CNA Seminar
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Grain boundaries in minimal planar N-partitions for large N

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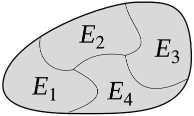
Summary

- Minimal partitions and Hales's Honeycomb Theorem
- Uniform energy distribution for minimal partitions
- Towards a description of the structure of minimal partitions: emergence of grains

Partitions

Let Ω be a two-dimensional domain with finite area.



N=4

An N-*partition* of Ω is a collection $\mathcal{E} = \{E_1, \ldots, E_N\}$ of closed sets in Ω (called *cells* of the partition)

- with pairwise disjoint interiors and union Ω,
- with equal area $|E_i| = |\Omega|/N$,
- and sufficiently regular boundaries.

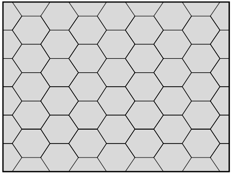
The *perimeter* of a partition $\mathcal{E}$ is

$$\begin{aligned}\mathrm{Per}(\mathcal{E}) &:= \mathrm{length}\big(\partial E_1 \cup \cdots \cup \partial E_n\big) \\ &= \frac{1}{2}\sum_{i=1}^{N} \mathrm{length}(\partial E_i) + \frac{1}{2}\mathrm{length}(\partial \Omega) \\ &= \frac{1}{2}\sum_{i=1}^{N} \mathrm{Per}(E_i) + \frac{1}{2}\mathrm{Per}(\Omega)\end{aligned}$$

- Ω admits a minimal N-partition for every integer N;
- the local structure of minimal N-partitions is simple;
- computing minimal N-partitions is challenging.

Hales's Honeycomb Theorem [Hales 2001]

Let Ω be a flat torus which admits a regular hexagonal N-partition $\mathcal{E}_{\mathrm{hex}}$.



N=48

Then $\mathcal{E}_{\mathrm{hex}}$ is the unique minimal N-partition of Ω.

- Not all flat tori admit a regular hexagonal partition.
- No counterpart in higher dimension!

Key tool: Hales's isoperimetric inequality

Simplified version (polygons only):

- let E be an n-polygon with area 1,
- let R_n be the regular n-polygon with area 1,
- let $H = R_6$ be the regular hexagon with area 1.

Then:

$$\mathrm{Per}(E) \geq \mathrm{Per}(R_n) \geq \mathrm{Per}(H) - c(n-6)$$

because $n \mapsto \mathrm{Per}(R_n)$ is a convex function (!)

We can do better:

$$\mathrm{Per}(E) \geq \mathrm{Per}(H) - c(n-6) + \delta\big(\mathrm{dist}(E,H)\big)^2$$

where $\mathrm{dist}(E,H)$ is *for example* the Hausdorff distance of E from the closest regular hexagon H.

Idea of proof of Hales's theorem

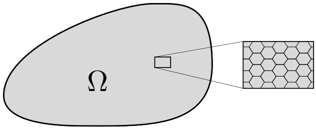
Let $\mathcal{E}$ be an N-partition of Ω with E_i an n_i-polygon:

$$\begin{aligned}
\mathrm{Per}(\mathcal{E}) &= \frac{1}{2}\sum_i \mathrm{Per}(E_i) \\
&\geq N \cdot \frac{1}{2}\mathrm{Per}(H) - \frac{c}{2}\sum_i (n_i - 6) + \frac{\delta}{2}\sum_i \big(\mathrm{dist}(E_i, H)\big)^2 \\
&= N \cdot \frac{1}{2}\mathrm{Per}(H) + \frac{\delta}{2}\sum_i \big(\mathrm{dist}(E_i, H)\big)^2 \\
&\geq N \cdot \frac{1}{2}\mathrm{Per}(H) = \mathrm{Per}(\mathcal{E}_{\mathrm{hex}})\,.
\end{aligned}$$

We set $\sigma := \frac{1}{2}\mathrm{Per}(H) = \sqrt[4]{12}$.

Minimal N-partitions of planar domains

We let Ω be an arbitrary planar domain with finite area,



and consider a minimal N-partition $\mathcal{E}$ of Ω with very large N.

- We expect that most cells are close to regular hexagons $\longrightarrow$ local hexagonal patterns.
- We expect some "disturbance" close to the boundary of Ω. Does such disturbance decay away from the boundary?
- Is the orientation of the local hexagonal pattern "constant"? If not, is it "piecewise constant"? $\longrightarrow$ emergence of "grains"

Uniform energy distribution

- We can prove a uniform distribution of energy in the spirit of [A+Choksi+Otto 2009].
- Purpose: show that "most" cells are close to be regular hexagons in a quantified way.
- It is convenient to replace N with the length parameter

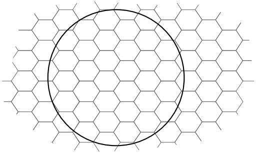
$$\varepsilon := \sqrt{|\Omega|/N}\,;$$

ε^2 is the area of the cells of the partition, now called ε-partition.

Theorem (Energy distribution of the hexagonal partition)

Let $\mathcal{E}_{\text{hex}}$ be the regular hexagonal partition of the plane with cells of area 1. Then the "energy density" of $\mathcal{E}_{\text{hex}}$ is $\sigma := \sqrt[4]{12}$. More precisely, for every disc $B = B(x, r)$ with $r \gg 1$:

$$\mathrm{Per}_B(\mathcal{E}_{\text{hex}}) = \sigma\,\mathrm{area}(B) + O(r^{2/3})$$



- Statement similar to Gauss's Circle Theorem.
- Proof by Fourier transform.

By scaling: if $\mathcal{E}_{\text{hex}}^{\varepsilon}$ is the regular hexagonal ε-partition of the plane, then for every disc $B = B(x, r)$ with $r \gg \varepsilon$

$$\mathrm{Per}_{B}(\mathcal{E}_{\text{hex}}^{\varepsilon}) = \frac{\sigma}{\varepsilon}\,\mathrm{area}(B) + O(\varepsilon^{1/3} r^{2/3})$$

Theorem (Uniform distribution of energy)
Let $\mathcal{E}_\varepsilon$ *be minimal* ε*-partitions of* Ω*. Let* $B_\varepsilon = B(x_\varepsilon, r_\varepsilon)$ *be discs in* Ω *with* $r_\varepsilon \gg \varepsilon$ *and* $\mathrm{dist}(B_\varepsilon, \partial\Omega) \gg \varepsilon$*. Then*

$$\mathrm{Per}_{B_\varepsilon}(\mathcal{E}_\varepsilon) = \frac{\sigma}{\varepsilon}|B_\varepsilon| + O(r_\varepsilon)\,.$$

- Proof of lower bound based Hales inequality.
- Proof of upper bound based on “cut and paste” technique.
- A precise statement depends on the variant of the problem considered.

Towards a precise description of minimal ε-partitions

- Recall the questions: *In the limit $\varepsilon \to 0$, is the orientation of the local hexagonal pattern constant?*
  Is it piecewise constant?

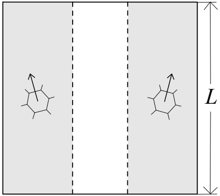
- Consider the "excess energy":

$$F_\varepsilon(\mathcal{E}) := \varepsilon \operatorname{Per}(\mathcal{E}) - \sigma|\Omega|\,;$$

  ideally, we would like to write the Γ-limit of F_ε as $\varepsilon \to 0$.
  What should be the variable of such Γ-limit?
  Guess: the "limit" of the orientation of the local hexagonal patterns.

- We did not write the Γ-limit, but we identified and partially addressed two key questions ("cell problems").

Excess energy due to change of orientation



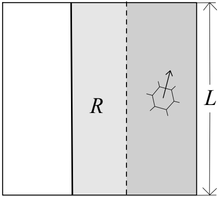
- Consider a square of side-length $L \gg 1$;
- consider all 1-partitions $\mathcal{E}$ which are prescribed in the grey zone, as in the picture, and satisfy suitable periodic conditions at bottom and top;
- $\theta :=$ angle between imposed orientations.

Define

$$\Phi(\theta) := \liminf_{L \to +\infty} \frac{1}{L} \Big\{ \inf_{\mathcal{E}} F_1(\mathcal{E}) \Big\} .$$

- Explicit construction gives $\Phi(\theta) = O\big(\theta |\log \theta|\big)$;
- Is $\Phi(\theta) > 0$? Presumably yes, partial proof.
- Is $\Phi(\theta)$ superlinear in $\theta = 0$? Presumably yes, no proof.

Excess energy due to boundary



- Consider a square of side-length $L \gg 1$ and the rectangle R as in the picture;
- consider all 1-partitions of R which are prescribed in the grey zone, as in the picture;
- $\theta :=$ angle between the imposed orientation and the vertical direction.

Define

$$\Phi_b(\theta) := \liminf_{L \to +\infty} \frac{1}{L} \left\{ \inf_{\mathcal{E}} F_1(\mathcal{E}) \right\}.$$

- Hales isoperimetric inequality gives $C \leq \Phi_b \leq C'$.
- Does $\Phi_b(\theta)$ depends on θ? Presumably yes, no proof.

Conclusions

- If $\Phi_b(\theta)$ does NOT depend on θ, then minimal ε-partitions of Ω have constant orientation (in the limit $\varepsilon \to 0$).
- If $\Phi_b(\theta)$ depends on θ, and Φ is strictly positive then minimal ε-partitions of Ω may not have constant orientation.
- If in addition $\Phi(\theta)$ is super-linear at 0 then the orientation of minimal ε-partitions is "piecewise constant".
  $\longrightarrow$ emergence of "grains".

Thanks for the attention!

Essential references

[A+Caroccia+DelNin] Giovanni Alberti; Marco Caroccia; Giacomo Del Nin. Paper in preparation.

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[A+Choksi+Otto 2009] Giovanni Alberti, Rustum Choksi, Felix Otto: Uniform energy distribution for an isoperimetric problem with long-range interactions. *J. Amer. Math. Soc.*, 22 (2009), 569-605.

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[Müller+Scardia+Zeppieri 2003] Stefan Müller, Lucia Scardia, Caterina Ida Zeppieri: Geometric rigidity for incompatible fields, and an application to strain-gradient plasticity. *Indiana Univ. Math. J.* 63 (2014), 1365-1396.