On ill- and well-posedness of dissipative martingale solutions to stochastic 3D Euler equations

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based on a joint work with R. Zhu and X. Zhu

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- flow of an inviscid fluid
- system of PDEs derived from the basic physical principles
  - conservation of mass + conservation of linear momentum

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p &= f \\ \operatorname{div} u &= 0 \qquad\qquad x \in \mathbb{T}^3, t \in (0,T) \\ u(0) &= u_0 \end{aligned}$$

- velocity $u\colon [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$, pressure $p\colon [0,T] \times \mathbb{T}^3 \to \mathbb{R}$
- source term of external force $f\colon [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$

Existence? Uniqueness?

- strong solutions exist only locally in time and are unique
- weak solutions can be constructed by convex integration, exist globally in time, non-unique
- weak–strong uniqueness principle

- consider a vanishing viscosity approximation (through a Navier–Stokes system)

$$\begin{aligned} \partial_t u^\nu + \mathrm{div}\,(u^\nu \otimes u^\nu) + \nabla p^\nu &= \nu \Delta u^\nu \\ \mathrm{div}\, u^\nu &= 0 \qquad\qquad x \in \mathbb{T}^3, t \in (0,T) \\ u^\nu(0) &= u_0 \end{aligned}$$

- assume u^ν is smooth – test the equation by u^ν

$$\langle \partial_t u^\nu, u \rangle + \langle \mathrm{div}\,(u^\nu \otimes u^\nu), u^\nu \rangle + \langle \nabla p^\nu, u^\nu \rangle = \nu \langle \Delta u^\nu, u^\nu \rangle$$

$$\Rightarrow \qquad \frac{1}{2}\partial_t \|u^\nu\|^2_{L^2_x} + \nu \|\nabla u^\nu\|^2_{L^2_x} \leqslant 0$$

- Leray solutions to NSE exist globally in time and satisfy the **energy inequality**
- uniform bounds in $L^\infty(0,T;L^2_x)$, resp. $C_{\mathrm{weak}}(0,T;L^2_x)$
- compactness (up to a subsequence) in $L^\infty(0,T;L^2_x)$ with weak-star topology
- $u^\nu \otimes u^\nu$ bounded in $L^\infty(0,T;L^1_x)$ converges weak-star in $L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}}))$
- the limit is a **dissipative solution** to Euler

De Lellis–Szekelyhidi Jr. '10 *There exists a bounded and compactly supported divergence-free vector field* u_0 *giving raise to infinitely many weak solutions satisfying the energy inequality.*

Daneri–Szekelyhidi Jr. '17 *The set of initial data giving raise to infinitely many weak solutions satisfying the energy inequality and having Hölder regularity* $1/5-\varepsilon$ *is dense in* L^2_x.

Buckmaster–De Lellis–Szekelyhidi Jr.–Vicol '18 *For every* $\beta < 1/3$ *and a given smooth energy profile* $e\colon [0,T] \to (0,\infty)$ *there exists a weak solution satisfying*

$$e(t) = \int_{\mathbb{T}^3} |v(t,x)|^2 dx$$

and having Hölder regularity β*. (Onsager's conjecture)*

Daneri–Runa–Szekelyhidi Jr. '20 *For every* $\beta < 1/3$*, non-uniqueness holds for a dense set of initial data and weak solutions satisfying the energy inequality and having Hölder regularity* β*.*

- weak-solutions with energy inequality satisfy the weak-strong uniqueness principle

Stochastic perturbations

$$\begin{aligned} \mathrm{d}u + [\mathrm{div}\,(u \otimes u) + \nabla p]\mathrm{d}t &= G(u)\mathrm{d}W \\ \mathrm{div}\, u &= 0 \end{aligned} \qquad x \in \mathbb{T}^3, t \in (0,T)$$

- a Brownian motion W, suitable coefficient G
- hope in the SPDE community that a noise can help with the well-posedness issue

Regularization by noise:

- damping – no explosion with large probability (Glatt-Holtz–Vicol '14)

$$G(u) = \alpha u$$

- but this system can be transformed to a deterministic setting by simple transformation
  - negative results from previous slide apply

- a unified solution theory to study well-posedness from various perspectives

Dissipative martingale solutions

1. **Global existence** - relies on a compactness argument and Skorokhod representation
2. **Weak–Strong uniqueness** - analytically strong solution coincides with all dissipative martingale starting from the same initial condition
3. **Non-uniqueness in law** - based on convex integration and general probabilistic extension of solutions from [HZZ19]
4. **Existence of a strong Markov solution** - relies on the Markov selection procedure by Krylov
5. **Non-uniqueness of strong Markov solutions** - combination of 3. and 4.

- more information than just velocity and pressure needed
- rewrite the Euler system as

$$\mathrm{d}u + \mathrm{div}\,\mathfrak{R}\,\mathrm{d}t + \nabla p \mathrm{d}t = G\mathrm{d}W, \qquad \mathrm{div}\,u = 0, \qquad u(0) = u_0$$

- a new matrix-valued variable $\mathfrak{R}$ - part of the solution with compatibility condition

$$\mathfrak{N} := \mathfrak{R} - u \otimes u \geqslant 0$$

- towards energy inequality: introduce a new variable z

$$\mathrm{d}z = 2\langle u, G\mathrm{d}W\rangle + \|G\|^2_{L_2(U,L^2)}\mathrm{d}t, \qquad z(0) = z_0$$

  - for regular weak solutions $z(t) = \|u(t)\|^2_{L^2_x}$
- postulate compatibility condition

$$\int_{\mathbb{T}^3} \mathrm{d}\,\mathrm{tr}\mathfrak{R}(t) = \|u(t)\|^2_{L^2_x} + \int_{\mathbb{T}^3} \mathrm{d}\,\mathrm{tr}\mathfrak{N}(t) \leqslant z(t)$$

Dissipative martingale solution

- is a probability law of

$$\left(u, y(\cdot) := y_0 + \int_0^{\cdot} \mathfrak{R}(s)\mathrm{d}s, z\right)$$

  satisfying the above conditions
- for **weak–strong uniqueness**: $z(0) = \|u(0)\|_{L^2}^2$ needed (no initial energy sink)
- for **Markov selection**: the general case of $z(0) = z_0 \geqslant \|u(0)\|_{L^2}^2$ needed
- **convex integration**: infinitely many probabilistically strong and analytically weak solutions up to a stopping time with energy inequality
- **probabilistic construction**: extension to $[0, \infty)$ as dissipative martingale solutions

$$\mathrm{d}u + [\mathrm{div}\,(u \otimes u) + \nabla p]\mathrm{d}t = G\mathrm{d}W\,, \qquad \mathrm{div}\, u = 0$$

- consider a stopping time τ to control certain norms of the noise GW
- let $W_\tau(\cdot) = W(\cdot \wedge \tau)$ and $v = u - GW_\tau$ then

$$\partial_t v + \mathrm{div}\,((v + GW_\tau) \otimes (v + GW_\tau)) + \nabla p = 0, \qquad \mathrm{div}\, v = 0$$

- solved on $[0, T]$, coincides with the original equation on $[0, \tau]$
- the desired energy equality

$$\frac{1}{2}\|(v + GW_\tau)(t)\|^2_{L^2_x} = \frac{1}{2}\|u_0\|^2_{L^2_x} + \int_0^t \langle v + GW_\tau, G\mathrm{d}W_\tau\rangle + \left(\frac{1}{2} - \frac{1}{\ell}\right)(t \wedge \tau)\|G\|^2_{L_2(U, L^2)}$$

H., Zhu, Zhu '20 *There exists $u_0 \in L^2_{\mathrm{div}}$ giving raise to infinitely many probabilistically strong and analytically weak solutions $v = v_\ell$ satisfying the above energy equality a.s. for a.e. $t \in (0, T)$, for $t = 0$ and for any given strictly positive stopping time σ.*

Convex integration solutions:

- exist on a given probability space
- existence up to a stopping time τ not very satisfactory
- how to extend these solutions to $[0,\infty)$?
- convex integration does not solve the Cauchy problem! – cannot connect them

Dissipative martingale solutions obtained by compactness:

- exist for every given $u_0 \in L^2_{\mathrm{div}}$ but not on a given probability space

Idea:

- work on the level of induced probability laws $P = \mathrm{Law}[u, y, z]$
- transfer the convex integration solutions to the canonical path space (difficulty for τ)
- measurable selection from all the dissipative martingale solutions

H., Zhu, Zhu '20 *Let $T > 0$ be given. Dissipative martingale solutions are not unique, i.e. non-uniqueness in law holds on $[0,T]$.*

Ideas:

- for every $\ell \in [2,\infty]$ there is a dissipative martingale solution P_ℓ on $[0,\infty)$ such that P_ℓ-a.s.

$$\frac{1}{2}\|u(t)\|_{L^2_x}^2 = \frac{1}{2}\|u_0\|_{L^2_x}^2 + \int_0^t \langle u, G\,\mathrm{d}W_\tau\rangle + \left(\frac{1}{2} - \frac{1}{\ell}\right)(t\wedge\tau)\|G\|_{L_2(U,L^2)}^2$$

- hence they are different!

Thanks for your attention!