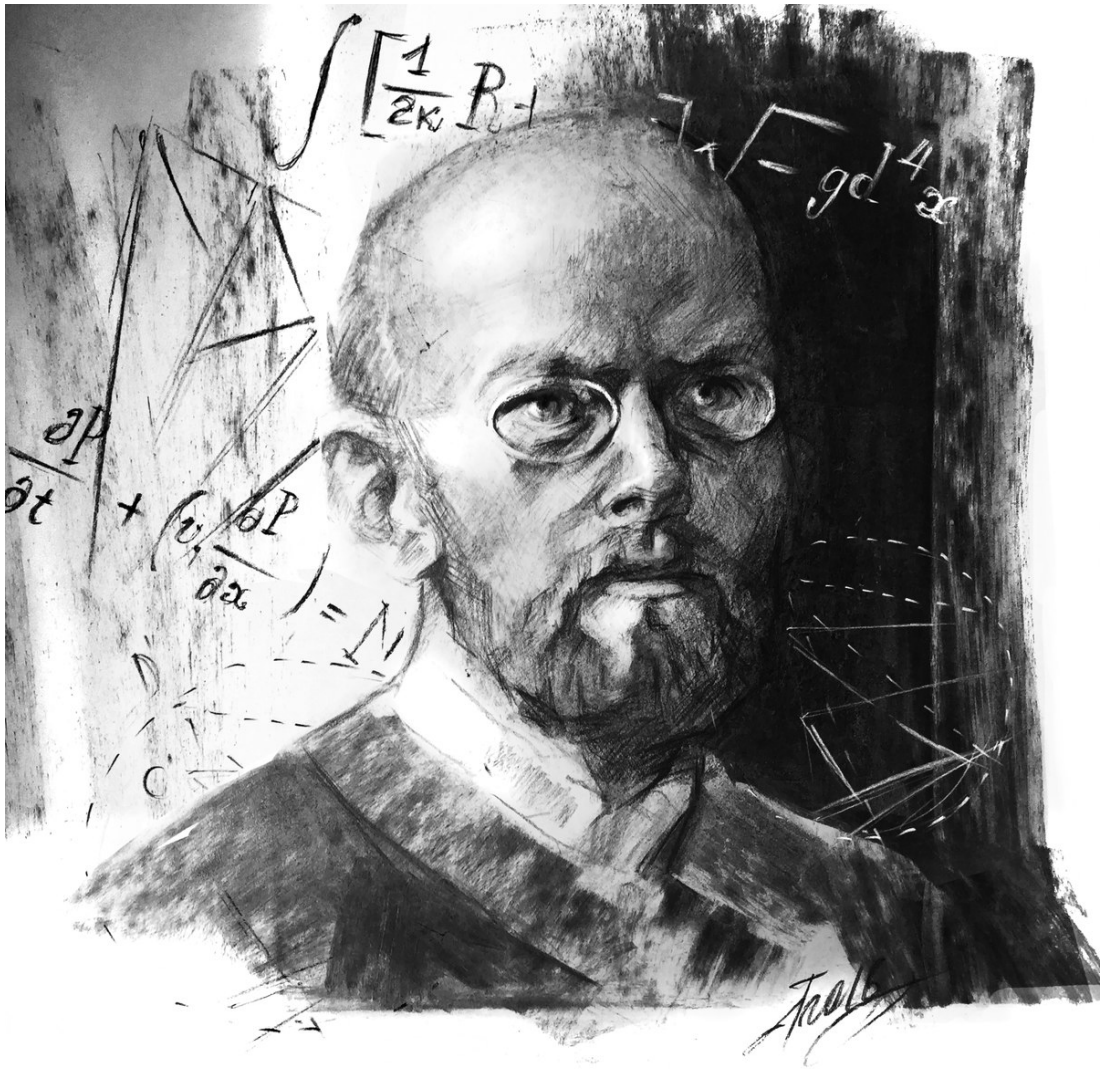
Hilbert's Sixth Problem

1900, ICM.

David Hilbert proposed 23 major mathematical problems.



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$$\begin{array}{c}\text{"atomistic view"}\\ \text{N-body system}\end{array} \Longrightarrow \begin{array}{c}\text{"Statistical Physics"}\\ \text{Kinetic theory}\end{array} \Longrightarrow \begin{array}{c}\text{"motion of a continuum"}\\ \text{Fluid mechanics}\end{array} \tag{2}$$

Unified theory of the gas dynamics including **different levels of descriptions** from a mathematical standpoint by connecting the behavior of solutions to the **mesoscopic kinetic equations** to solutions of **microscopic** models and **macroscopic** models that arise in formal limits

Kinetic Theory

The object: the modelling of a gas (or plasma, or any system made up of a large number of particles) by a distribution function in the particle phase space.

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This phase space includes macroscopic variables, i.e. the position in physical space, but also microscopic variables, which describe the state of the particles: the velocity.

Distribution Function

Consider the dynamics of rarified gas in a bounded domain $\Omega \subset \mathbb{R}^3$

$$F(t, x, v) \geq 0 \quad \text{defined on} \quad [0, T] \times \Omega \times \mathbb{R}^3.$$

Here, $\mathbb{R}^3$ is the space of velocities.

The quantity $F(t, x, v)$ stands for the density of particles at time t, position x, and with velocity v.

No external force, No interaction between particles (collision)

$$\partial_t F + v \cdot \nabla_x F = 0 \quad \text{(transport equation)}$$

Collision: The Boltzmann equation

Maxwell (1866), Boltzmann (1872) proposed the Boltzmann equation: PDE for the dilute gas taken into account interactions between particles.

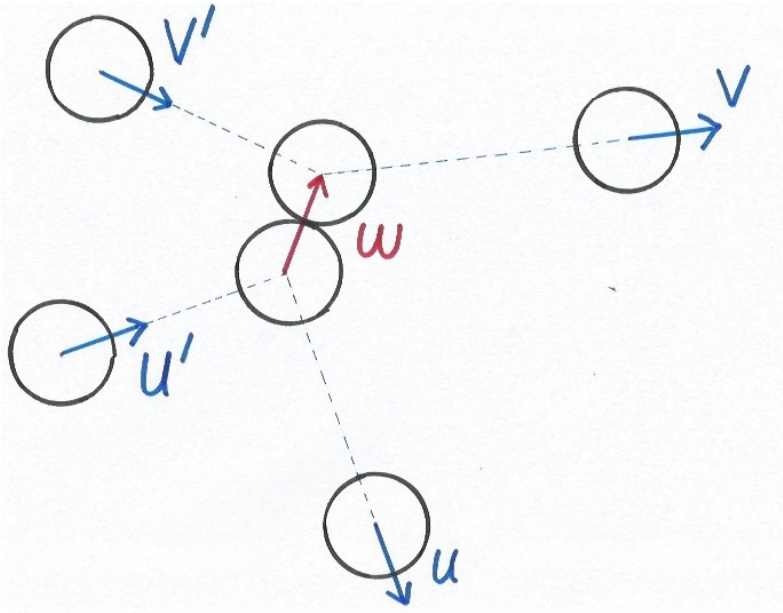
$$\partial_t F + v \cdot \nabla_x F = Q(F, F)$$

Collision operator

$$\begin{aligned} Q(F_1, F_2)(v) :=& Q_{\text{gain}}(F_1, F_2) - Q_{\text{loss}}(F_1, F_2) \\ =& \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \omega)[F_1(v')F_2(u') - F_1(v)F_2(u)] d\omega du \end{aligned}$$

- Hard sphere $B(v-u,\omega) = |(v-u)\cdot\omega|$
- 6 unknowns v', u' and 4 equations : 2 dimensional $\omega \in \mathbb{S}^2$

$$v' + u' = v + u \quad \text{(Conservation of Momentum)}$$
$$|v'|^2 + |u'|^2 = |v|^2 + |u|^2 \quad \text{(Conservation of Energy)}$$



- $v' = v - [(v-u)\cdot\omega]\omega$ and $u' = u + [(v-u)\cdot\omega]\omega$.

The dimensionless Boltzmann equation

$$\mathbf{St}\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathbf{Kn}} Q(F, F) \tag{3}$$

- Strouhal number: $\mathbf{St} \sim 1/\text{time scale}$
- Knudsen number: $\mathbf{Kn} \sim$ mean free path

Formally, as $\mathbf{Kn} \to 0$,

$$Q(F, F) \to 0. \tag{4}$$

On the other hand, $Q(F, F) = 0$ has a generic family of solutions (Maxwellian):

$$M_{R,U,T}(v) = \frac{R}{(2\pi T)^{3/2}} e^{-\frac{|v-U|^2}{2T}} \tag{5}$$

Therefore, formally,

$$F \to M_{R,U,T}(v) \text{ as } \mathbf{Kn} \to 0, \tag{6}$$

where (R, U, T) is the hydrodynamic variables.

Global Maxwellian and Mach number

Global Maxwellian

$$\mu(v) := M_{1,0,1}(v) = \frac{1}{(2\pi)^{3/2}} \exp\left\{-\frac{|v|^2}{2}\right\}. \tag{7}$$

Mach number: $\mathbf{Ma} \sim$ bulk velocity $\sim$ fluctuations around the reference state $\mu(v)$

$$F \to M_{1+\mathbf{Ma}\rho,\mathbf{Ma}u,1+\mathbf{Ma}\theta}(v) \tag{8}$$

$$\begin{aligned}&\Big(\rho(t,x), u(t,x), \theta(t,x)\Big)\\ &= \lim_{\mathbf{Ma}\downarrow 0} \frac{1}{\mathbf{Ma}} \int_{\mathbb{R}^3} \{F(t,x,v) - \mu(v)\}\Big(1, v, \frac{|v|^2-3}{\sqrt{6}}\Big)\mathrm{d}v\end{aligned}. \tag{9}$$

Reynolds number and Inviscid limit

Reynolds number **Re**:

$$\frac{1}{\mathbf{Re}} = \frac{\mathbf{Kn}}{\mathbf{Ma}} \quad \text{(von Karman relation)} \tag{10}$$

- $\mathbf{Re} \sim 1$: Viscous fluid as limit
  - $\mathbf{Kn} \sim \mathbf{Ma} \sim \mathbf{St} \to 0$: Incompressible Navier-Stokes-Fourier
- $\mathbf{Re} \gg 1$: Invisicd fluid as limit
  - $\mathbf{Kn} \ll \mathbf{Ma} \sim \mathbf{St} \to 0$: Incompressible Euler

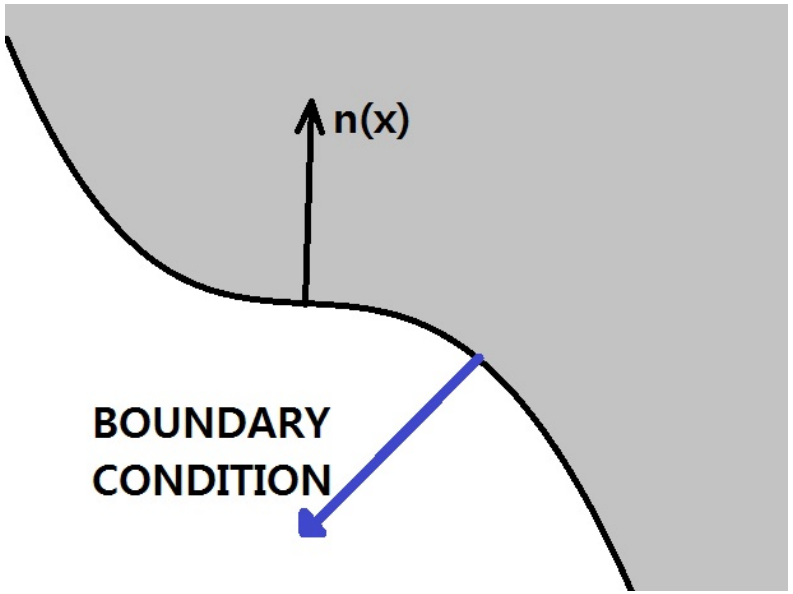
Cf. $\mathbf{Kn} \to 0$ but $\mathbf{Ma} \sim \mathbf{St} \sim 1$: Compressible Euler

Boundary conditions for kinetic equations

Incoming set:

$$\gamma_- := \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}$$

where $n(x)$ is the outward normal at $x \in \partial\Omega$.



What kind of BCs would we impose?

1879, James Clerk Maxwell: Basic Boundary Conditions

- Diffuse reflection BC:

$$F(t,x,v)|_{\gamma_-} = c_\mu \mu(v) \int_{n(x)\cdot \mathfrak{u}>0} F(t,x,\mathfrak{u})\{n(x)\cdot \mathfrak{u}\} d\mathfrak{u}, \quad (11)$$

  c_μ is chosen to be $\sqrt{2\pi}$ so that

$$c_\mu \int_{n(x)\cdot \mathfrak{u}>0} \mu(\mathfrak{u})(n(x)\cdot \mathfrak{u}) d\mathfrak{u} = 1.$$

- Null flux: $\int_{\mathbb{R}^3} F(t,x,v)\{n(x)\cdot v\}\mathrm{d}v = 0, \ x \in \partial\Omega.$
- When the particles hit the boundary and are re-emitted to the domain, they reach the equilibrium instantaneously. One can view this boundary condition as one of the ideal scattering model.

Boundary and the Major Difficulty

Kinetic BC: diffuse reflection BC (instantaneous thermal equilibration)

$$F(t,x,v)|_{\gamma_-} = c_\mu \mu(v) \int_{n(x)\cdot \mathfrak{u}>0} F(t,x,\mathfrak{u})\{n(x)\cdot \mathfrak{u}\} d\mathfrak{u}, \qquad (12)$$

for $(x,v) \in \{\partial\Omega \times \mathbb{R}^3 : n(x)\cdot v < 0\}$

- **Mismatch** occurs at the boundary even in the formal level when one seeks a limit toward an inviscid fluid with the no-penetration boundary.
- In general with $u \cdot n = 0$ on $\partial\Omega$:

$$M_{1+\mathbf{Ma}\rho,\mathbf{Ma}u,1+\mathbf{Ma}\theta}(v)$$
$$\neq c_\mu \mu(v) \int_{n(x)\cdot \mathfrak{u}>0} M_{1+\mathbf{Ma}\rho,\mathbf{Ma}u,1+\mathbf{Ma}\theta}(\mathfrak{u})\{n(x)\cdot \mathfrak{u}\} d\mathfrak{u}$$

- Fluid boundary mismatch in the inviscid limit
  - Viscous fluid: No-slip boundary condition $u|_{\partial\Omega} = 0$
  - Inviscid fluid: No-penetration boundary condition $u \cdot n|_{\partial\Omega} = 0$
  - Layer formation: Prandtl layer, ...

The incompressible Euler limit with heat transfer from the Boltzmann equation with diffuse boundary has not been established in any framework yet.

- Bouchut-Golse-Pulvirenti-Desvillettes (2000): renormalized solutions of the Boltzmann equation $\to$ dissipative solutions of the incompressible Euler, no boundary, no heat transfer
- Lions-Masmoudi (2001)
- Saint-Raymond (2003, 2009): heat transfer, Boltzmann with specular reflection BC: there is no mismatch at least in the formal level
- Bardos-Golse-Paillard (2012):

$$\text{Maxwell BC} = \alpha \times \text{Diffuse BC} + (1-\alpha) \times \text{Specular BC},$$

  where $\alpha \to 0$ as $\mathbf{Kn} \to 0$.
- Jang-Kim (submitted): Diffuse BC, no heat transfer

Intermediary approximation via Navier-Stokes-Fourier

$$\text{Boltzmann} \xrightarrow{\mathbf{Kn}\to 0} \text{Navier-Stokes-Fourier} \xrightarrow{\mathbf{Re}=\frac{\mathbf{Ma}}{\mathbf{Kn}}\to\infty} \text{Euler}$$

- Boltzmann:

$$F = \mu + \left(\mathbf{Ma} + \mathbf{Ma}^2\right)\{\text{NSF-Correction}\} + \mathbf{Ma}^{3/2}\{\text{Remainder}\}$$

- Navier-Stokes-Fourier with **Re** and no-slip BC

$$\partial_t u + u \cdot \nabla_x u - \frac{1}{\mathbf{Re}}\Delta u + \nabla_x p = 0 \quad \text{in } \Omega, \tag{13}$$

$$\nabla_x \cdot u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{14}$$

$$\partial_t \theta + u \cdot \nabla_x \theta - \frac{1}{\mathbf{Re}}\Delta\theta = 0 \quad \text{in } \Omega, \tag{15}$$

$$\rho + \theta = 0 \quad \text{in } \Omega, \quad \theta = 0 \quad \text{on } \partial\Omega. \tag{16}$$

- We can avoid the boundary mismatch
- We need the inviscid limit: Navier-Stokes-Fourier $\to$ Euler with heat transfer

Inviscid limit: $(u, \theta) \to (u_E, \theta_E)$ in real analytic spaces

Both u and u_E are divergence-free and

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \frac{1}{\mathbf{Re}} \Delta u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$
$$\partial_t \theta + u \cdot \nabla_x \theta - \frac{1}{\mathbf{Re}} \Delta \theta = 0 \text{ in } \Omega, \quad \theta = 0 \text{ on } \partial\Omega,$$
$$\partial_t u_E + u_E \cdot \nabla_x u_E + \nabla_x p_E = 0 \text{ in } \Omega, \quad u_E \cdot n = 0 \text{ on } \partial\Omega,$$
$$\partial_t \theta_E + u_E \cdot \nabla_x \theta_E = 0 \text{ in } \Omega.$$

- strong regularity such as analyticity at least near the boundary
  - Prandtl expansion: Sammartino-Caflisch (1998)
  - Green's function approach using the boundary vorticity formulation of Maekawa→Kato's creterion: Nguyen-Nguyen (2018), Kukavica-Vicol-Wang (2020)
- cf. certain symmetry assumption on the domain and data

Main result (C-Jang-Kim Informal statement)

We consider a half space in 3D

$\Omega := \mathbb{T}^2 \times \mathbb{R}_+ \ni (x_1, x_2, x_3)$, where $\mathbb{T}$ is a periodic interval of $(-\pi, \pi)$.

For some choice of **Ma** and **Kn** $=$ **Kn**(**Ma**), there exists a large set of initial data u_{in}, θ_{in} (in real analytic space), F_{in} such that a unique solution $F(t, x, v)$ satisfies, for $0 < T \ll 1$,

$$\frac{F(t,x,v) - \mu(v)}{\mathbf{Ma}\sqrt{\mu(v)}} \longrightarrow$$
$$\left(-\theta_E(t,x) + u_E(t,x) \cdot v + \theta_E(t,x)\frac{|v|^2 - 3}{2}\right)\sqrt{\mu(v)}$$

in $L^\infty((0, T); L^2(\Omega \times \mathbb{R}^3))$ as $\mathbf{Ma} \to 0$.

Key#1 of Boltzmann to NS: New Hilbert expansion around the global maxwellian

- **Ma** $= \varepsilon =$ **St** and **Kn** $= \kappa\varepsilon$ with $\kappa \downarrow 0$ as $\varepsilon \downarrow 0$
  - $\varepsilon\partial_t F + v \cdot \nabla_x F = \frac{1}{\kappa\varepsilon} Q(F, F)$
  - For $\mu = M_{1,0,1}$

$$F = \mu + \varepsilon f_1 \sqrt{\mu} + \varepsilon^2 f_2 \sqrt{\mu} + \varepsilon^{3/2} f_R \sqrt{\mu} \tag{17}$$

- $Lf = \frac{-2}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f)$ and $\Gamma(f, g) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} g)$.
  - $\mathcal{N}$: Null space of L is spanned by $\{\sqrt{\mu}, v\sqrt{\mu}, \frac{|v|^2-3}{\sqrt{6}}\sqrt{\mu}\}$. Denote $\mathbb{P}g$ to be projection of g in $\mathcal{N}$.
  - Image of L and $\Gamma \perp \mathcal{N}$, L is self-adjoint
  - L is Fredholm: $Lf = g$ is solvable if $g \in \mathcal{N}^\perp$ (and uniquely $f \in \mathcal{N}^\perp$)

$$
\begin{aligned}
&\partial_t f_R + \frac{1}{\varepsilon} v \cdot \nabla_x f_R + \frac{1}{\varepsilon^2 \kappa} L f_R - \frac{1}{\varepsilon^{1/2} \kappa} \Gamma(f_R, f_R) \\
&= -\frac{1}{\varepsilon^{5/2} \kappa} L f_1 && (18) \\
&\quad - \frac{1}{\varepsilon^{3/2}} \left\{ v \cdot \nabla_x f_1 - \frac{1}{\kappa} \Gamma(f_1, f_1) + \frac{1}{\kappa} L f_2 \right\} && (19) \\
&\quad - \frac{1}{\varepsilon^{1/2}} \left\{ \partial_t f_1 + v \cdot \nabla_x f_2 - \frac{2}{\kappa} \Gamma(f_1, f_2) \right\} && (20) \\
&\quad - \varepsilon^{1/2} \partial_t f_2 + \frac{2}{\varepsilon \kappa} \Gamma(f_1, f_R) + \frac{2}{\kappa} \Gamma(f_2, f_R) + \frac{\varepsilon^{1/2}}{\kappa} \Gamma(f_2, f_2). && (21)
\end{aligned}
$$

- $(18) = 0$: $f_1 = \left(\rho + u \cdot v + \theta \frac{|v|^2 - 3}{\sqrt{6}} \right) \sqrt{\mu}$
- $(19) = 0$: $v \cdot \nabla_x f_1 \in \mathcal{N} \to \nabla_x \cdot u = 0$ and $\nabla(\rho + \theta) = 0$
  $(\mathbb{I} - \mathbb{P}) f_2 = -\kappa L^{-1}(v \cdot \nabla_x f_1) + L^{-1}(\Gamma(f_1, f_1))$

- $$\begin{aligned}\mathbb{P}(20) =&\mathbb{P}\big(\partial_t f_1 + v\cdot\nabla_x f_2\big)\\ =&\partial_t \mathbb{P} f_1 + v\cdot\nabla_x \mathbb{P} f_2 + v\cdot\nabla_x(\mathbb{I}-\mathbb{P})f_2\end{aligned} \tag{22}$$

- $$\begin{aligned}(\mathbb{I}-\mathbb{P})f_2 =& -\kappa L^{-1}\big(v\cdot\nabla_x f_1\big) + L^{-1}\big(\Gamma(f_1,f_1)\big)\\ \mathbb{P} f_2 =&(\rho_2 + u_2\cdot v + \theta_2\frac{|v|^2-3}{2})\sqrt{\mu}\end{aligned} \tag{23}$$

  with $\nabla\cdot u_2 = -\partial_t\rho$, $\rho_2+\theta_2-\frac{|u|^2}{3} = p$.

- $$\begin{aligned}\langle v\sqrt{\mu}, \partial_t f_1 + v\cdot\nabla_x f_2\rangle =&\partial_t u - \kappa\eta_0\Delta_x u + u\cdot\nabla_x u + \nabla_x p\\ \langle\frac{|v|^2-5}{2}\sqrt{\mu}, \partial_t f_1 + v\cdot\nabla_x f_2\rangle =&\partial_t\theta - \kappa\eta_1\Delta_x\theta + u\cdot\nabla_x\theta\end{aligned} \tag{24}$$

- $\mathbb{P}(20) = 0$ if (u,θ) solve the Navier-Stokes-Fourier system

Remark in the Hilbert expansion

- $F = \mu + \varepsilon f_1 \sqrt{\mu} + \varepsilon^2 f_2 \sqrt{\mu} + \varepsilon^{3/2} f_R \sqrt{\mu}$
- $f_1 = \left(\rho + u \cdot v + \theta \frac{|v|^2 - 3}{2}\right) \sqrt{\mu}$
- boundary condition: $f_R = \text{Diffuse BC}(f_R) + O(\varepsilon^{1/2})$
- $$\begin{aligned} &\partial_t f_R + \frac{1}{\varepsilon} v \cdot \nabla_x f_R + \frac{1}{\varepsilon^2 \kappa} L f_R - \frac{1}{\varepsilon^{1/2} \kappa} \Gamma(f_R, f_R) \\ &= - \frac{1}{\varepsilon^{1/2}} (\mathbb{I} - \mathbb{P}) \left(v \cdot \nabla_x f_2 - \frac{2}{\kappa} \Gamma(f_1, f_2) \right) + (21) \end{aligned}$$
- (21) contains $\nabla_x u$, $\nabla_x \theta$, $\partial_t \nabla_x u$, and $\partial_t \nabla_x \theta$.

Key#2: $\mathbf{P}f_R$ in $L^\infty_t L^6_x$

- $\partial_t f_R + \frac{1}{\varepsilon} v \cdot \nabla_x f_R + \frac{1}{\varepsilon^2 \kappa} L f_R - \frac{1}{\varepsilon^{1/2}\kappa}\Gamma(f_R, f_R) = \frac{2}{\varepsilon\kappa}\Gamma(f_1, f_R) + \cdots$
- If we expand $F = \mu + \varepsilon f_1 \sqrt{\mu} + \varepsilon^2 f_2 \sqrt{\mu} + \varepsilon^{3/2} f_R \sqrt{\mu}$ more then singularity of the nonlinear term would become less significant.
- But it would cost another boundary layer since $(\mathbb{I} - \mathbb{P}) f_2$ is already determined and f_2 likely does not honor the diffuse BC.

$$
\begin{aligned}
&\|f_R(t)\|^2_{L^2_{x,v}} + \frac{1}{\varepsilon^2\kappa}\int_0^t \|(\mathbb{I}-\mathbb{P})f_R\|^2_{L^2_{x,v}} \\
\lesssim &\frac{\varepsilon^{1/2}}{\kappa^{1/2}}\|\mathbb{P}f_R\|_{L^\infty_t L^6_{x,v}}\|\mathbb{P}f_R\|_{L^2_t L^3_{x,v}}\frac{1}{\varepsilon\kappa^{1/2}}\|(\mathbb{I}-\mathbb{P})f_R\|_{L^2_{t,x,v}} \\
&+\frac{1}{\kappa^{1/2}}\|f_1\|_{L^\infty_{x,v}}\|f_R\|_{L^2_{t,x,v}}\frac{1}{\varepsilon\kappa^{1/2}}\|(\mathbb{I}-\mathbb{P})f_R\|_{L^2_{t,x,v}}
\end{aligned}
\tag{25}
$$

- $\|\mathbb{P}f_R\|_{L^2_t L^3_{x,v}}$: Average lemma $H^{1/2} \subset L^3$ in 3D
- $\|\mathbb{P}f_R\|_{L^\infty_t L^6_{x,v}}$: $v\cdot\nabla_x f_R = -\varepsilon\partial_t f_R - \frac{1}{\varepsilon\kappa}Lf_R + \cdots$
  - test function method
  - $\partial_t f_R$-estimate requires $\partial_t^2\nabla_x u$, $\partial_t^2\nabla_x\theta$ estimate
- Gronwall's inequality: $\|f_R(t)\|^2_{L^2_{x,v}} \lesssim e^{t/\kappa}$.
- By choosing κ such that $\varepsilon^{1/2}e^{1/\kappa}\downarrow 0$ as $\varepsilon\downarrow 0$,

$$
\frac{1}{\varepsilon}|F-(\mu+\varepsilon f_1\sqrt{\mu})| \lesssim \varepsilon^{1/2}|f_R| \downarrow 0 \tag{26}
$$

NS to Euler

- NS $\to$ Euler can be justified in real analytic space (Sammartino-Caflisch 1998)
- Vorticity formulation of Maekawa:
  $\omega = \nabla \times u, \quad u = \nabla \times (-\Delta)^{-1}\omega$

$$
\begin{aligned}
\partial_t \omega - \kappa\eta_0 \Delta\omega &= -u \cdot \nabla\omega + \omega \cdot \nabla u \ \text{ in } \ \Omega, \\
\omega\,|_{t=0} &= \omega_{in} \ \text{ in } \ \Omega, \\
\kappa\eta_0(\partial_{x_3} + \sqrt{-\Delta_h})\omega_h &= [\partial_{x_3}(-\Delta)^{-1}(-u \cdot \nabla\omega_h + \omega \cdot \nabla u_h)] \\
\omega_3 &= 0 \ \text{ on } \ \partial\Omega,
\end{aligned}
$$

where $\sqrt{-\Delta_h} = |\nabla_h|$ is defined as

$$
\sqrt{-\Delta_h} g(x_h, x_3) = \sum_{\xi \in \mathbb{Z}^2} |\xi| g_\xi(x_3) e^{i x_h \cdot \xi}. \tag{27}
$$

- Green's function approach (Nguyen-Nguyen)

$$\partial_t \omega_\xi - \kappa\eta_0 \Delta_\xi \omega_\xi = N_\xi \quad \text{in } \mathbb{R}_+, \tag{28}$$

$$\kappa\eta_0(\partial_{x_3} + |\xi|)\omega_{\xi,h} = B_\xi, \quad \omega_{\xi,3} = 0 \quad \text{on } x_3 = 0, \tag{29}$$

$$\partial_t \theta_\xi - \kappa\eta_c \Delta_\xi \theta_\xi = M_\xi \text{ in } \mathbb{R}_+, \ \theta_\xi = 0 \text{ on } x_3 = 0, \tag{30}$$

with $\omega_\xi|_{t=0} = \omega_{0\xi}$, $\theta_\xi|_{t=0} = \theta_{0\xi}$ for $\xi \in \mathbb{Z}^2$. Here $\Delta_\xi = -|\xi|^2 + \partial_{x_3}^2$.

$$N_\xi := (-u \cdot \nabla\omega + \omega \cdot \nabla u)_\xi (t, x_3), \ M_\xi := (-u \cdot \nabla\theta)_\xi (t, x_3),$$

$$B_\xi := (\partial_{x_3}(-\Delta_\xi)^{-1} N_{\xi,h}(t))|_{x_3=0}$$

$$\begin{aligned} \omega_\xi(t, x_3) = &\int_0^\infty G_\xi(t, x_3, y)\omega_{0\xi}(y)\mathrm{d}y \\ &+ \int_0^t \int_0^\infty G_\xi(t-s, x_3, y) N_\xi(s, y)\mathrm{d}y\mathrm{d}s \\ &- \int_0^t G_\xi(t-s, x_3, 0)(B_\xi(s), 0)\mathrm{d}s. \end{aligned} \tag{31}$$

- Key#1: Higher regularity of ω with the aid of boundary layer weight

$$\phi_\kappa(z) := \frac{1}{\sqrt{\kappa}}\phi(\frac{z}{\sqrt{\kappa}}) \text{ with } \phi(z) = \frac{1}{1+|\mathrm{Re}\, z|^{\mathfrak{r}}} \text{ for some } \mathfrak{r} > 1, \tag{32}$$

and initial-boundary layer

$$\phi_{\kappa t}(z) = \frac{1}{\sqrt{\kappa t}}\phi(\frac{z}{\sqrt{\kappa t}}). \tag{33}$$

- Key#2: Conormal derivative of $\theta \implies \frac{1}{x_3}$ singularity. Directly target the ∂_{x_3} derivative + analytic recovery lemma gives

$$\|\partial_t \nabla_x^2 \theta\|_{L^\infty_{t,x}} \sim \frac{1}{\kappa}.$$

Thank you !